Modular Properties of Generalised Gibbs Ensembles

- **Based on:**
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  - 2508.16258 with Max Downing, Gerard Watts, Tanmoy Sengupta and Adarsh Sudhakar,
  - 2601.XXXXX with Gerard Watts.

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Motivation
2d CFTs and GGEs

$$\mathrm{Tr}(e^{-\beta H}) \longrightarrow \mathrm{Tr}(e^{-\sum_i \beta_i H_i})$$

- *Statistical Physics* → *Generalised Eigenstate Thermalisation Hypothesis*
  - *Second-order phase transitions.*
- *AdS/CFT correspondence.* → *Higher Spin Black Holes*
  - *E.g. Black Hole Thermodynamics*
  - *KdV Black Holes, W(3) Black Holes...*
- *Pure Maths* → *Generalised Power Partitions*
  - *(e.g. monstrous moonshine, Langlands program, spectral geometry, automorphic forms etc...)*

Conformal Field Theory

- A (chiral) **2d CFT** is a QFT whose spacetime symmetry is the Virasoro algebra, given by:

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{c}{12}(m^3 - m)\delta_{m+n,0}$$

- Which is equivalent to a formulation in terms of an Operator Product Expansion. $T(z) := \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}$$

- Mode expansion of a field with scaling dimension h is

$$\phi(z) = \sum_{n=-\infty}^{\infty} \phi_n z^{-n-h}$$

State-Operator Correspondence

- There is a one-to-one correspondence between fields and states.

$$|\phi\rangle = \phi(0)|0\rangle = \phi_{-h}|0\rangle$$

- Doing normal ordering and derivatives can also be understood in terms of states too.

$$(\phi^{(1)}(\phi^{(2)}\phi^{(3)}))(z) \leftrightarrow \phi^{(1)}_{-h_1}\phi^{(2)}_{-h_2}\phi^{(3)}_{-h_3}|0\rangle$$

$$\partial^n\phi(z) \leftrightarrow L^n_{-1}\phi_{-h}|0\rangle = n!\phi_{-h-n}|0\rangle$$

- In general, the ***vacuum*** state satisfies

$$\phi_k|0\rangle = 0 \qquad k > -h$$

Representations (1/3)

- Representations are given by primary states, which correspond to primary fields, and their descendants.
- A field ϕ is called ***primary*** if its corresponding state satisfies:

$$L_{n>0}|\phi\rangle = 0 \qquad\qquad L_0|\phi\rangle = h|\phi\rangle$$

- Acting with negative modes will give ***descendants***.
- In general, L_0 grades a representation.

$$L_0\left(L_{-n_1}...L_{-n_k}\right)|\phi\rangle = \left(h + \sum_i n_i\right)\left(L_{-n_1}...L_{-n_k}\right)|\phi\rangle$$

- We call the sum of indices the ***level*** of the state, which determines the conformal weight of the field/state.
- We call a field ***quasi-primary*** if its state satisfies:

$$L_1|\phi\rangle = 0$$

Representations (2/3)

An example of a ***quasi-primary field*** of weight 4 is

$$\Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T(z)$$

Which corresponds to a ***level 4 quasi-primary state***

$$|\Lambda\rangle = (L^2_{-2} - \frac{3}{5}L_{-4})|0\rangle$$

One can check that:

$$L_1|\Lambda\rangle = 0$$

Representations (3/3)

- We demand that every representation is ***irreducible***.
- That means we quotient out states which are both descendants and primaries (set them to zero). These are ***null states***.
- At special rational values of central charge, this constrains the number of representations to be finite.
- e.g. Lee-Yang Minimal model. Allowed representations are:

$$c = -22/5 \qquad |0\rangle \quad \&(descendants) \qquad |-1/5\rangle \quad \&(descendants)$$

- An example of a null state, which constrains the representations is:

$$\left(L^2_{-2} - \tfrac{3}{5} L_{-4}\right)|0\rangle = 0$$

Characters of Representations

- Denote by V_{h_i} a representation with highest weight state $|h_i\rangle$.
- Suppose in our theory there are finitely many such representations i=1,...,N
- The characters (“Gibbs Ensembles”) of the representations are:

$$\chi_i(\tau) := \mathrm{Tr}_{V_{h_i}}(q^{L_0 - c/24}) \qquad q := e^{2\pi i \tau}$$

- They transform in a vector representation of the modular group. $\mathrm{SL}(2,\mathbb{Z})$
- For us, this just means:

$$\chi_i(-\tfrac{1}{\tau}) = S_{ij}\chi_j(\tau)$$

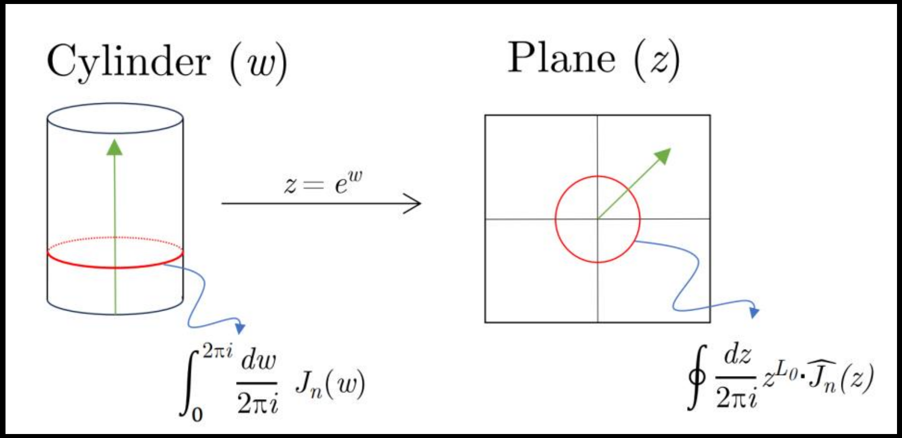
- Can build ***Partition Function*** with characters. (Modular *Invariant*)

Local Integral of Motion

- An integral motion of spin s is the integral, on a fixed spatial slice of a cylinder, of a level $s+1$ quasi-primary descendant of the identity field.
- This integral can be expressed as one on the plane by transforming the field in an algorithmic way. In that way we extract modes.
- For integrability, we demand that infinitely many integrals of motion ***commute*** with one another.

$$\mathbf{Q}_s = \int_0^{2\pi i} \frac{dw}{2\pi i} J_{s+1}(w)$$

$$= \oint \frac{dz}{z 2\pi i} \tilde{J}_{s+1}(z)$$



Example: The KdV Charges

- These charges can be defined at any central charge. They are expressed only in terms of Virasoro modes.
- They have odd-spin. Here are the first 3.

$$\mathbf{I}_1 = \int_0^{2\pi i} \frac{dw}{2\pi i} T(w) = L_0 - \frac{c}{24}$$

$$\mathbf{I}_3 = \int_0^{2\pi i} \frac{dw}{2\pi i} \Lambda(w) = \left(2 \sum_{k=1}^{\infty} L_{-k} L_k + L_0^2 - \frac{c+2}{12} L_0 + \frac{c(5c+22)}{2880} \right)$$

$$\mathbf{I}_5 = \sum_{k_1+k_2+k_3=0} : L_{k_1} L_{k_2} L_{k_3} : + \sum_{k=1}^{\infty} \left(\frac{c+11}{6} k^2 - 1 - \frac{c}{24} \right) L_{-k} L_k$$
$$+ \frac{3}{2} \sum_{k=1}^{\infty} L_{1-2k} L_{2k-1} - \frac{c+4}{8} L_0^2 + \frac{(c+2)(3c+20)}{576} L_0 - \frac{c(3c+14)(7c+68)}{290304}$$

Thermal Correlation Functions (1/2)

- Thermal correlators of IoMs have known modular properties.
- Multi-point functions transform as quasi-modular forms. Here we have one of weight $s_1 + s_2 + \ldots + s_N$ and depth $N-1$.

$$\mathrm{Tr}_i(\mathbf{Q}_{s_1-1}\ldots\mathbf{Q}_{s_N-1}q^{L_0-c/24})$$

- We can write them as Modular Linear Differential Operators (MLDO). E.g.

$$\mathrm{Tr}_i(\mathbf{I}_3^2 q^{L_0-c/24}) = \left(D^4 + \tfrac{3(c/24)+5}{90}E_4D^2 - \tfrac{72(c/24)+11}{1080}E_6D + \tfrac{(c/24)(1221(c/24)+500)}{75600}E_4^2\right)\chi_i$$
$$+E_2\left(\tfrac{2}{3}D^3 + \tfrac{72(c/24)+11}{1080}E_4D - \tfrac{(c/24)(12(c/24)+5)}{756}E_6\right)\chi_i$$

- Here, D is the covariant serre derivative, which acts on a weight r modular form, and gives back a weight $r+2$ modular form, and the E 's are Eisenstein series.

$$D = q\partial_q - \tfrac{r}{12}E_2 \qquad\qquad E_{2k}(\tau) = 1 + \tfrac{2}{\zeta(1-2k)}\textstyle\sum_{n=0}^{\infty}\frac{n^{2k-1}q^n}{1-q^n}$$

Thermal Correlation Functions (2/2)

- Generically $\tau \longrightarrow -\frac{1}{\tau}$:

$$E_4^a E_6^a D^c \chi_i \longrightarrow (\tau^{4a+6b+2c}) S_{ij} E_4^a E_6^a D^c \chi_j \qquad E_2 \longrightarrow \tau^2 E_2 - \frac{6i\tau}{\pi}$$

- One-point functions transform simply:

$$\mathrm{Tr}_i(\mathbf{Q}_s q^{L_0 - c/24}) \longrightarrow \tau^{s+1} S_{ij} \mathrm{Tr}_j(\mathbf{Q}_s q^{L_0 - c/24})$$

- When we do modular transformations, we pick up additional IoMs from outside of a particular hierarchy. E.g. KdV:

$$\mathrm{Tr}(\mathbf{I}_3^2 q^{L_0 - c/24}) \longrightarrow \tau^8 \mathrm{Tr}(\mathbf{I}_3^2 q^{L_0 - c/24}) - \frac{i\tau^7}{\pi} \left(4\mathrm{Tr}(\mathbf{I}_5 q^{L_0 - c/24}) + \frac{5}{54}(c+2)\mathrm{Tr}(\mathbf{J}_5 q^{L_0 - c/24})\right)$$

$$[\mathbf{I}_5, \mathbf{J}_5] \neq 0$$

GGEs

- Suppose we have and infinite set of commuting IoMs.

$$\{\mathbf{Q}_s\}_{s=1}^{\infty} \quad [\mathbf{Q}_i, \mathbf{Q}_j] = 0$$

- By extending the characters, we obtain the GGE, with chemical potentials α_s .

$$\mathrm{Tr}_i(q^{L_0 - c/24} e^{\sum_s \alpha_s \mathbf{Q}_s})$$

- Convergent if $\mathrm{Re}(\alpha_{2n\text{-}1}) \leq 0$, and if $\mathrm{Re}(\alpha_{2n}) = 0$.
- We know that characters and thermal correlators transform nicely under modular transformations. What about the GGE?

$$\tau \longrightarrow -\frac{1}{\tau} \qquad \mathrm{Tr}_i(q^{L_0 - c/24} e^{\sum_s \alpha_s \mathbf{Q}_s}) \longrightarrow ???$$

What are our options?

- ☛ We could expand in the chemical potentials and transform each thermal correlation function in the expansion.
  - ☛ General, but this is ***asymptotic*** at best
- ☛ We could take advantage of integrability techniques like the Thermodynamic Bethe Ansatz
  - ☛ Works only in cases where the integrable hierarchy can be understood as coming from a relevant perturbation of the theory.
- ☛ We could look for cases where the transformation can be done exactly.
  - ☛ 100% correct and unambiguous, but not general.

General Asymptotic Formula (1/3)

- Consider a GGE with a single charge of spin s inserted.

$$\mathrm{Tr}(q^{L_0-c/24}e^{\alpha \mathbf{Q}_s}) \sim \sum_{k=0}^{\infty} \mathrm{Tr}(q^{L_0-c/24}\mathbf{Q}_s^k)\frac{\alpha^k}{k!}$$

- We can transform each term separately. This is straightforward but can only go up to a few orders in chemical potential.
- We get charges from outside any given hierarchy too. E.g. KdV

$$\mathrm{Tr}(\mathbf{I}_3^2 q^{L_0-c/24}) \longrightarrow \tau^8 \mathrm{Tr}(\mathbf{I}_3^2 q^{L_0-c/24}) - \frac{i\tau^7}{\pi}\left(4\mathrm{Tr}(\mathbf{I}_5 q^{L_0-c/24}) + \frac{5}{54}(c+2)\mathrm{Tr}(\mathbf{J}_5 q^{L_0-c/24})\right)$$

- Can we re-sum this expression into a new exponential, even though these new charges do not commute?
- We showed that at lowest non-trivial order**, yes you can!** E.g. Lee-Yang GGE with spin-5 KdV charge:

$$\mathrm{Tr}(\hat{q}^{L_0-c/24}e^{\alpha \mathbf{I}_5}) \sim \mathrm{Tr}(q^{L_0-c/24}e^{\alpha_5 \mathbf{I}_5 + \beta_9 \mathbf{J}_9 + \alpha_{13}\mathbf{I}_{13} + \beta_{13}\mathbf{J}_{13} + \dots})$$

General Asymptotic Formula (2/3)

- Can we determine exactly which charges appear in the modular transformation?
- W_3 algebra (Extension of Virasoro by weight-3 Field $W(z)$).

- Spin-2 IM: $W_0 = \int_0^{2\pi i} \frac{dw}{2\pi i} W(w)$
- Conjecture: $\mathrm{Tr}(q^{L_0 - c/24} e^{\alpha W_0}) \sim \mathrm{Tr}(q^{L_0 - c/24} e^{\alpha \tau^3 \mathcal{W}_0})$

- The new charge is the IM of an infinite sum of local fields.

$$\mathcal{W}(z) := \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\alpha \tau^2}{4\pi i}\right)^n \left[W^{n+1}\right](z) \qquad \mathcal{W}_0 = \int_0^{2\pi i} \frac{dw}{2\pi i} \mathcal{W}(w)$$

Generic Asymptotic Formula (3/3)

- The fields in the sum are determined recursively.

$$[W^n] = \sum_{m=1}^{n-1} \binom{n-2}{m-1} \left([W^m]\,[W^{n-m}] \right)_2 \,, \qquad [W] = W$$

- Here, $(AB)_2(z)$, denotes the 2nd order pole in the OPE between $A(w)$ and $B(z)$.
- Using the MLDO approach, we tested this conjecture up to $O(\alpha^7)$ at generic central charge.
- We showed also that is true to all orders in α at $c =$ -2.
- It is easy to generalise this conjecture to an arbitrary number of charges inserted.

Integrability Approach (1/4)

- We can ***formally*** define the KdV hierarchy in any 2d CFT.
- We can understand this hierarchy as ***integrable*** when it preserved by a relevant perturbation.
  - Always possible in a minimal model $M(p,q)$, perturb action as follows:

$$M(p,q) + \lambda \int d^2 z \phi_{1,3}(z) \quad , \quad h_{1,3} = 2(p/q) - 1$$

- Then the modular transformation can be understood as the ***mirror transformation*** of the ***Thermodynamic Bethe Ansatz (TBA)***.
- This is can replicate the asymptotic analysis, but also one can find non-asymptotic solutions to it, which one must include in the transformation.

Integrability Approach (2/4)

- We looked at the Lee-Yang GGE with a spin-5 KdV charge inserted.

$$\mathrm{Tr}(\hat{q}^{L_0-c/24}e^{\alpha \mathbf{I}_5}) \sim \mathrm{Tr}(q^{L_0-c/24}e^{\alpha_5 \mathbf{I}_5+\beta_9 \mathbf{J}_9+\alpha_{13}\mathbf{I}_{13}+\beta_{13}\mathbf{J}_{13}+\dots})$$

- The TBA equation is:

$$\epsilon(\theta) = e^{\theta} + \frac{\alpha C}{L^4}e^{5\theta} - \int \varphi(\theta-\theta')\log(1+e^{-\epsilon(\theta')})\frac{d\theta'}{2\pi}$$

- Each state in the theory has an associated TBA equation. Choose different integration contours to pick them out.
- The spectrum is given by: $E(L) = -\frac{1}{L}\int e^{\theta}\log(1+e^{-\epsilon(\theta)})\frac{d\theta}{2\pi}$
- Can recover asymptotic results by expanding:

$$E \sim \mathcal{E}_0 + \alpha\mathcal{E}_1 + \alpha^2\mathcal{E}_2 + \alpha^3\mathcal{E}_3 + \dots$$

Integrability Approach (3/4)

- We solved the TBA equations perturbatively and numerically, then showed they matched the charges that appear in the transformation.
- Excited state eigenvalues are difficult to directly calculation, but TBA gives them to high accuracy.
- E.g. Ground sate Eigenvalues:

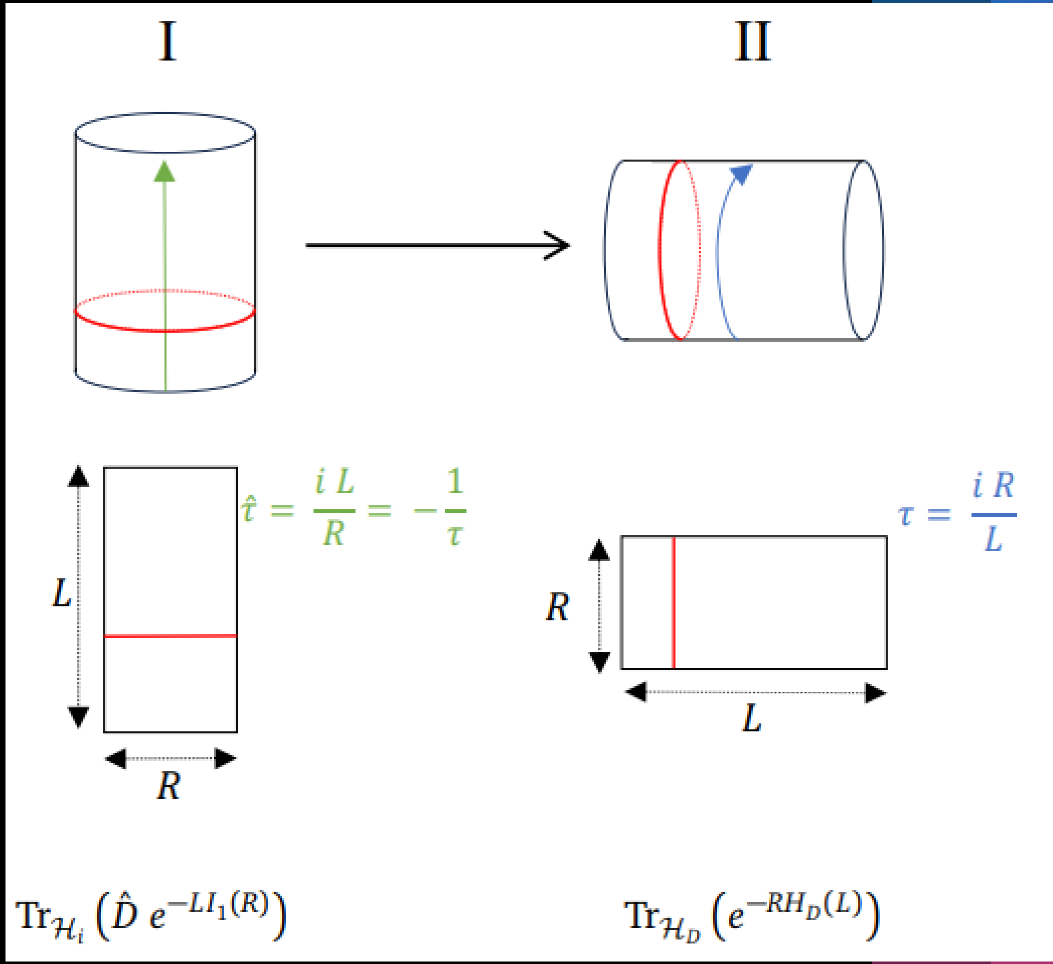
Eigenvalue	Numerical Value (TBA)	Analytic Value (Direct Calculation)
$\mathbf{I}_1 = \mathcal{E}_0$	−0.0166666666666666	$-\frac{1}{60}$= −0.0166666666666666
$\mathbf{I}_5 = \mathcal{E}_1$	0.00011772486772486771 89	$\frac{89}{756000}$ = 0.00011772486772486772
$\mathbf{J}_9 = \mathcal{E}_2$	-0.0000800462962962962 3	$-\frac{1729}{21600000}$= −0.0000800462962962963
$\alpha_{13}\mathbf{I}_{13} + \beta_{13}\mathbf{J}_{13} = \mathcal{E}_3$	-0.000415888503086419 04	$-\frac{1077983}{2592000000}$ = -0.000415888503086419 75

Integrability Approach (4/4)

- We showed that the TBA has additional, non asymptotic solutions.
- Requires a fractional power expansion:

$$\epsilon(\theta) = \sum_{n=0}^{\infty} \epsilon_{n/4}(\theta) \left(\frac{\alpha}{L^4}\right)^{n/4}$$

- Excited state energies of this form vanish as $\alpha \to 0$.
- These energies cannot be attributed to any state in the CFT, yet we must include them to reconstruct the GGE.
  - → Interpret GGE as a Defect (Not yet fully worked out except for free fermion at c=1/2).



Exact approach (1/4)

- At $c = 1/2$ and $c =$ -2, we can construct hierarchies explicitly.
- At $c =$ -2 we have the symplectic fermion theory.

$$\chi^{\pm}(z)\chi^{\mp}(w) = \frac{\pm 1}{(z-w)^2} + O(1)$$

- We can build a hierarchy of charges which contain KdV.

$$\mathbf{Q}_{n-1} = \int_0^{2\pi i} \frac{dw}{2\pi i} (\partial^{n-2}\chi^- \chi^+)(w) \qquad \mathbf{Q}_{2n-1} = \mathbf{I}_{2n-1}$$

- We can obtain explicit expressions in a λ -twisted Fock module

$$\mathbf{Q}_{n-1} = (-1)^n \sum_{j \in \mathbb{Z}+\lambda} j^{n-2} : \chi_j^- \chi_{-j}^+ : -\mathbf{c}_{n-1}(\lambda). \quad , \quad \lambda = 0, \tfrac{1}{2}.$$

- $\mathbf{c}_{n\text{-}1}$ is a regularisation constant.
- The currents B_n associated to each $\mathbf{Q}_{n\text{ -}1}$ satisfies:

$$(B_n B_m)_2 = (m+n \text{ -}2) B_{n+m\text{-}2}$$

Exact approach (2/4)

- The GGE can be determined exactly, since the theory is free and the modules are simple fock spaces.

$$\mathrm{Tr}_{\lambda,s}(q^{L_0-c/24}e^{\sum_{k=2}^{N}\alpha_k \mathbf{Q}_k}) = e^{-\sum_n \alpha_n \mathbf{c}_n(\lambda)} \prod_{n\in\mathbb{N}-\lambda}(1-e^{2\pi i s}e^{\sum_k \alpha_k n^k}) \times(1-e^{2\pi i s}e^{\sum_k (-1)^k \alpha_k n^k})$$

- We can apply techniques from analytic number theory [D. Zagier, 2021], to our situation, and find the modular transformation of this exactly.

Exact approach (3/4)

- The Exact transformation of this GGE is a big mess, but its exact and correct. It clearly not a GGE of the same CFT.

$$\boxed{\begin{aligned}\Psi_{\lambda,s}(\hat{\tau},\alpha) = \frac{2^{\delta_{s,0}\delta_{\lambda,1/2}}}{2^{\delta_{s,1/2}}2^{\delta_{\lambda,0}}}\left(\frac{\tau}{i}\right)^{\delta_{\lambda,0}\delta_{s,0}} \prod_{\pm} \exp\left\{\int_0^\infty (\log(1-e^{2\pi i s}e^{2\pi i(\sum_{n=1}^N (\pm 1)^{n+1}\alpha_n \hat{\tau}^n x^n)}))dx\right\} \\ \times \left(\prod_{n\in\mathbb{Z}-s}\ \prod_{z_j^\pm(n)\in\mathbb{H}^+} (1-e^{2\pi i\lambda}e^{2\pi i z_j^\pm(n)})\right),\end{aligned}} \tag{5.74}$$

with roots satisfying

$$\boxed{z = z_j^\pm(\omega):\quad \pm\sum_{n=1}^N (\pm 1)^n(\hat{\tau}^n\alpha_n z^n) = \omega.} \tag{5.75}$$

Exact approach (4/4)

- Looking at the case with just a spin-2 charge inserted, we can check our conjecture from earlier in the talk to all orders.
- Expanding around $\alpha \to 0$, one finds that

$$\mathrm{Tr}(q^{L_0 - c/24} e^{\alpha \mathbf{Q}_2}) \longrightarrow \mathrm{Tr}(q^{L_0 - c/24} e^{\alpha \mathcal{Q}_2})$$

$$\mathcal{Q}_2 = \sum_{k=0}^{\infty} \left(\left(\frac{\alpha \tau^2}{4\pi i} \right)^k \frac{1}{k!} \frac{2^{k+1}(2k+1)!}{(k+2)!} \mathbf{Q}_{k+2} \right)$$

- This is exactly what one expects to get from the recursive definition from earlier in the talk.

$$\mathcal{W}(z) := \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\alpha \tau^2}{4\pi i} \right)^n \left[W^{n+1} \right](z)$$

Outlook

- Can we apply other integrability techniques to this transformed GGE?
  - DDV equation? ODE/IM correspondence? Quantum Spectral Curve approach?
- What actually **is** the GGE defect?
  - Worked out at $c=1/2$, could be worked out at $c=-2$.
- Can we prove that the asymptotic formula works at any central charge?
- Cardy formula for GGE? Higher Spin Black Hole entropy?

Fin