Lattice for Precision Physics

(Justus) Tobias Tsang

j.t.tsang@liverpool.ac.uk

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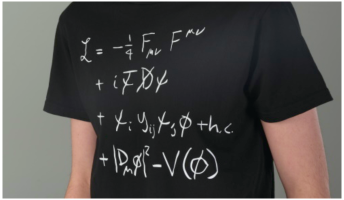


Disclaimers

We would like to invite you to speak at the meeting with a keynote review of "Lattice for Flavour Physics" (or a topic that you find most interesting)

Disclaimer 1

At Lattice2023 the traditional parallel stream "Weak decays and matrix elements" became "Quark and lepton flavour physics"
"Lattice for Flavour Physics" $\Rightarrow$ "Lattice for Precision Physics"

Disclaimer 2

My aim is to give an overview of where the field stands, rather than detailing specific calculations. I will (attempt to) stay non-technical where possible.

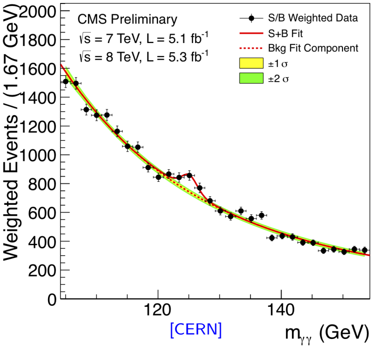
Outline

Indirect searches and Lattice QCD

Searches for New Physics

Direct searches

NP is directly observed as "bump in the spectrum".



[CERN]

Can probe up to Λ_{exp}

Indirect searches

NP visible via quantum corrections of low energy observables.

- theory $\equiv$ SM
- experiment $\equiv$ SM + NP

significant discrepancy $\equiv$ **NP**

Establish NP via

$\Rightarrow$ high precision experiments

$\Rightarrow$ control over all theory uncertainties

Evidence of NP found *below* $\Lambda_{\rm NP}$

$\Downarrow$

Can probe beyond Λ_{exp}

The challenge: Quantum Chromodynamics and colour confinement

Theory

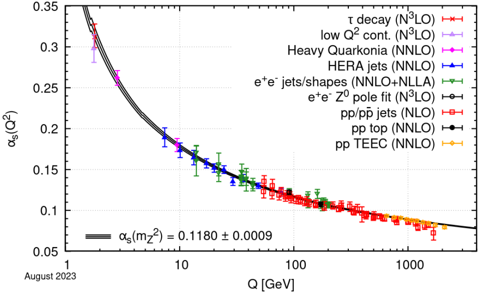
quarks and gluons

$$\mathcal{L}_{\mathrm{QCD}} = \bar{\psi}\left(i\gamma^{\mu} D_{\mu} - m\right)\psi - \tfrac{1}{4} G^{a}_{\mu\nu} G^{\mu\nu,a}$$

Experiment

hadronic bound states

[Wikipedia]



[PDG]

perturbative at high Q
$\Rightarrow$ "standard" methods work

non-perturbative at low Q
$\Rightarrow$ "standard" methods fail

Consequence: hadronic uncertainties dominate in many situations

Lattice QCD simulations provide **first principle predictions**.

Lattice QCD in a nutshell

Based on the **Path Integral** formulation.

$$\langle \mathcal{O} \rangle_M = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\psi, \bar{\psi}, U] \, \mathcal{O}[\psi, \overline{\psi}, U] \, e^{iS[\partial si, \bar{\psi}, U]}$$

Minkowski: Highly oscillatory, infinite dimensional integral. ✗

⇒ Wick rotate to Euclidean (i.e. imaginary) time ($t \to i\tau$).

$$\langle \mathcal{O} \rangle_E = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\psi, \bar{\psi}, U] \, \mathcal{O}_E[\psi, \overline{\psi}, U] \, e^{-S_E[\psi, \bar{\psi}, U]}$$

Euclidean: Exponentially decaying, infinite dimensional integral. ✗

⇒ Discretise space-time and **interpret as a probability distribution**.

- Lattice spacing a (UV regulator)
- Box of length L (IR regulator)
- $\int \to \sum$, $\partial \to$ finite differences
- Evaluate **numerically**

Lattice: Exponentially decaying and finite dimensional ✓

Parameters of QCD and scale setting

Lattice QCD $\equiv$ **first principles QCD**. Defined by parameters in the action:

- bare **coupling constant** g
- bare **quark masses** m_f

$$S_{\rm QCD}[\psi,\overline{\psi},U] = S_G[U] + S_F[\psi,\overline{\psi},U]$$
$$= \int \mathrm{d}^4x\, \frac{1}{2g^2}\mathrm{Tr}\left[F_{\mu\nu}F_{\mu\nu}\right] + \sum_{N_f} \overline{\psi}_f\left(\gamma_\mu D_\mu + m_f\right)\psi_f$$

Simulations takes bare quantities, but want to reproduce real world (example $N_F = 2+1$):

- want to determine 3 parameters: $m_l (= m_u = m_d)$, m_s and $a(g)$
- need to sacrifice 3 predictions: for example m_π, m_K and m_Ω

Vary the quark masses m_l and m_s until

$$\left(\frac{am_\pi}{am_\Omega}\right)_{\rm lat} \equiv \left(\frac{m_\pi}{m_\Omega}\right)_{\rm PDG} \quad \text{and} \quad \left(\frac{am_K}{am_\Omega}\right)_{\rm lat} \equiv \left(\frac{m_K}{m_\Omega}\right)_{\rm PDG}$$

Require $(m_\Omega)_{\rm lat} = (m_\Omega)_{\rm PDG}$

$$a^{-1} = \frac{(m_\Omega)_{\rm PDG}}{(am_\Omega)_{\rm lat}}$$

Everything else is a (QCD) prediction

Lattice QCD workflow

1. Generate gauge ensembles
2. Measure n-point correlation functions
3. Extract masses and matrix elements from correlation functions

   Repeat steps 1. - 3. for different choices of lattice spacing a, volumes L and quark masses m_q
4. Perform limits to recover "the real world": $a \to 0$, $L \to \infty$, $m_q \to m_q^{\mathrm{phys}}$, ...
5. Compare to/ Combine with experiment

Infinite volume limit

$\Rightarrow$

Continuum limit

$\Rightarrow$

Making choices and universality

- many ways to discretise action
- freedom to add any term that vanishes when $a \to 0$
  $\Rightarrow$ different fermion actions
  (Wilson, Domain Wall, Twisted Mass, Staggered, $\cdots$)
- Different (lattice) actions have very different properties:
  - discretisation effects
  - renormalisation
  - symmetries
  - numerical cost

  $\Rightarrow$ dominated by different sources of uncertainty
- Universality $\equiv$ they must all agree when all limits have been taken!

☺ **Different lattice calculations are often highly complementary!** ☺

$g-2$

The anomalous magnetic moment of the muon

- Spin magnetic moment μ is related to spin $\boldsymbol{S}$ by g:

$$\boldsymbol{\mu} = g \frac{e}{2m} \boldsymbol{S}$$

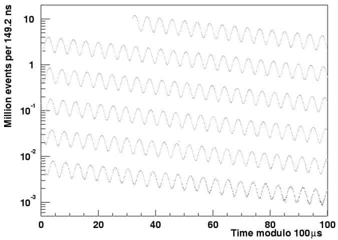
- At tree-level: $g = 2$
- $g \neq 2$ due to quantum corrections. Define

$$a_\mu \equiv \frac{g-2}{2}$$

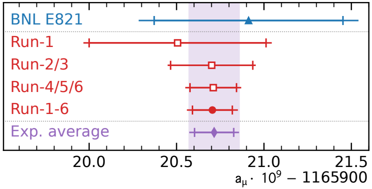
- Sensitive to effects due to QED, weak and strong interactions (and NP?)
- precisely measurable as a frequency:

$$a_\mu^{\mathrm{exp,combined}} \times 10^{11} = 116592071 5(145)\ [124\,\mathrm{ppb}]$$

[E821 experiment at BNL, PRD **73** (2006) 072003]



[Fermilab experiment, PRL **135** (2025) 101802]



$g-2$: Theory [Numbers based on Theory Initiative white paper 1: Phys. Rep. **887** (2020) 1]

$$\frac{g-2}{2} \equiv a_\mu = a_\mu^{\mathrm{QED}} + a_\mu^{\mathrm{weak}} + a_\mu^{\mathrm{strong,HLbL}} + a_\mu^{\mathrm{strong,HVP}}$$

- QED (differing input for α) [PRL **109** 111808, Atoms **7** 28]

$$a_\mu^{\mathrm{QED}} \times 10^{11} = 116584718.931(104)$$

$$a_\mu^{\mathrm{QED}} \times 10^{11} = 116584718.842(106)$$

- Weak [PRD **67** 073006, PRD **73** 119901, PRD **88** 053005]

$$a_\mu^{\mathrm{weak}} \times 10^{11} = 153.6(1.0)$$

- Strong (Had. Light-by-Light) [PRD **70**, 113006, PRD **95**, 054026, JHEP **04** 161, JHEP **10** 141, PRD **100** 034520, PLB **798** 134994, JHEP **03** 101, EPJC **74** 3008, PRD **95** 014019, STMP **274** 1, PLB **787** 111, PRD **101** 054015, PRD **101** 074019, PLB **735** 90, PRL **124** 132002, EPJC **81** (2021) 651; EPJC 82 (2022) 664,...] – hadronic models, data-driven & lattice: all consistent

$$a_\mu^{\mathrm{HLbL}} \times 10^{11} = 92(18)$$

- Strong (Hadronic vacuum polarisation) ???

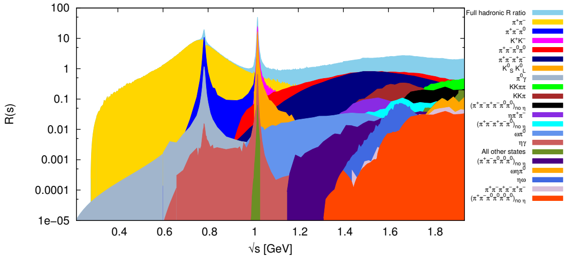
Computing the HVP: Data driven method (in 2018)

$$a_\mu^{HVP} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} \mathrm{d}s \frac{R_{\mathrm{had}}(s)\hat{K}(s)}{s^2}$$

[KNT, PRD 97 (2018), 114025]

data driven

- based on dispersive methods
- dominated by low s region
- $\hat{K}(s)$ - known analytic function
- $R_{\mathrm{had}}(s) = \frac{3s}{4\pi(\alpha(s))^2}\sigma(e^+e^- \to \text{hadrons})$
- "exclusively" add up experimental channels



$$a_\mu^{HVP,LO} = 6847(24) \times 10^{-11}$$

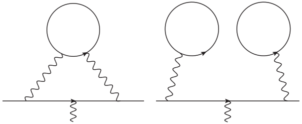
Computing the HVP: From the lattice (in 2018)

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}t \tilde{K}(t) G(t)$$

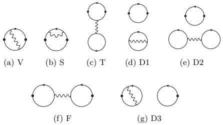
lattice

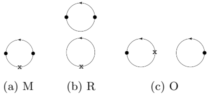
- $\tilde{K}(t)$ - known analytic function
- $G(t)$: lattice 2-point function(s)
- Automatically sums over all hadronic channels ("inclusive")
- Decompose into flavours:
  - **Isospin symmetric world**: $(ud) + (s) + (c) +$ quark-disconnected
  - **isospin breaking corrections**: QED corrections + sIB corrections

quark-connected disconnected



QED-corrections sIB corrections





$$a_\mu^{HVP,LO} = 7154(187) \times 10^{-11}$$

[RBC/UKQCD 18, PRL **121** 022003]

2018: 2.6% precision vs 0.4% (R-ratio) $\Rightarrow$ Good but **need to improve!**

“data driven” vs “lattice” at WP1

data driven

$$a_\mu^{HVP} = \left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{m_{\pi^0}^2}^{\infty} \mathrm{d}s \frac{R_{\mathrm{had}}(s)\hat{K}(s)}{s^2}$$

- dominated by low s region
- $\hat{K}(s)$ - known analytic function
- $R_{\mathrm{had}}(s) = \frac{3s}{4\pi(\alpha(s))^2}\sigma(e^+e^- \to \text{hadrons})$
- based on experimental data

White paper 1 [taken for Theory prediction]:

$$\mathbf{a}_\mu^{HVP} \times \mathbf{10^{11}} = \mathbf{6845(40)}$$

lattice

$$a_\mu^{\mathrm{HVP}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^{\infty} \mathrm{d}t \tilde{K}(t) G(t)$$

- dominated by light-quark contributions
- $\tilde{K}(t)$ - known analytic function
- $G(t)$: lattice 2-point function(s)
- Independent of experimental inputs (other than m_μ and scale setting)

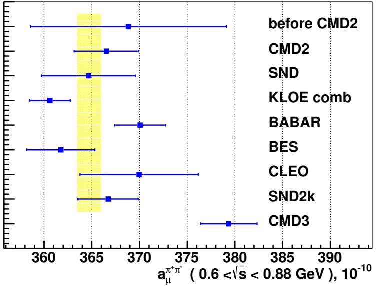
White paper 1:

$$a_\mu^{HVP} \times 10^{11} = 7116(184)$$

Two game-changing new results

Input for data-driven

[Ignatov et al. (CMD-3 Collab.), PRD 109 (2024) 11, 112002]

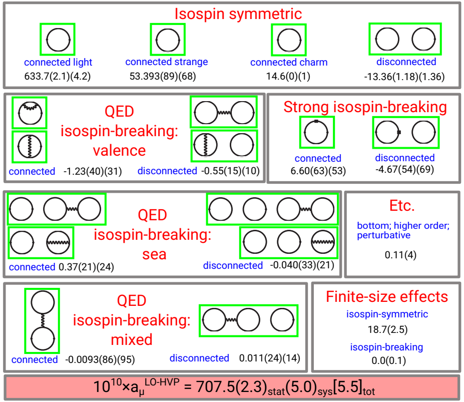


Tensions between experiments (**inputs for data driven method**) in dominant region

- **Kloe** vs **BaBar**
- **CMD3** result yet to be understood

Complete lattice calculation

[BMWc 20, Nature **593** (2021) 7857, 51-55]



- Impressive result: sub-percent precision
- Needs independent verification

Contributions to the HVP: by windows [RBC/UKQCD 18, PRL **121** 022003]

Particularly challenging:

small t region: discretisation effects **large t region: finite volume effects**

$$a_\mu^{\mathrm{HVP,X}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}t \qquad \tilde{K}(t)G(t)$$

Q: How can we scrutinise these difficult calculations?

Contributions to the HVP: by windows [RBC/UKQCD 18, PRL **121** 022003]

Particularly challenging:

small t region: discretisation effects **large t region: finite volume effects**

$$a_\mu^{\mathrm{HVP,X}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}t\, T_X(t; t_0, t_1, \Delta)\tilde{K}(t)G(t)$$

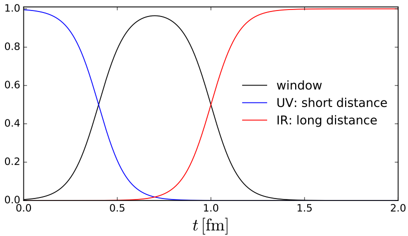
Q: How can we scrutinise these difficult calculations? **A:** Separate these regions!

By integration range [RBC/UKQCD 18]

$$a_\mu^{\mathrm{HVP}} = a_\mu^{\mathrm{HVP,SD}} + a_\mu^{\mathrm{HVP,W}} + a_\mu^{\mathrm{HVP,LD}}$$

- SD Short distance ($T_{SD} = 1 - \Theta(t, t_0, \Delta)$)
- W Intermediate window ($T_W = \Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)$)
- LD Long distance ($T_{LD} = \Theta(t, t_1, \Delta)$)

$$\Theta(t_0, t_1, \Delta) = \frac{(1 + \tanh(t_0 - t_1)/\Delta)}{2}$$

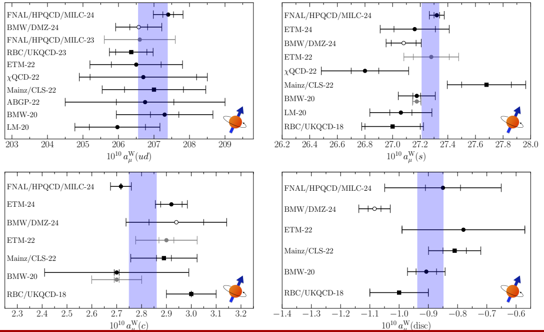


$\Rightarrow$ **Use windows for scrutiny**

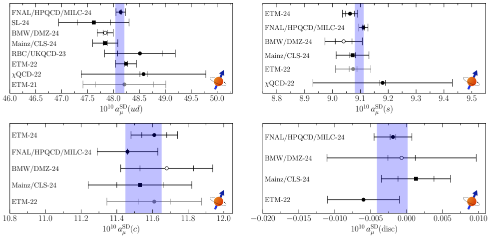
New results for WP2

- Many new results
- Many of them from blinded analyses
- Many ways to build averages of these
- Highly complementary

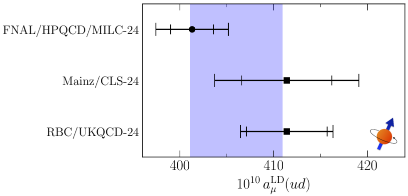
W



SD

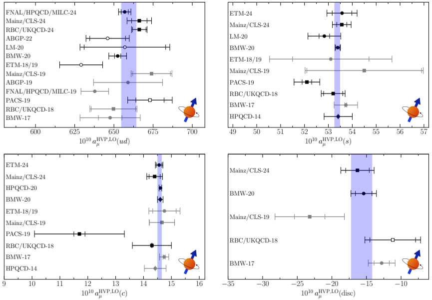


LD

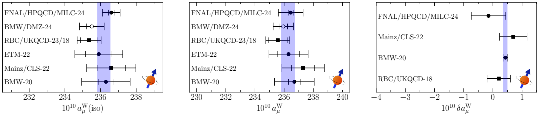


White Paper 2 - Consolidating averages

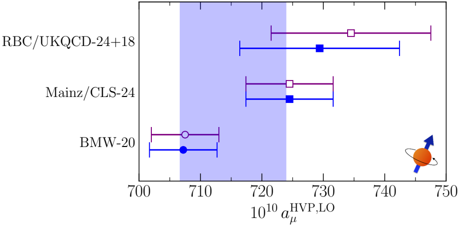
Summing by windows:



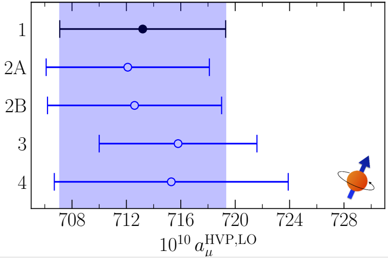
Summing by flavours:



"Complete" calculations

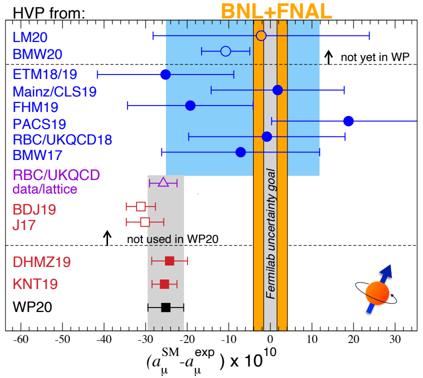


Final averages



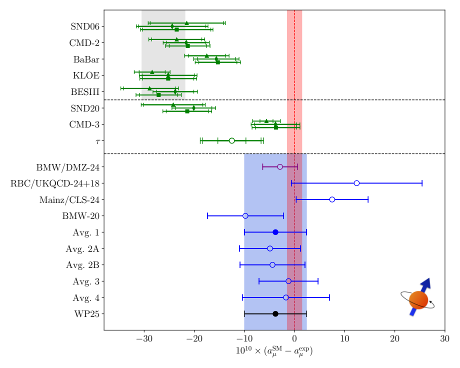
HVP through the (recent) ages

Snowmass (post WP1) 2203.15810



$$a_\mu^{HVP,\text{data driven}} \times 10^{11} = 6845(40)$$

2025 - WP2



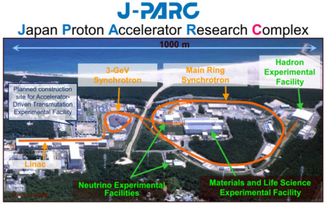
$$\mathbf{a_\mu^{HVP,lattice} \times 10^{11} = 7132(61)}$$

a huge success for the lattice!

Outlook $g-2$

A huge step forward, but more work needs to be done:

- (hopefully) more experimental **and highly complementary** results:



$+$

- Data driven vs Lattice tensions need to be understood!
- More experimental inputs for $e^+e^- \to$ hadrons to scrutinise data-driven predictions
- Hybrid approach by BMW+DMZ'24: Supplements very noisy large-t tail with experimental data.
- Further reduction from lattice expected: more collaborations, finer ensembles, larger volumes, ...

Lattice for flavour (a.k.a. CKM) physics

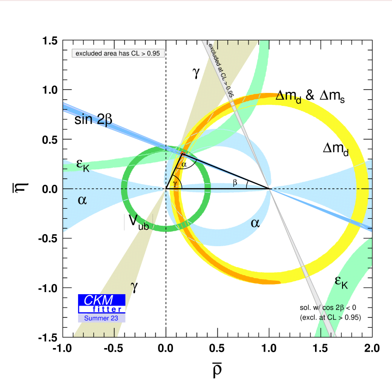
Indirect NP searches – CKM

Goal: *Lead SM into a contradiction*

C[Cabibbo '63]KM[Kobayashi, Maskawa '73]

- parameterises transitions up-type $\leftrightarrow$ down-type
$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$
- **unitary matrix in the SM**
- **fundamental SM parameters**
- bands: different constraints on the same parameters ($\rightarrow$)
- unitarity $\Leftrightarrow$ meet in one apex

Unitarity Triangle [CKMfitter'05 + web updates]



Non-unitarity of CKM $\Leftrightarrow$ New Physics Beyond the SM

CKM: Relating theory and experiment

$$\text{experiment} \approx \left|V_{qq'}\right|^n \sum_i \text{kinematic \& perturbative} \times \text{non-perturbative}$$

$$\text{CKM} = \text{theory} \otimes \text{experiment}$$

experiment measures

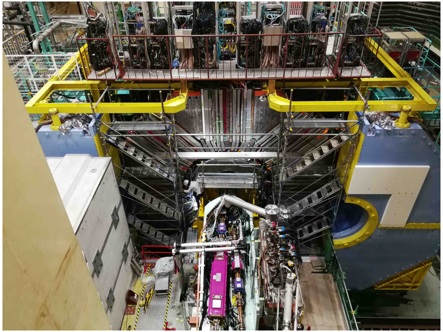
- Leptonic decay rates
- differential decay decay rates
- angular distributions
- mass and width differences, mixing parameters
- ...

Lattice QCD predicts

- leptonic decay constants
- semi-leptonic form factors
- bag parameters
- inclusive decay rates
- long distance mixing parameters
- ...

CKM-precision depends on knowledge of theory AND experiment.

Large flavour physics experiments



left: Belle II at KEK; *right*: LHCb at CERN,

⇒ ***B*-factory**; **hadron machine**
Very complementary

\+ BES-III, other LHC experiments & older data from BaBar, Belle, Cleo, . . .

\+ NA62 at CERN, KOTO II at KEK

\+ High-Lumi LHC (2030s)

\+ Future colliders? (FCC-ee, FCC-hh, · · ·) (⇒ talk by Jonathan Butterworth)

Over-constraining the Standard Model: CKM

- Determine individual elements from (many) different decays
- SM CKM-matrix is unitary

$\Rightarrow$ Different determinations compatible?

$\Rightarrow$ Does unitarity hold?

CKM = theory $\otimes$ experiment

$$\text{experiment} \approx \left|V_{qq'}\right|^n \sum_i \text{kinematic} \times \text{non-perturbative}$$

$$\Gamma(B \to \ell\nu) \approx |V_{ub}|^2 \, \mathcal{K} f_B$$

$$\frac{\mathrm{d}\Gamma(B \to \pi\ell\nu)}{\mathrm{d}q^2} \approx |V_{ub}|^2 \left(\mathcal{K}_1 f_+^{B\to\pi}(q^2) + \mathcal{K}_2 f_0^{B\to\pi}(q^2) \right)$$

$$\Delta m_d \approx |V_{tb}^* V_{td}|^2 \, \mathcal{K} f_{B_d}^2 \hat{\mathcal{B}}_{B_d}^{(1)}$$

"best" channel depends on availability of experimental and theoretical precision.

Non-perturbative lattice observables

Leptonic decays

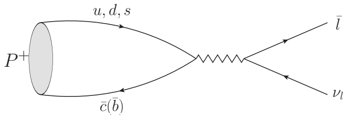
- Decay constants
- e.g. f_B
- Scalar quantities
- 2-point functions
- Large experimental uncertainties
- V_{cd}, V_{cs}, V_{ub}

Neutral meson mixing

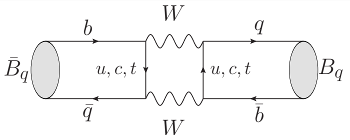
- bag parameters
- e.g. $\mathcal{B}_{B_d}$
- Scalar quantities
- 3-point functions
- 4-quark operators
- Experimentally very precise
- V_{td}, V_{ts}

Semi-leptonic decays

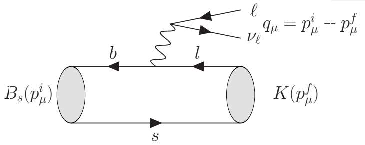
- form factors
- e.g. $f_+^{B_s \to K}(q^2)$
- functions of q^2
- 3-point functions
- Many experimental channels
- V_{cd}, V_{cs}, V_{ub}, V_{cb}



e.g. $B^+ \to \mu^+ \nu$

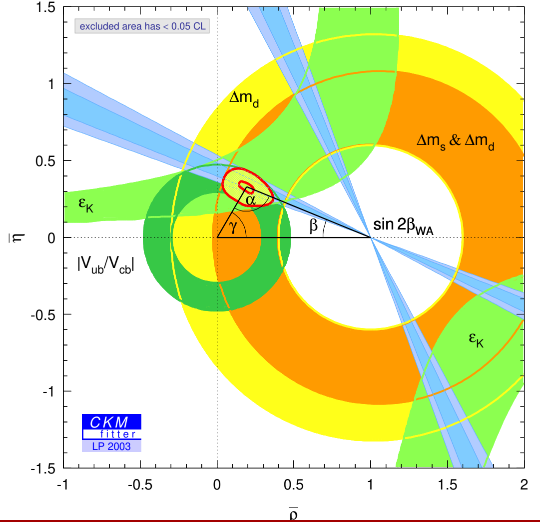


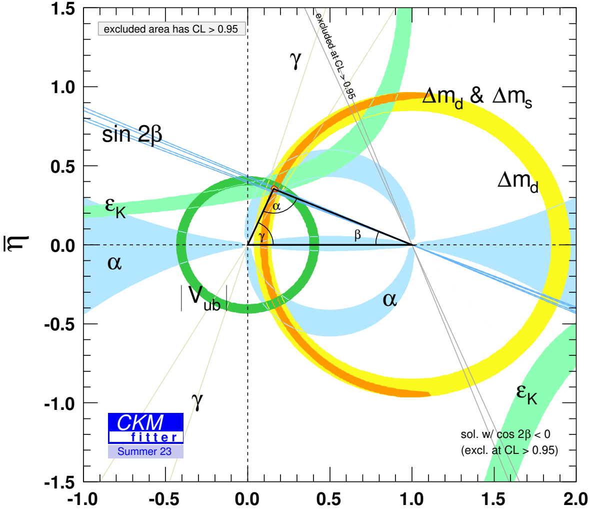
e.g. $B_d^0 \leftrightarrow \overline{B_d^0}$



e.g. $B_s^0 \to K^- \mu^+ \nu$

The last two decades of CKM





Preferred (lattice) determinations

$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

V_{ud} $\frac{K \to \ell\nu}{\pi \to \ell\nu}$ $(+\ V_{us})$
V_{us} $K \to \pi\ell\nu$

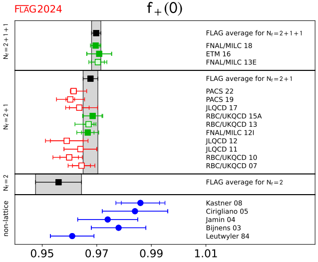
V_{cd} $D \to \ell\nu$, $D \to \pi\ell\nu$
V_{cs} $D_s \to \ell\nu$, $D \to K\ell\nu$

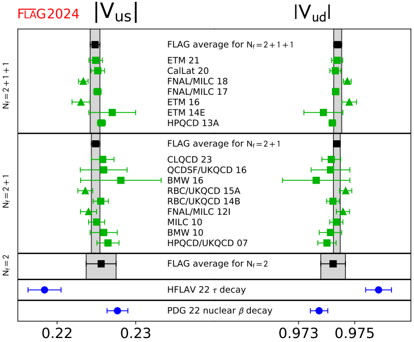
V_{ub} $B \to \pi\ell\nu$
V_{cb} $B \to D^{(*)}\ell\nu$

V_{td} $B^0 \to \bar{B}^0$ $(+\ V_{tb})$
V_{ts} $B_s^0 \to \bar{B}_s{}^0$ $(+\ V_{tb})$

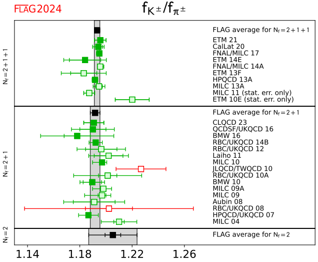
Determining V_{ud} and V_{us}

$K \to \pi \ell \nu$





$K \to \ell \nu / \pi \to \ell \nu$

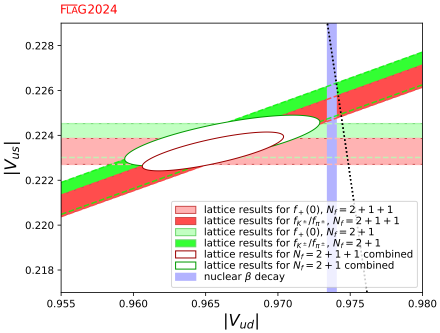


- ✓ different gauge ensembles, fermion and gauge actions
- ✓ different leading systematics
- ✓ independent collaborations

- (✓) "historically" isospin symmetric limit only
- (✓) separation between isoQCD and QED+QCD not uniquely defined

- $|V_{ud}| = 0.97373(31)$ [nuclear β decay]
- $|V_{us}|\, f_+^{K\to\pi\ell\nu}(0) = 0.21654(41)$
- $\frac{|V_{us}|}{|V_{ud}|}\frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.27599(41)$ (IB-breaking correction from χPT [Cirigliano,Neufeld'11])

  $\frac{|V_{us}|}{|V_{ud}|}\frac{f_{K^\pm}}{f_{\pi^\pm}} = 0.27683(29)_e(20)_t[35]$ (IB-correction from direct calculation [Di Carlo, ETMC '19])



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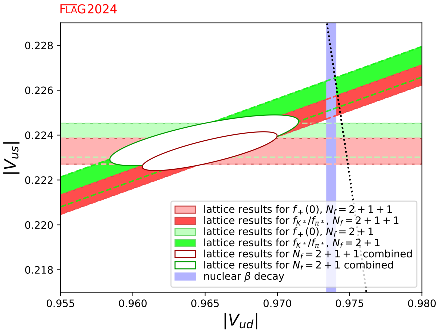


A lot of activity for QED corrections

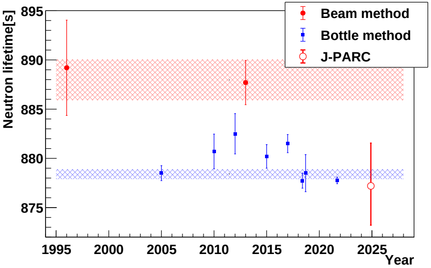
- Different lattice formulations of QED: QED$_L$, QED$_m$, QED$_{C^*}$, QED$_\infty$, QED$_r$
  $\Rightarrow$ Complementarity and strong cross checks
- “Edinburgh consensus” for scheme separation (isoQCD vs QCD+QED)
  $\Rightarrow$ easier comparison between calculations

"you win some, you lose some": V_{ud}, neutron β decay and g_A

$$\tau_n[1 + 3(g_A/g_V)^2]\ |V_{ud}|^2 = 4902.3(1.3)\,\mathrm{s}$$

- previous experimental tensions (bottle vs beam) in τ_n appear to have disappeared [J-PARC,Fuwa et al.,2022]
- Current lattice predictions for g_A have sub-percent precision (in pure QCD)
- Up to 2% structure dependent QED corrections previously overlooked [Cirigliano et al.,2022]

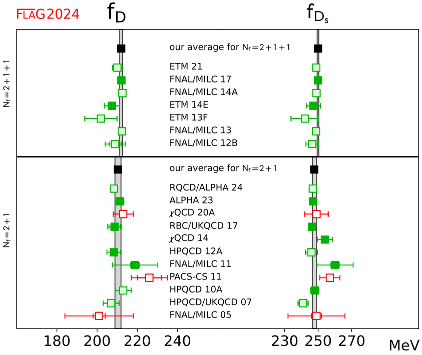
$\Rightarrow$ New lattice calculations needed to address this



Leptonic charm decays

$$D_{(s)} \to \ell\nu$$

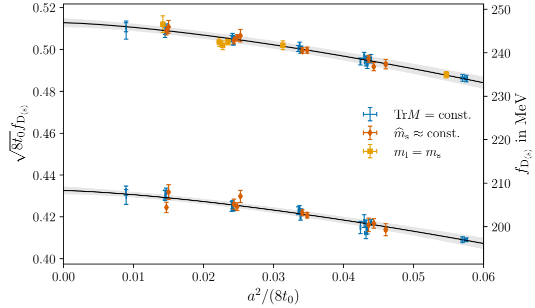
$$f_D\,|V_{cd}| = 45.82(1.10)\,\mathrm{MeV}\ \ [2.4\%]$$
$$f_{D_s}\,|V_{cs}| = 243.5(2.7)\,\mathrm{MeV}\ \ [1.1\%]$$



✓ many independent results + good agreement

new ALPHA'23 [Bussone et at.,2309.14154]

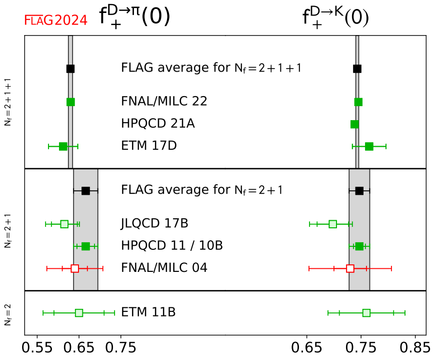
new RQCD/ALPHA'24 [Kuberski et al, 2405.04506]:
$O(50)$ gauge ensembles, 6 lattice spacings
$a \in [0.039, 0.098]\,\mathrm{fm}$, including m_π^{phys}



semi-leptonic charm decays

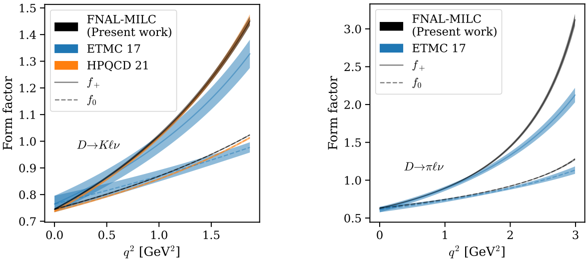
$$D \to \pi/K\ell\nu$$

$$f_+^{D\to\pi}(0)\,|V_{cd}| = 0.1426(18)\ [1.2\%]$$
$$f_+^{D\to K}(0)\,|V_{cs}| = 0.7180(33)\ [0.5\%]$$



✗ Far fewer results and some tensions ↓

[FNAL/MILC'22]

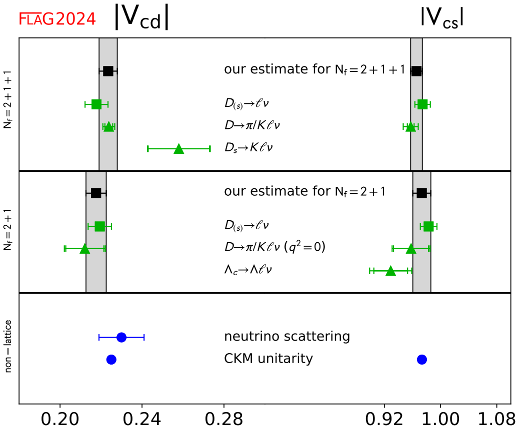


✗ Impact on CKM "somewhat mitigated" since using ff at $q^2 = 0$ (but this is a small comfort)

✗ More results needed to resolve this!

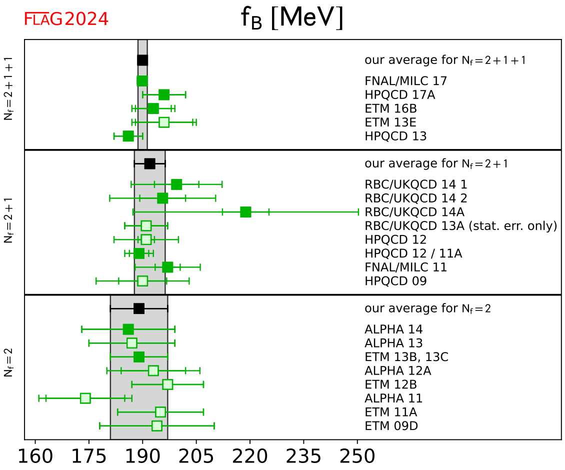
✓ Ongoing efforts by several groups

Determining V_{cd} and V_{cs}



- broad agreement
- leptonic determinations based on wealth of consistent results
- semi-leptonic determinations require further scrutiny (serval ongoing works)
- $D_s \to \phi\ell\nu$ experimentally available + single prediction [HPQCD'14]
- further baryonic results needed (single calculation for $\Lambda_c \to \Lambda\ell\nu$ [Meinel 2016], $\Xi_c \to \Xi\ell\nu$ in progress)
- predictions for $B_c \to B_s$ and $B_c \to B_d$ exist [HPQCD'20], but no exp. data yet

B-mesons: leptonic decays



- ✓ Good agreement between different decay constant determinations.
- ✗ Experimentally not competitive: e.g. $|V_{ub}|\, f_B = 0.77(12)\,\mathrm{MeV}$
- ✓ Provides valuable benchmark quantity for new lattice calculations

Instead: determine V_{ub} and V_{cb} from the many different semi-leptonic decay channels with $b \to u$ and $b \to c$ transitions.

semi-leptonic b-decays: a wealth of observables (incomplete list)

$b \to u$ (tree)

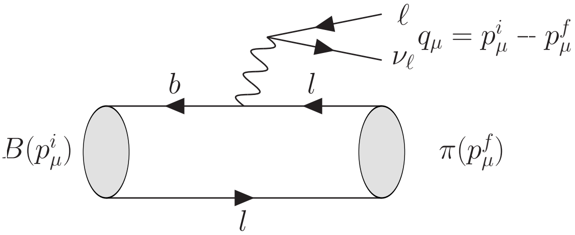
- $B \to \pi \ell \nu$
- $B_s \to K \ell \nu$
- $\Lambda_b \to p \ell \nu$
- $\dots$

$b \to c$ (tree)

- $B_{(s)} \to D^{(*)}_{(s)} \ell \nu$
- $\Lambda_b \to \Lambda_c \ell \nu$
- $B_c \to J/\Psi \ell \nu$
- $\dots$

$b \to s$ (loop)

- $B \to K^{(*)} \ell \ell$
- $B \to \rho \ell \ell$
- $B_s \to \phi \ell \ell$
- $\dots$

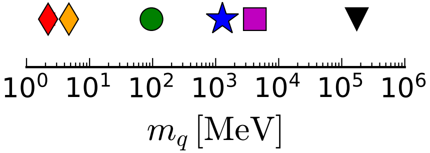


- In the SM:
  - PS $\to$ PS (tree): 2 ffs
  - PS $\to$ V (tree): 4 ffs
  - PS $\to$ PS (loop): 3 ffs
  - PS $\to$ V (loop): 7 ffs
- Each ff depends on momentum transfer q^2 to the lepton pair

A lattice interlude: Heavy quarks and multiple scales

To control effects of IR (finite volume) and UV (discretisation) regulators we need

$$m_\pi L \gtrsim 4 \qquad m_\pi^{\mathrm{phys}} \sim 0.140\,\mathrm{GeV}$$
$$a^{-1} \gg \text{scale of interest} \qquad \overline{m}_b(m_b) \sim 4.2\,\mathrm{GeV}$$



$$\Rightarrow L \gtrsim 5.6\,\mathrm{fm} \quad \text{and} \quad a^{-1} \gg 4.2\,\mathrm{GeV} \approx (0.05\,\mathrm{fm})^{-1}$$

Requires $N \equiv L/a \gg 120 \Rightarrow N^3 \times (2N) \gg 4 \times 10^8$ lattice sites.

very expensive to satisfy both constraints simultaneously...

... needs to be repeated for different values of a.

First results near physical quark masses are becoming available

$\Rightarrow$ Expect to see more fully-relativistic results in the future

Lattice b-physics: a compromise (at least for now)

Effective action for b

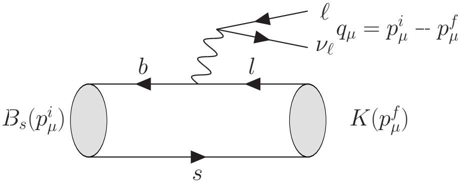
- Can tune to $m_b \sim m_b^{\mathrm{phys}}$
- comes with systematic errors which are hard to estimate/reduce

Relativistic action for b

- Theoretically cleaner and systematically improvable
- $m_b < m_b^{\mathrm{phys}}$ + extrapolation to m_b^{phys}

- relativistic will win in the long term
- for now, settle on a compromise (and pursue both!)
- very different systematics $\Rightarrow$ complementary results!

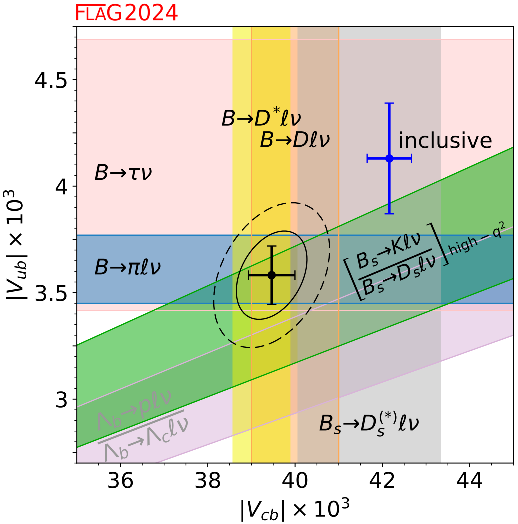
Complication: semi-leptonic form factors depend on momentum transfer q^2:



$$q^2 = (E_{B_s} - E_K)^2 - (\vec{p_i} - \vec{p_f})^2$$

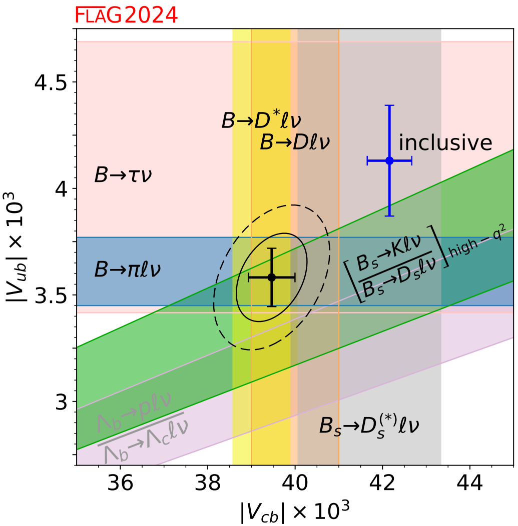
Interplay between varying m_b in q^2, M_{B_s} and $(am_b)^n$ highly non-trivial!

Status of V_{ub} and V_{cb}



- $B \to \tau\nu$
- $B \to \pi\ell\nu$
- $B_s \to K\ell\nu$
- $B \to D\ell\nu$
- $B \to D^*\ell\nu$
- $B \to K\ell\ell$
- $\Lambda_b \to p\ell\nu$, $\Lambda_b \to \Lambda_c\ell\nu$

Status of V_{ub} and V_{cb}



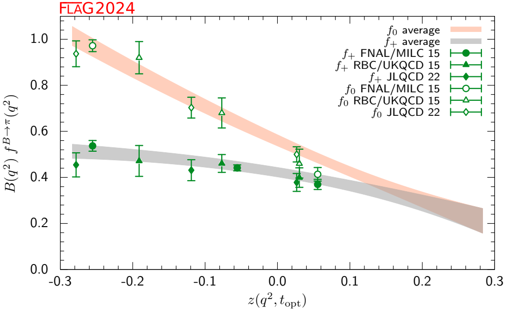
- $B\to\tau\nu$ ✓
- $B\to\pi\ell\nu$ (χ^2-inflation ✗)
- $B_s\to K\ell\nu$ (χ^2-inflation ✗)
- $B\to D\ell\nu$ ✓
- $B\to D^*\ell\nu$ (✓)
- $B\to K\ell\ell$ (χ^2-inflation ✗)
- $\Lambda_b\to p\ell\nu$, $\Lambda_b\to\Lambda_c\ell\nu$ (single calculations, red rating ✗)

FLAG plot to be taken with a grain of salt...
BUT the situation is not terrible:

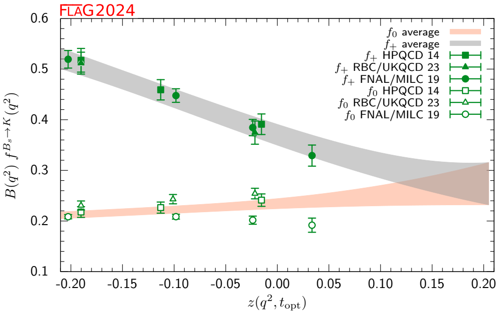
- ...adding stat+sys errors in quadrature
- ...typically $\sqrt{\chi^2/\mathrm{dof}}\lesssim 2$

V_{ub} from semi-leptonic decays

$B \to \pi \ell \nu$



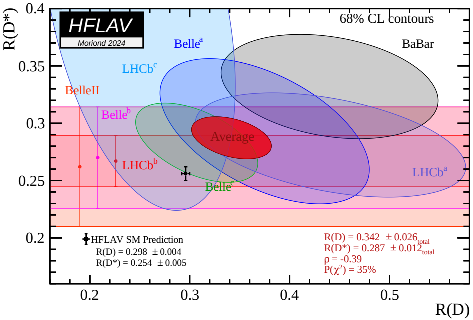
$B_s \to K \ell \nu$



- ✓ f_+ in good agreement
- ✗ f_0 has some tensions
- ✓ Only f_+ relevant for extraction of $|V_{ub}|$
- ✗ Extrapolation over full q^2 range usually imposes the constraint $f_+(0) = f_0(0)$.
- ✓ Some origin of tension identified [JTT, RBC/UKQCD, PRD 107 (2023) 11, 114512]

What about $b \to c$? $B \to D^*\ell\nu$ and $R(D^*)$

$$R(D^{(*)}) = \frac{\int \mathrm{d}q^2 \mathrm{d}\Gamma(B \to D^{(*)}\tau\nu)/\mathrm{d}q^2}{\int \mathrm{d}q^2 \mathrm{d}\Gamma(B \to D^{(*)}\ell\nu)/\mathrm{d}q^2}$$



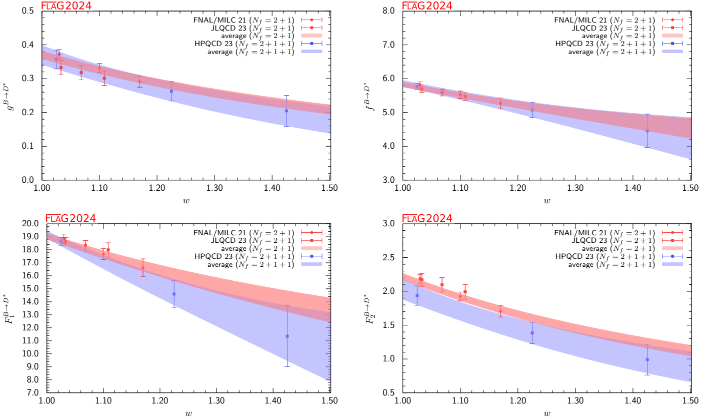
- Long standing "tension"
- SM here not from lattice (yet!)
- $B \to D^*$ also important for V_{cb}

There is more than just $R(D^*)$. Compare shapes?

- ☺ 3 recent results away from q^2_{max} [FNAL/MILC'21, HPQCD'23, JLQCD'23]
- ☺ different ensembles, different actions, different analyses
- ☺ jointly fittable with $p > 5\%$ [Bordone, Jüttner '24].
- ☺ Quoted $R(D^*)$ values:
  0.265(13) [FNAL/MILC'21]
  0.273(15) [HPQCD'23]
  0.252(22) [JLQCD'23]
- ? using known constraints: [Martinelli, Simula, Vittorio'23] [Bordone, Jüttner '24]

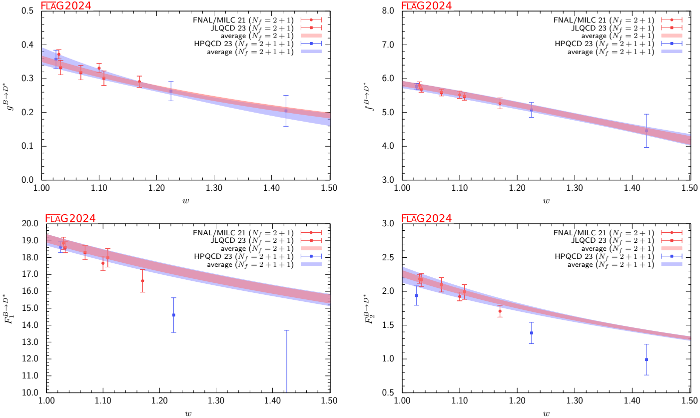
	FNAL/MILC'21	HPQCD'23	JLQCD'23
[*MSV*]	0.275(8)	0.266(12)	0.247(8)
[*BJ*]	0.2748(89)	0.270(13)	0.2482(81)

$B \to D^*$ semi-leptonic decays (lattice only)



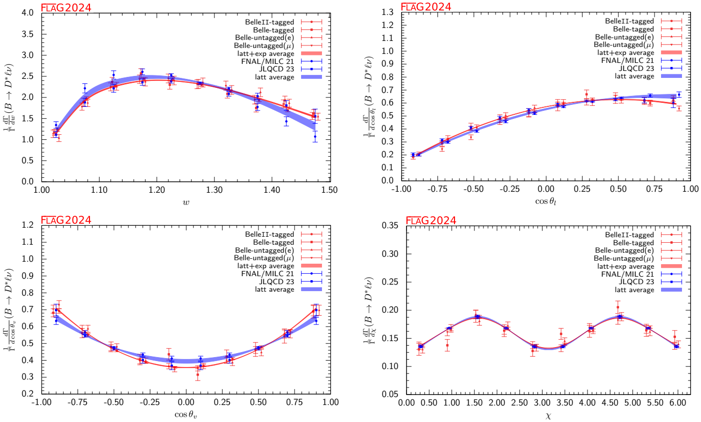
- 3 works away from q^2_{max}
  - ✓ Different gauge ensembles
  - ✓ Different actions
  - ✓ Completely independent
- Some "small-ish" tensions
- subject to a lot of scrutiny
- Several ongoing works

$B \to D^*$ semi-leptonic decays (lattice+experiment)



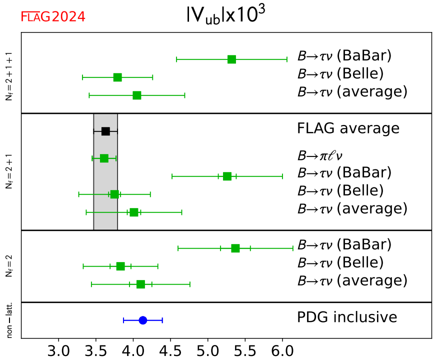
- 3 works away from $q^2_{\max}$
  - ✓ Different gauge ensembles
  - ✓ Different actions
  - ✓ Completely independent
- Some "small-ish" tensions
- subject to a lot of scrutiny
- Several ongoing works
- some shifts for t+e

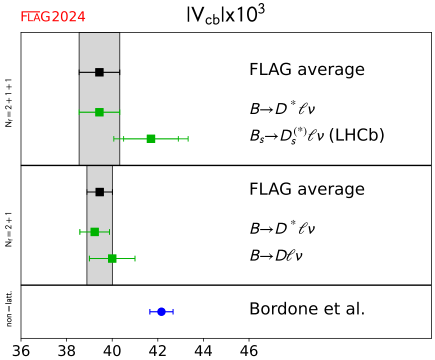
$B \to D^*$ comparison with experiment



- this is for $2+1f$
- similar picture for $2+1+1$.
- For further pheno exploitation see also joint analysis using
  - ✓ Bayesian Inference frame work [Bordone,Jüttner,'24]
  - ✓ Dispersive matrix method [Fedele et al.'23, Martinelli,Simula,Vittorio '24]

Determining $|V_{ub}|$ and $|V_{cb}|$

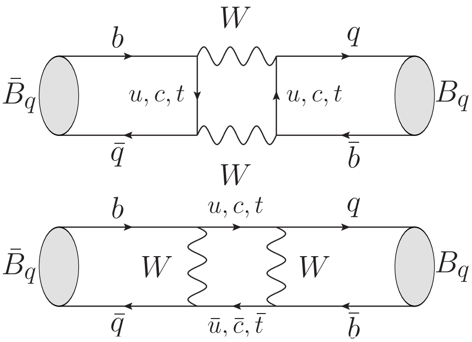




$\Rightarrow$ We need further lattice (and experimental) results to resolve the various small-ish tensions

Neutral meson mixing - background e.g. $B_{(s)}$

Neutral mesons oscillate:



where $q = d, s$

mass eigenstate $\neq$ flavour eigenstate

$$|B_{L,H}\rangle = p\,|B_q^0\rangle \pm q\,|\bar{B}_q^0\rangle$$

$\Rightarrow$ **splittings** in mass eigenstates:

- mass splitting $\Delta m_q \equiv m_H - m_L$
- width splitting $\Delta\Gamma_q \equiv \Gamma_L - \Gamma_H$
- in particular Δm_q experimentally very precisely measured!

Time dependence:

$$|B_q^0(t)\rangle = g_+(t)\,|B_q^0\rangle + \frac{q}{p} g_-(t)\,|\bar{B}_q^0\rangle$$

$$|\bar{B}_q^0(t)\rangle = \frac{p}{q} g_-(t)\,|B_q^0\rangle + g_+(t)\,|\bar{B}_q^0\rangle$$

Operator Product Expansion

Short distance dominated for K^0, B_d^0, B_s^0 $\Rightarrow$ OPE factorises short distance part into

- **Perturbative model-dependent Wilson coefficients** $C_i(\mu)$
- **Non-perturbative model-independent matrix elements**

$$\left\langle P^0 \middle| \mathcal{H}^{\Delta F=2} \middle| \bar{P}^0 \right\rangle = \sum_i C_i(\mu) \left\langle P^0 \middle| \mathcal{O}_i^{\Delta F=2}(\mu) \middle| \bar{P}^0 \right\rangle$$

- 5 independent (parity even) operators $\mathcal{O}_i$, only $\mathcal{O}_1$ relevant for Δm:

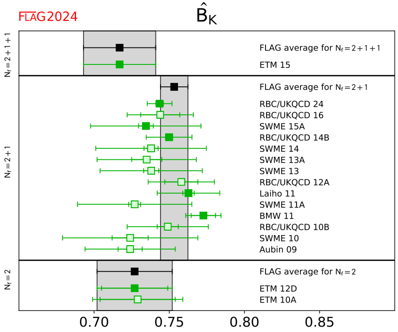
$$\mathcal{O}_1 = \left(\bar{b}_a \gamma_\mu \left(\mathbb{1} - \gamma_5\right) q_a\right) \left(\bar{b}_b \gamma_\mu \left(\mathbb{1} - \gamma_5\right) q_b\right) = \mathcal{O}_{VV+AA}$$

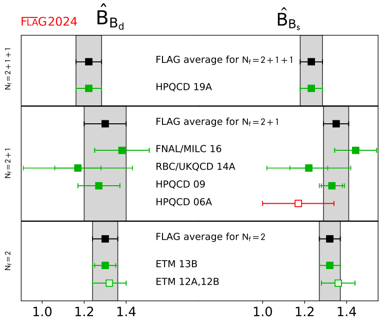
- Define bag parameters: $B_i = \left\langle \bar{B}_q^0 \middle| \mathcal{O}_i \middle| B_q^0 \right\rangle / \left\langle \bar{B}_q^0 \middle| \mathcal{O}_i \middle| B_q^0 \right\rangle_{VSA}$

$$\Delta m_q = \left|V_{tb}^* V_{tq}\right|^2 \times f_{B_q}^2 \hat{B}_{B_q}^{(1)} \times m_{B_q} \mathcal{K}$$

$\Rightarrow$ **Non-perturbative matrix elements calculable on the lattice**

Lattice status for neutral meson mixing

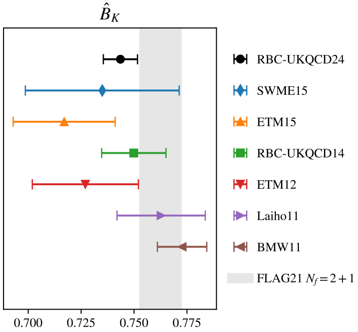


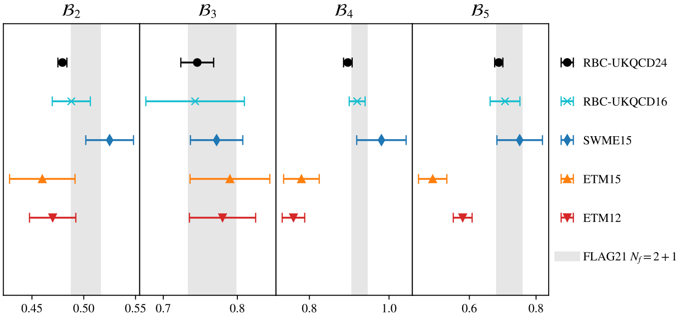


- many more results for K than $B_{(s)}$
- (slow) ongoing efforts by RBC/UKQCD + JLQCD [JTT, RBC/UKQCD + JLQCD PoS LATTICE2021 (2022) 224]
- new result for $K^0 - \bar{K}^0$ in 2024 [JTT, RBC/UKQCD, PRD 110 (2024) 3, 034501]
- new proposed method for long distance contribution [Di Carlo, Erben, Hansen, JHEP 07 (2025) 229]

Neutral Kaon mixing

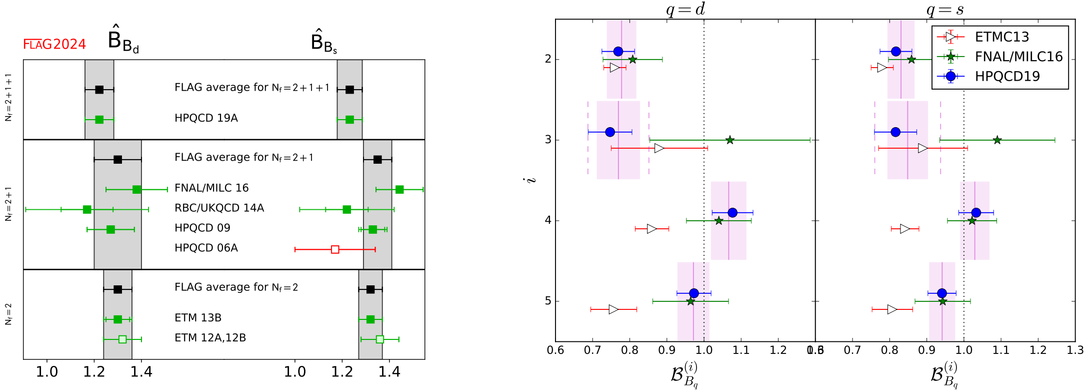
- Some tensions for BSM bag parameters $\mathcal{B}_4$, $\mathcal{B}_5$ (recently confirmed! [JTT, RBC/UKQCD, 2024]).
- Full error budget based on 3 lattice spacings, 2 m_π^{phys} ensembles, domain-wall fermions + RI/SMOM, comprehensive estimates of all sources of uncertainties, several cross checks for continuum limit,...





$\Rightarrow$ Foundation for $B_{(s)} - \overline{B}_{(s)}$ mixing programme [JTT, RBC/UKQCD + JLQCD PoS LATTICE2021 (2022) 224]

Literature for mixing at the $B_{(s)}$



[JTT + Della Morte, EPJ-ST 233 (2024) 2]

- Fewer results in the *b*-sector
- similar tension of $\mathcal{B}_4$ and $\mathcal{B}_5$ as for kaons
- no new results yet (but we are working on it – kaon mixing was a first step!).

Status of CKM determinations from the lattice

"Simple" quantities

- Quark masses, decay constants, g_A under good control in isoQCD world.
- Sub-percent isoQCD precision

$\Rightarrow$ need IB corrections!

- A lot of activity!

$\Rightarrow$ corrections available for light mesons

$\Rightarrow$ first calculations under way for heavy mesons

"Less simple" quantities

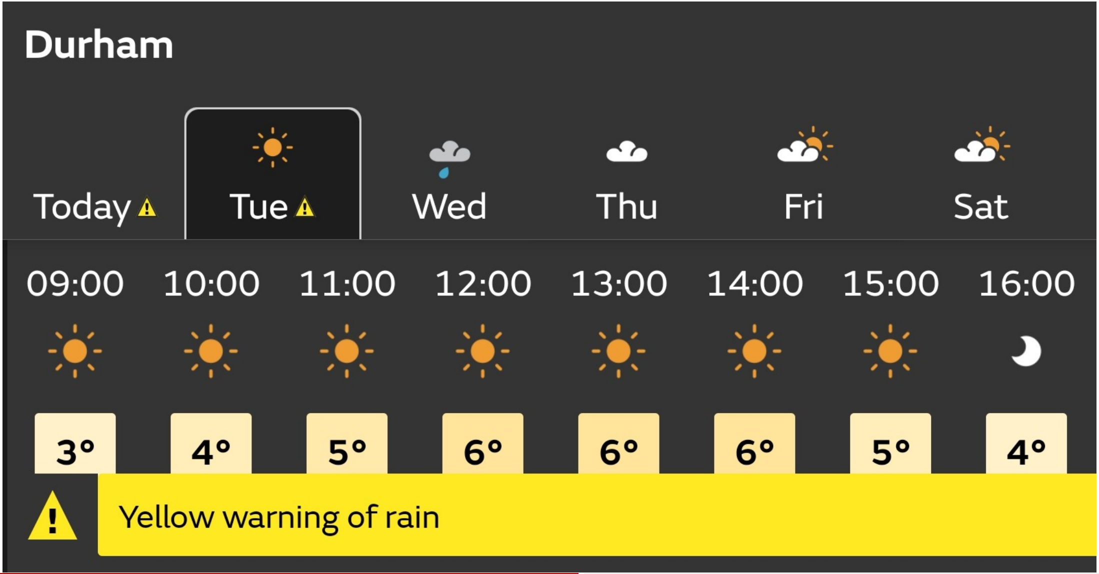
- Calculations for semi-leptonic form factors well understood
- More difficult due to extra kinematic dependence
- Few-percent precision
- Some (small) tensions emerging:

$\Rightarrow$ entering precision era

$\Rightarrow$ many calculations under way

Methods vs Tensions

Addressing systematic uncertainties



Addressing systematic uncertainties

- analyses are **hard** and very (particularly human) time consuming!
- many dependencies (sources of systematics) to consider:
  - **excited states (particularly when approaching $M_\pi^{\rm phys}$ ensembles)**
  - chiral (M_π)
  - heavy quark (m_b)
  - kinematic (q^2)
  - **discretisation, improvement and renormalisation (a)**
- In an ideal world: only change one thing at a time. E.g. data at fixed $m_h; M_\pi; q^2(E_i(m_h), E_f)$ and only vary a.
  BUT: often limited data to control all of these
- particularly challenging for "fully relativistic" (varying b-quark mass) approach
- many choices to make and/or parameters to fit

How can we scrutinise results?

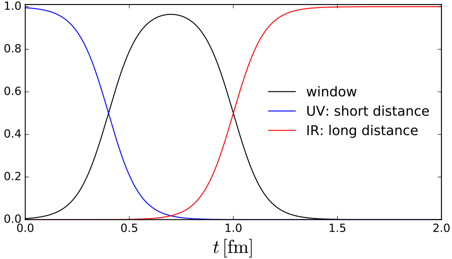
Learning from our community: Similarities with $g-2$

- the stakes are high ✓
- if final results disagree, it is very hard to pin down why ✓
- potential for an "analyst bias" ✓
- not so "easy" to blind: (normalisation vs shapes!) ✗

  but we should still do it!

Profit from **simpler quantities**, less susceptible to some systematics

["windows" - JTT, RBC/UKQCD, PRL 121 (2018) 2, 022003]



$\Rightarrow$ **publish enough information to reproduce the analyses**

$\Rightarrow$ **break calculation into smaller pieces that can be compared more easily**

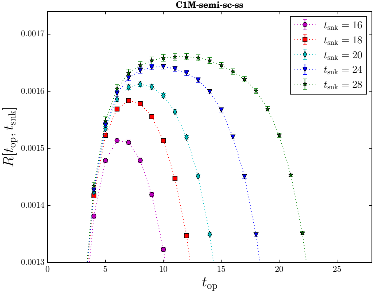
$\Rightarrow$ **suggestion of benchmark quantities** [JTT & Della Morte, EPJ-ST 233 (2024) 2, 253-27]

$\Rightarrow$ **improve methods for crucial steps in lattice calculations**

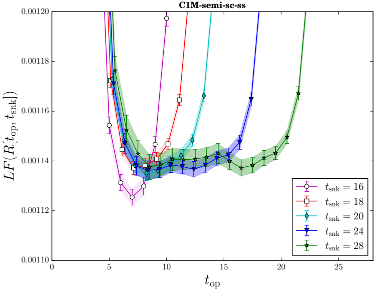
Laplace filtering ($D_\lambda C(t) = -C(t-1) - C(t+1) + (2+\lambda^2)C(t)$) preserves exponentials:

$$D_\lambda \sum_i A_i e^{-E_i t} = \sum_i \left[\lambda^2 - 2(\cosh(E_i) - 1)\right] A_i e^{-E_i t} \equiv \sum_i \left[\lambda^2 - \tilde{E}_i^2\right] A_i e^{-E_i t}$$

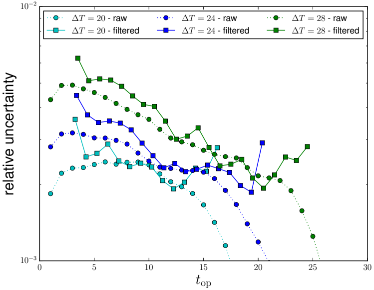
- Free parameter λ can be chosen to eliminate excited states (i.e. $\lambda_1 = \tilde{E}_1$)
- multiple source-sink separations in agreement $\Leftrightarrow$ successfully reduced excited states



"raw"



"filtered"



"relative uncertainties"

Example: Taming discretisation effects via massive NPR

Strategy:

1. renormalise $\mathcal{O}_{\mathrm{bare}}$ in *regularisation independent* scheme
2. take the continuum limit
3. perturbatively match from this scheme to $\overline{\mathrm{MS}}$

Caveat: Typical schemes (RI/(S)MOM) defined in massless limit of QCD

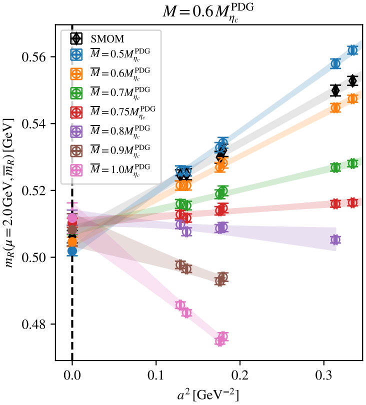
- mass independent renormalisation constants
- introduces discretisation effects scaling with $(am_q)^n$.
- on typical lattices $am_c \sim 0.2$ $am_b \lesssim 1$. Large cut-off effects!

for HQ: a $\to$ 0 often limit hardest to control:

Way out: RI/mSMOM (defined at finite $\overline{m} \geq 0$)? [Boyle et al., PRD 95 (2017) 5, 054505]

ADVANTAGE: Different masses at which the scheme is defined.
Tunable! - different approaches to $a \to 0$?
Possible to choose this to reduce cut-off effects?

modified continuum approach [JTT, RBC/UKQCD, 2024]: the charm quark mass

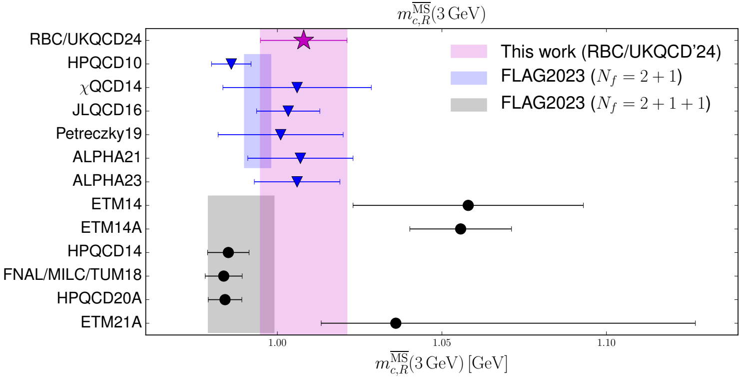


First mSMOM application

[JTT, RBC/UKQCD, PRD 110 (2024) 5, 054512]

← Very different CL approaches

- Free choice of tunable parameter



⇒ **Many possible applications in flavour and beyond!**

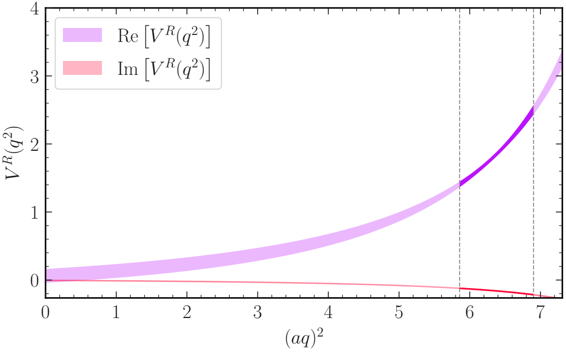
New developments (a selection)

Treating vector final states on the lattice

- ρ, K^*, ϕ, D^* all QCD-unstable.
- Typically treated as stable (narrow width approximation)
  - $\Gamma(\rho) \approx O(100)\,\mathrm{MeV}$
  - $\Gamma(\phi) \approx O(1)\,\mathrm{MeV}$
  - $\Gamma(K^*) \approx O(100)\,\mathrm{MeV}$
  - $\Gamma(D^*) \approx O(10^-1)\,\mathrm{MeV}$

  Probably reasonable for ϕ and D^* (✓), less so for ρ and K^* (✗).
- Better to do it "properly" by including e.g. $K^* \to K\pi$ in the lattice calculation ($1 + \mathcal{J} \to 2$ transition)
- Formalism developed [Briceno et al, 2016], based on finite volume scattering amplitudes [Lellouch-Lüscher formalism]

[Leskovec et al.,2025]



$B \to \rho(\to \pi\pi)\ell\nu$

- single lattice spacing and pion mass
- first exploratory calculations underway

Spectral densities and their reconstruction: the rough idea

Long distance mixing in charm [Di Carlo, Erben, Hansen, JHEP 07 (2025) 229]

$$\mathcal{M}^{\mathrm{LD}}_{D^0\to\bar{D}^0} = \frac{1}{2}\int \mathrm{d}^4x \left\langle \bar{D}^0, \vec{p}_D \right| \mathcal{T}\left\{\mathcal{H}_W(x)\mathcal{H}_W(0)\right\} \left| D^0, \vec{p}_D \right\rangle = \lim_{\epsilon\to 0}\int \frac{\mathrm{d}\omega}{2\pi}\frac{\rho(\omega)}{\omega - E_D - i\epsilon}$$

$$C_L(t) = \int \frac{\mathrm{d}\omega}{2\pi} e^{-\omega t}\rho_L(\omega)$$

Inclusive decays [Hashimoto'17, Hashimoto, Gambino '20, Barone et al '23+'25, ETMC'25, $\cdots$]

$$\frac{\Gamma}{\left|V_{fg}\right|^2} = G_F^2 \int \frac{\mathrm{d}^3 p_\nu}{(2\pi)^3}\frac{\mathrm{d}^3 p_\ell}{(2\pi)^3}\frac{L_{\mu\nu}(p_\ell, p_\nu)H^{\mu\nu}(\omega)}{4M_H^2 e_\ell e_\nu}$$

$$C^{\mu\nu}(t,\omega) = \int \mathrm{d}\omega_0 e^{-(m_H\omega_0 t)} H^{\mu\nu}(\omega_0, \omega)$$

Ill-posed inverse problem to go from correlation function to desired hadronic information.

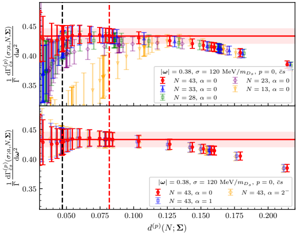
- Finite volume spectral densities (distributions) $\Rightarrow$ smear them for simulations
- approximate smearing kernels in some convergent way [HLT, Chebyshev,...]
- reconstruct and take appropriate ordered limits $(\lim_{\epsilon\to 0}(\lim_{L\to\infty} X))$

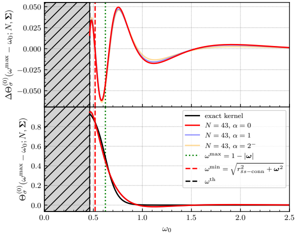
Inclusive decays on the lattice [ETMC, 2504.06063 & 2504.06064]

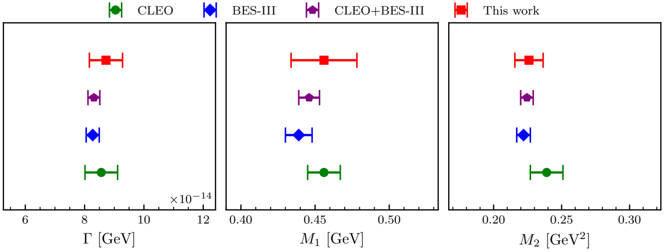
- First full calculation ($D_s \to X\ell\nu$)
- 4 lattice spacings, m_π^{phys}
- $|V_{cs}| = 0.951(35)$
- statistically dominated
- in agreement with experimental determinations

✓ Proof of concept

several other groups also active







Conclusions and Outlook

Summary - lattice precision observables

✓ $g-2$

✓ Leptonic decays

(✓) Neutral meson mixing - agreement but theory uncertainties “large”

✓ form factors

- ☹ Emergence of “niggling” tensions in some observables
- ☺ Sign that we are in a precision era

BUT:

- Systematically improvable ⇒ **“only a matter of time”**.
- Existing literature is **highly complementary**.
- **Many works in progress** ⇒ might resolve tensions/provide understanding.
- **benchmark quantities** to scrutinise intermediate results [JTT, Della Morte, EPJ-ST 233 (2024) 2, 253-27].
- Expect **improvement and scrutiny** of all aspects of these calculations (HQET-inspired vs fully relativistic; excited states; chiral-continuum extrapolation; z-expansions,...).

Summary - the future

MANY ongoing calculations in the community.

"Consolidation"

- Many new fine ensembles recently generated for $g-2$.
  $\Rightarrow$ They are just waiting to be exploited for flavour!
- Removal of approximation (iso-spin symmetry), effective heavy quark actions

New observables

- inclusive decays
- radiative corrections
- D-mixing
- radiative decays
- charming penguin contributions
- $\cdots$

Expect the lattice to weigh in in many places.

ADDITIONAL SLIDES

The bigger RBC/UKQCD heavy-flavour physics programme

MANY observables; plan to address tensions:

- experiment vs theory ($R(D^{(*)})$, form factor shapes, ...)
- different types of determinations (e.g. exclusive vs inclusive)
- between theory calculations (!)

V_{cd} $D \to \pi\ell\nu$, $D_s \to K\ell\nu$, $D \to \ell\nu$

V_{cs} $D \to K\ell\nu$, $D_s \to \phi\ell\nu$, $D_s \to \ell\nu$

V_{ub} $B \to \pi\ell\nu$, $B_s \to K\ell\nu$, $B \to \ell\nu$

V_{cb} $B_{(s)} \to D^{(*)}_{(s)}\ell\nu$

V_{td} $B^0 - \overline{B^0}$ mixing

V_{ts} $B^0_s - \overline{B^0_s}$ mixing

$+$ rare decays

Supplemented by development of new methods addressing

- truncation effects (extrapolation to $q^2 = 0$) [Flynn, Jüttner, JTT, JHEP 12 (2023) 175]
- discretisation effects via massive renormalisation schemes [JTT, RBC/UKQCD, PRD 110 (2024) 5, 054512]
- design of benchmark quantities [JTT & Della Morte, EPJ-ST 233 (2024) 2, 253-27]
- correlator fitting strategies [JTT & Portelli, PRD 112 (2025), 094512]

Correlation functions

Consider a pion which is created and annihilated again

$$
\begin{aligned}
C_2(t) &= \sum_{\mathbf{x}} \langle 0| \mathcal{T}\left[\bar{u}(0,\mathbf{0})\gamma_5 d(0,\mathbf{0})\right]\left[\bar{d}(t,\mathbf{x})\gamma_5 u(t,\mathbf{x})\right]|0\rangle \\
&= \frac{1}{Z}\int \mathcal{D}[\psi,\bar{\psi},U]e^{-S_E[\psi,\bar{\psi},U]}\sum_{\mathbf{x}} \bar{u}(0,\mathbf{0})\gamma_5 d(0,\mathbf{0})\bar{d}(t,\mathbf{x})\gamma_5 u(t,\mathbf{x}) \\
&= -\frac{1}{Z}\int \mathcal{D}[U]e^{-S_E[U]}\sum_{\mathbf{x}} \mathrm{tr}\left(\gamma_5 S_u(U;t,\mathbf{x};0,\mathbf{0})\gamma_5 S_d(U;0,\mathbf{0};t,\mathbf{x})\right)
\end{aligned}
$$

$$
\begin{aligned}
C_2(t) &= \sum_{\mathbf{x}} \langle 0| \pi(t,\mathbf{x})\pi^\dagger(0,\mathbf{0})|0\rangle \\
&= \sum_{\mathbf{x}}\sum_{n} \langle 0| e^{Ht-i\mathbf{p}\cdot\mathbf{x}}\pi(0,\mathbf{0})e^{-Ht+i\mathbf{p}\cdot\mathbf{x}}|n\rangle\langle n|\pi^\dagger(0,\mathbf{0})|0\rangle \\
&= \sum_{n} \left|\langle 0|\pi(0,\mathbf{0})|_{\mathbf{p}=0}|n\rangle\right|^2 e^{-E_n t}
\end{aligned}
$$

where the *propagator* $S(U,x,y)$ is solution of $D_{a,b;\alpha,\beta}(U,z,y)S_{b,c;\beta,\gamma}(U,y,x) = \delta_{ac}\delta_{\alpha\gamma}\delta(z-x)$

1. Get correlation functions from

2. Get E_n and matrix elements from data analysis.

Size of the numerical problem

Using $N_{\text{spin}} = 4$, $N_{\text{colour}} = 3$ and defining $|\Lambda| = (L/a)^3 \times (T/a)$

- Gauge fields $\rightarrow$ *link variables* $U_\mu(n)$: $SU(3)$ matrix linking sites: $9 \times |\Lambda|$ complex entries.
- Fermion fields turn into vectors of size $4 \times 3 \times |\Lambda|$.
- Operators $\rightarrow$ matrices. E.g. Dirac operator: $(4 \times 3 \times |\Lambda|)^2$ entries.

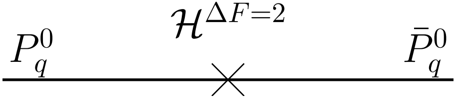
(former) BG/Q in Edinburgh

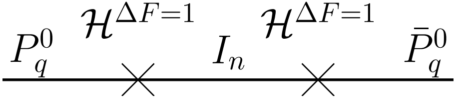
Small volume: $L/a = 16, T/a = 32$: $2.5 \times 10^{12}\,\text{entries}$

State of the art: $L/a = 96, T/a = 192$: $4.2 \times 10^{18}\,\text{entries}$

Correlation functions built from propagators $\equiv$ inverse of the Dirac operator...

Neutral meson mixing - theory





$$\left\langle P_q^0 \right| \mathcal{H}_{\text{eff}} \left| \bar{P}_q^0 \right\rangle \propto \underbrace{\left\langle P_q^0 \right| \mathcal{H}^{\Delta F=2} \left| \bar{P}_q^0 \right\rangle}_{\text{Short distance}} + \underbrace{\sum_n \frac{\left\langle P_q^0 \right| \mathcal{H}^{\Delta F=1} \left| n \right\rangle \left\langle n \right| \mathcal{H}^{\Delta F=1} \left| \bar{P}_q^0 \right\rangle}{E_n - M_{P_q}}}_{\text{Long distance}}$$

For K^0, B^0, B_s^0: **Short distance dominated** ('easy' on the lattice)

$$\text{short distance} \propto \left| \sum_{q'=u,c,t} \frac{m_{q'}^2}{M_W^2} V_{q'b} V_{q'q}^* \right|^2 \approx \frac{m_t^4}{M_W^4} \left| V_{tb} V_{tq}^* \right|^2$$

SD: Top enhanced: $m_t^2 V_{tb} V_{tq}{}^* \gg m_c^2 V_{cb} V_{cq}^* \gg m_u^2 V_{ub} V_{uq}^*$
LD: Only m_c, m_u in intermediate states: no top + CKM suppressed.