4-dimensional String Islands from Asymmetric Orbifolds

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Based on WIP with G. Aldazabal, E. Andrés, A. Font, K. Narain, I. Zadeh

Durham, 17 December 2025

Motivation

- String Theory contains gravity!
- Reproduces Einstein's gravity at low energy: hope to realise our world as a string vacuum

Problem: many internal geometries

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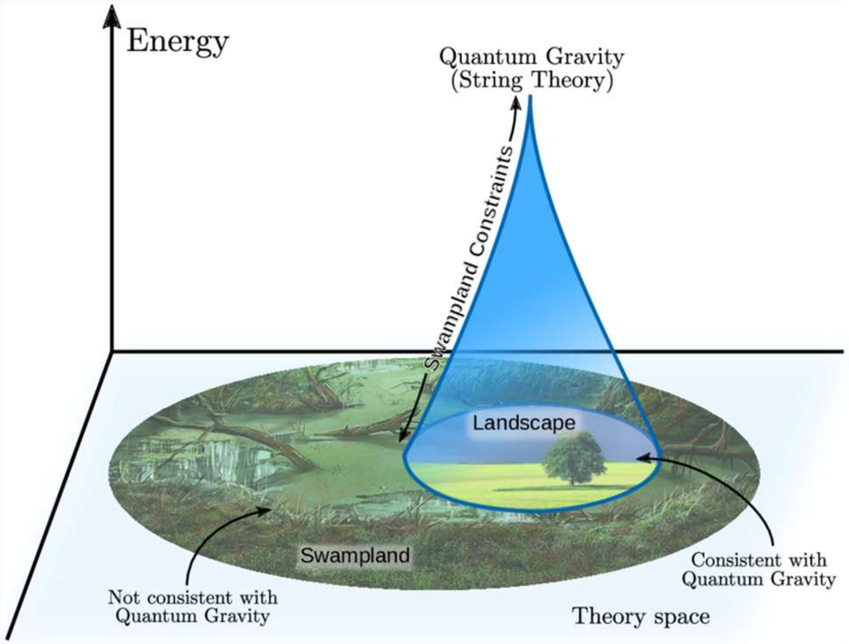


Picture: [van Beest, Calderón-Infante, Mirfendereski, Valenzuela; 20]

Motivation

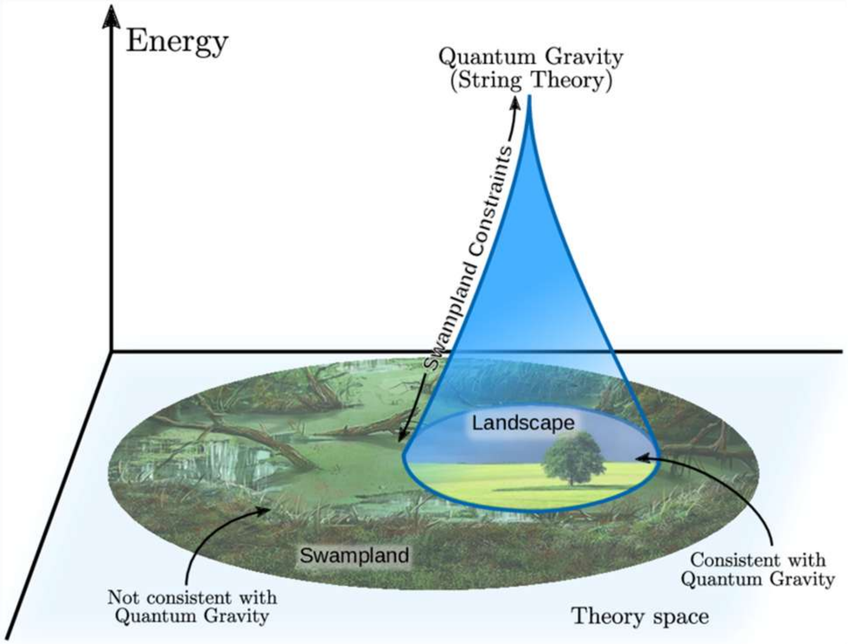
- String Theory contains gravity!
- Reproduces Einstein's gravity at low energy: hope to realise our world as a string vacuum

Problem: many internal geometries

Phenomenologically relevant theories:

- 4d
- Moduli
- Supersymmetry?
- Rank of the gauge group

[Acharya, Douglas; 06]



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Since the landscape is finite, it's worth studying → understand Quantum Gravity
↘ moduli space of CFTs

Roadmap:

1. Compactify (geometric orbifold) down to 4d ← many scalar fields (70 in type IIB on T^6) enhanced rank (heterotic)

2. SUSY breaking mechanism that gives mass to the scalars

3. Phenomenologically interesting theory

Scalars: continuous parameters of the moduli space

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instead →

Asymmetric Orbifolds

No moduli (except the (axi)dilaton)
Reduced Rank
Still some SUSY left

ISLANDS

In this talk:

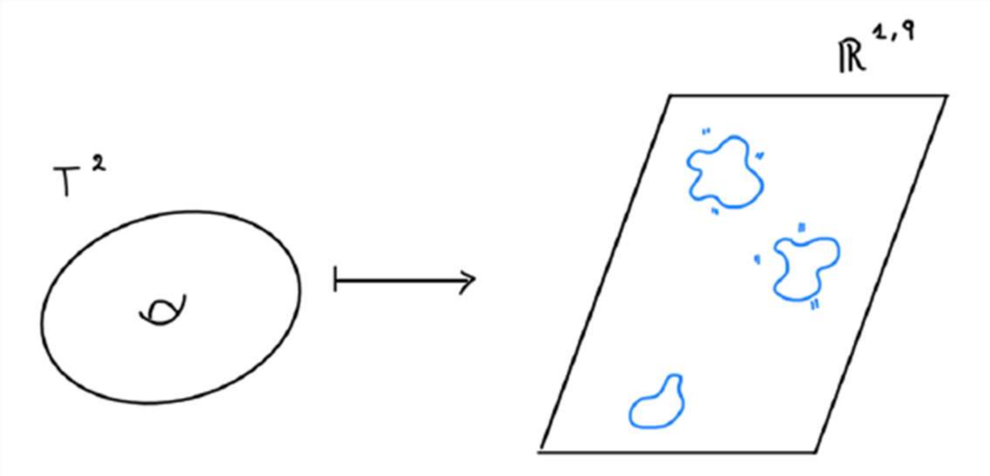
focus on type IIB on the torus, T^6/Z_9 asymmetric orbifolds

DH islands from Lie algebra lattices [Dabholkar, Harvey; 98]

Main result: archipelago of 3 islands!

Type IIB String Theory on the Torus

String theory is a sigma model: $\Sigma \mapsto R^{1,9}$



$$Z = \frac{1}{\left(\sqrt{\tau_2}\eta\bar{\eta}\right)^8}\frac{1}{(\eta\bar{\eta})^4}\left(\sum_{p,r\in v} - \sum_{p,r\in s}\right) q^{\frac{1}{2}p^2}\bar{q}^{\frac{1}{2}r^2} \qquad p,r \in SO(8)$$

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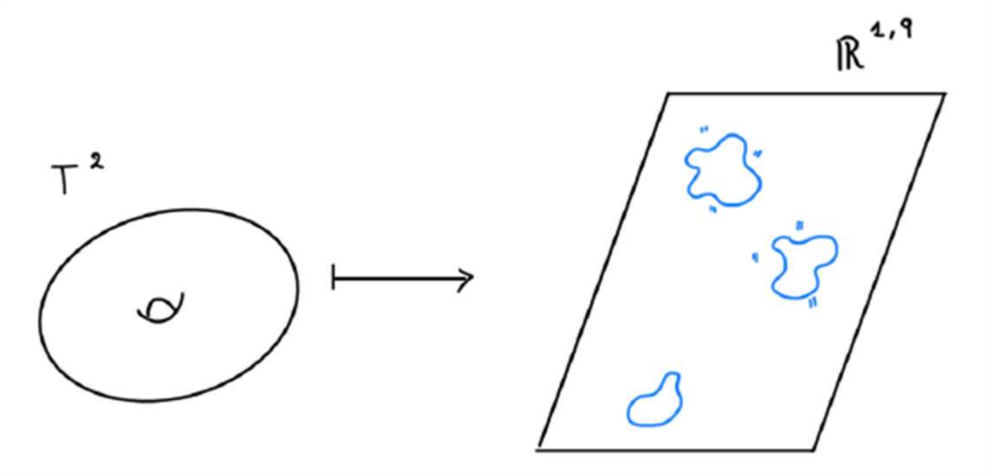


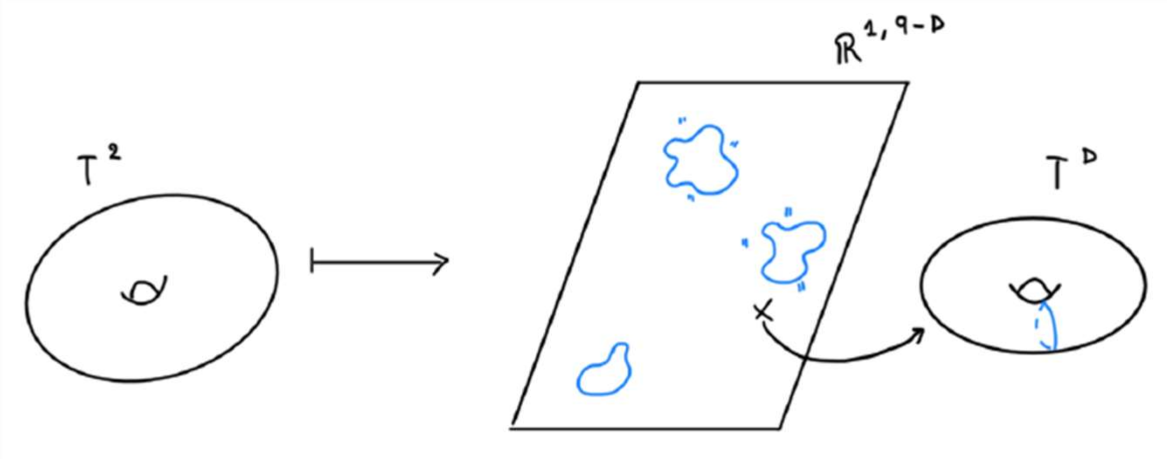
$$q = e^{2\pi i\tau},$$
$$\tau = \tau_1 + i\tau_2,$$
$$\eta = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

$$Z = \frac{1}{\left(\sqrt{\tau_2}\eta\bar{\eta}\right)^8}\frac{1}{(\eta\bar{\eta})^4}\left(\sum_{p,r\in v} - \sum_{p,r\in s}\right) q^{\frac{1}{2}p^2}\bar{q}^{\frac{1}{2}r^2}$$

$p, r \in SO(8)$

Toroidal Compactification

$$\Sigma \mapsto R^{1,9-\mathrm{D}} \times T^{D}$$



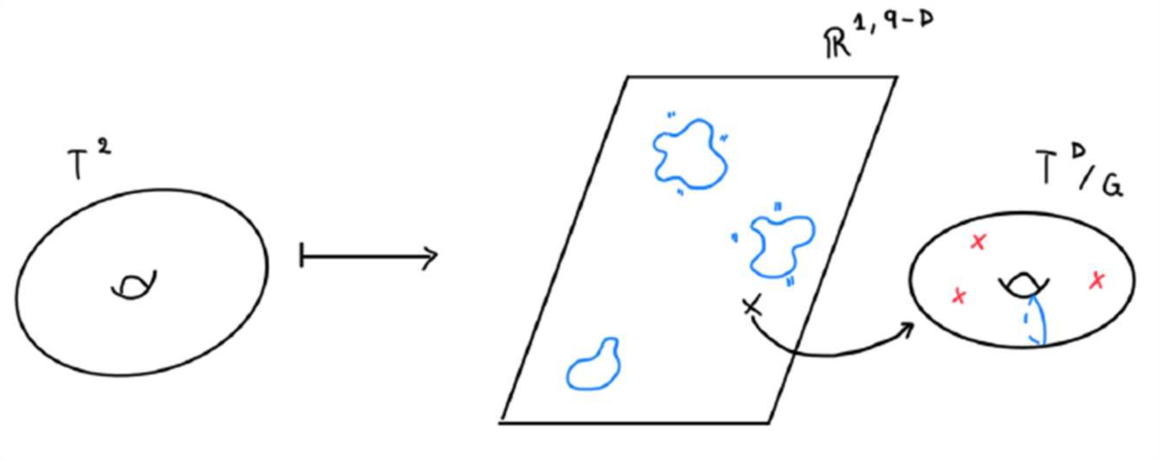
Strings propagate on $R^{1,9-\mathrm{D}}$ and have dofs that can wind and vibrate on T^{D}: changes the spectrum.

States with quantised momenta and winding, taking values in a Lorentzian evenself-dual lattice $\Gamma_{D,D}$

$$Z = \frac{1}{\left(\sqrt{\tau_2}\eta\bar{\eta}\right)^{8-\mathrm{D}}} \frac{1}{(\eta\bar{\eta})^4} \left(\sum_{p,r\in v} - \sum_{p,r\in s} \right) q^{\frac{1}{2}p^2} \bar{q}^{\frac{1}{2}r^2} \frac{1}{\eta^D \bar{\eta}^D} \sum_{(P_L,P_R)} q^{\frac{1}{2}\mathrm{P_L}^2} \bar{q}^{\frac{1}{2}\mathrm{P_R}^2} \qquad (P_L, P_R) \in \Gamma_{D,D}$$

Toroidal Orbifolds

$$\Sigma \mapsto R^{1,9-\mathrm{D}} \times T^{D}/G$$



Intuitively: all the closed string states invariant under G will survive, the others are projected out

We mod out by an automorphism G of T^D

= identify points on T^D

= gauging a global symmetry

But there is more than that!

Twisted sectors: strings that are open on T^D, but are closed on T^D/G.
These sectors can be pictured as living at the fixed points under the G-action: GS degeneracy $\mathcal{F}$.

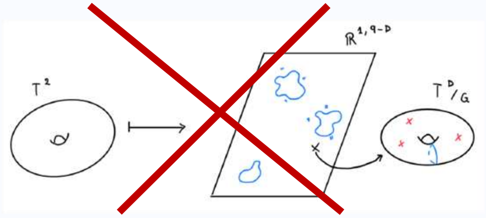
These states cannot move freely on T^D/G because of the twisting conditions.

$$\sum_{(P_L,P_R)} q^{\frac{1}{2}P_L{}^2}\,\bar{q}^{\frac{1}{2}P_R{}^2}$$

More precisely:

$$Z = \frac{1}{|G|}\sum_{k,l} Z\left(g^k, g^l\right) \qquad\qquad Z\left(g^k, g^l\right) = \mathrm{Tr}\left(g^l q^{L_0\left(g^k\right)}\,\bar{q}^{\bar{L}_0\left(g^k\right)}\right)$$

Asymmetric Orbifolds [Narain, Samradi, Vafa; 87]



No geometric meaning

Instead of modding by an isometry of T^D, we mod out by an isometry of $\Gamma_{D,D}$, that acts asymmetrically on L/R-movers

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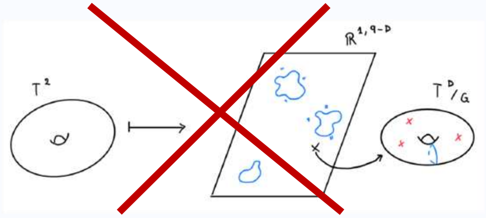

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DH island construction [Dabholkar, Harvey; 98]:

1. Type IIB on T^D: go to a point in the moduli space where $\Gamma_{D,D} = (P_L, P_R)$, $P_{L,R} \in \Lambda_w(\mathfrak{g})$ and $P_L - P_R \in \Lambda_r(\mathfrak{g})$. $\Lambda_{r,w}$ root/weight lattice of $\mathfrak{g}$ rank D simple Lie algebra

2. This means that the Weyl group $\mathcal{W}(\mathfrak{g})$ is an isometry of $P_{L,R}$ separately!

3. We choose a generator Θ of Z_n and act asymmetrically: $\Theta = (\Theta_L, 0)$

4. L-movers are completely rotated, but the sublattice $\mathrm{I} = (0, \Lambda_r)$ survives!

Action of Z_n on the lattice vectors defined up to a phase:

$$g : |P_L, P_R > \longrightarrow e^{2\pi i P\cdot v}|\Theta_L P_L, P_R >$$

$$g : \left|p > \longrightarrow e^{2\pi i p\cdot v_f}\right|p >$$

5. To get an island appropriate shift vectors v must be chosen

Non trivial Θ eigenvalues $e^{\pm 2\pi i t_a}$, $0 < t_a \leq \frac{1}{2}$, $a = 1, \ldots, \frac{D}{2}$. Action on complexified bosons

$$gX^a = e^{2\pi i t_a} X^a\ ,\ gX^{*a} = e^{-2\pi i\ \ _a} X^{*a}$$

University of Southampton

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Breaks half SUSY!

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Our work

[Dabholkar, Harvey; 98] claim that there is a Type IIB T^6/Z_9 island with $\mathfrak{g} = E_6$ leading to $\mathcal{N} = 4$ pure sugra in 4d.

$\mathrm{p} \cdot \mathrm{p}' = 2\mathrm{Z}$

$\mathrm{Vol}(\Gamma_{6,6}) = 1$

We built the Lorentzian even self-dual lattice using the gluing construction [Nikulin; 80]:

$$\Gamma_{6,6}(E_6) = \Lambda_r \oplus \Lambda_r(-1) + (\omega_1, \omega_1)$$

Modular Invariance

The Gram matrix

$$G\left(\Gamma_{6,6}\right) = \begin{pmatrix} 2 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 & 1 & -1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & -2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} , \quad \sqrt{\det(G\left(\Gamma_{6,6}\right))} = 1\,.$$

The partition function

$$Z = Z_X Z_{\bar{\psi}} Z_{\psi\Gamma}$$

Non-compact bosons

$$Z_X = \frac{1}{\left(\sqrt{\tau_2}\eta\bar{\eta}\right)^2}$$

R-moving fermions

$$Z_{\bar{\psi}} = \frac{1}{(\eta\bar{\eta})^4}\left(\sum_{r\in v} - \sum_{r\in s} \quad\right)\bar{q}^{\frac{1}{2}r^2}$$

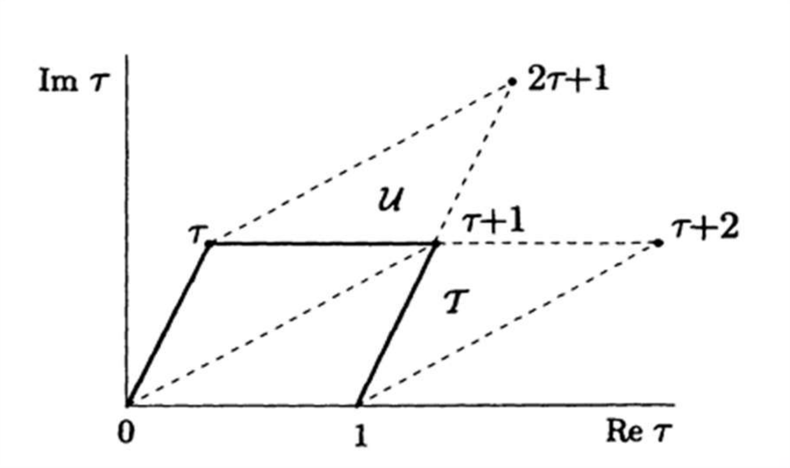
L-moving fermions + compact bosons

$$Z_{\psi\Gamma} = \frac{1}{9}\sum_{k,l} Z\left(g^k, g^l\right)$$

Orbifold action is only here

Compute the untwisted sector $Z\left(1, g^l\right)$ and then use modular invariance!

Picture: [Di Francesco, Mathieu, Sénécal; 80]



$$T\colon\ \tau \longrightarrow \tau + 1$$

$$S\colon \tau \longrightarrow \frac{1}{\tau}$$

$$U \;=\; TST$$

Coprime sectors $(l, 9) = 1$: $\quad Z\big(1, g^l\big) \xrightarrow{S} Z\big(g^l, 1\big) \xrightarrow{T^k} Z\big(g^l, g^{lk}\big)$

g^3, g^6 sectors: $\quad Z(1, g^3) \xrightarrow{S} Z(g^3, 1) \xrightarrow{T} Z(g^3, g^3) \xrightarrow{T} Z(g^3, g^6)$

$$Z(g, g^3) \xrightarrow{S} Z(g^3, g^{-1}) \xrightarrow{T} Z(g^3, g^2) \xrightarrow{T} Z(g^3, g^5) \xrightarrow{T} Z(g^3, g^8)$$

$$Z(g^2, g^3) \xrightarrow{S} Z(g^3, g^{-2}) \xrightarrow{T} Z(g^3, g) \xrightarrow{T} Z(g^3, g^4) \xrightarrow{T} Z(g^3, g^7)$$

$$Z(\mathbb{1},\mathbb{1}) = \left\{ \frac{1}{\eta^4} \left(\sum_{p\in V} - \sum_{p\in Sp} \right) q^{\frac{1}{2}p^2} \right\} \left\{ \frac{1}{\eta^6\bar{\eta}^6} \sum_{P\in\Gamma} q^{\frac{1}{2}P_L^2} \bar{q}^{\frac{1}{2}P_R^2} \right\}$$

$$Z(\mathbb{1},g^j) = \left\{ \frac{1}{\eta^4} \left(\sum_{p\in V} - \sum_{p\in Sp} \right) q^{\frac{1}{2}p^2} e^{2\pi i p\cdot j v_f} \right\} \left\{ \frac{1}{\eta^6\bar{\eta}^6} \prod_{a=1}^{3} \frac{2\sin\pi j t_a\, \eta^3}{\vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}-jt_a\end{bmatrix}} \sum_{P\in I} \bar{q}^{\frac{1}{2}P_R^2}\, e^{2\pi i P\cdot jv} \right\}$$

Untwisted sector

$$Z(g,g^j) = \left\{ \frac{e^{-i\pi j v_f^2}}{\eta^4} \left(\sum_{p\in V} - \sum_{p\in Sp} \right) q^{\frac{1}{2}(p+v_f)^2} e^{2i\pi(p+v_f).jv_f} \right\} \left\{ \frac{\mathcal{F}_1\, e^{2\pi i j E_1}}{\eta^6\bar{\eta}^6} \prod_{a=1}^{3} \frac{\eta^3}{\hat{\vartheta}\begin{bmatrix}\frac{1}{2}-t_a\\ \frac{1}{2}-jt_a\end{bmatrix}} \sum_{P\in I^*} \bar{q}^{\frac{1}{2}(P+v)_R^2} e^{i\pi j (P+v)^2} \right\}$$

Coprime twisted sectors

$$q = e^{2\pi i\tau}\,, \qquad \tau = \tau_1 + i\tau_2\,, \qquad \eta = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \eta e^{2\pi i\alpha\beta} q^{\frac{\alpha^2}{2}-\frac{1}{24}} \prod_{n=1}^{\infty}\left(1+q^{n+\alpha-\frac{1}{2}}e^{2\pi i\beta}\right)\left(1+q^{n-\alpha-\frac{1}{2}}e^{-2\pi i\beta}\right) \qquad \hat{\vartheta}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = e^{-2\pi i\alpha\beta}\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}$$

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Untwisted sector

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Coprime twisted sectors

Many consistency conditions (crystallographic action, modular invariance,…) give constraints on v, v_f, t_a. In the end we find 6 inequivalent shift vectors!

Consistency of the worldsheet CFT ⟹ $9v \in \Lambda_r$

Island (removing moduli) ⟹ $3v \notin \Lambda_w$

$$Z(\mathbb{1},\mathbb{1}) = \left\{ \frac{1}{\eta^4} \left(\sum_{p\in V} - \sum_{p\in Sp} \right) q^{\frac{1}{2}p^2} \right\} \left\{ \frac{1}{\eta^6\bar{\eta}^6} \sum_{P\in\Gamma} q^{\frac{1}{2}P_L^2} \bar{q}^{\frac{1}{2}P_R^2} \right\}$$

$$Z(\mathbb{1}, g^j) = \left\{ \frac{1}{\eta^4} \left(\sum_{p\in V} - \sum_{p\in Sp} \right) q^{\frac{1}{2}p^2} e^{2\pi i p\cdot j v_f} \right\} \left\{ \frac{1}{\eta^6\bar{\eta}^6} \prod_{a=1}^{3} \frac{2\sin\pi j t_a\, \eta^3}{\vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} - j t_a \end{bmatrix}} \sum_{P\in I} \bar{q}^{\frac{1}{2}P_R^2} e^{2\pi i P\cdot jv} \right\}$$

Untwisted sector

$$Z(g, g^j) = \left\{ \frac{e^{-i\pi j v_f^2}}{\eta^4} \left(\sum_{p\in V} - \sum_{p\in Sp} \right) q^{\frac{1}{2}(p+v_f)^2} e^{2i\pi(p+v_f).jv_f} \right\} \left\{ \frac{\mathcal{F}_1\, e^{2\pi i j E_1}}{\eta^6\bar{\eta}^6} \prod_{a=1}^{3} \frac{\eta^3}{\widehat{\vartheta}\begin{bmatrix} \frac{1}{2} - t_a \\ \frac{1}{2} - j t_a \end{bmatrix}} \sum_{P\in I^*} \bar{q}^{\frac{1}{2}(P+v)_R^2} e^{i\pi j (P+v)^2} \right\}$$

Coprime twisted sectors

Many consistency conditions (crystallographic action, modular invariance,…) give constraints on v, v_f, t_a. In the end we find 6 inequivalent shift vectors!

$$v_f = \frac{1}{9}(0,1,2,5)$$

$$t_a = \frac{1}{9}(1,2,5)$$

$$v_1 = \frac{1}{9}(1,0,1,0,1,0) \qquad v_4 = \frac{1}{9}(1,0,1,0,4,0)$$

$$v_2 = \frac{1}{9}(2,0,1,1,0,0) \qquad v_5 = \frac{1}{9}(2,0,1,0,2,1)$$

$$v_3 = \frac{1}{9}(0,1,0,1,0,2) \qquad v_6 = \frac{1}{9}(0,1,0,2,1,1)$$

(In the weight basis)

The Spectrum

The partition function can be expanded as a series:

$$Z = n_{1B} - n_{1F} + (n_{2B} - n_{2F})(q\bar{q})^{1/9} + (n_{3B} - n_{3F})(q\bar{q})^{2/9}$$

Importance of v for removing massless states!

Type IIB on T^6

$$(\pm 2) + 8(\pm 3/2) + 28(\pm 1) + 56(\pm 1/2) + \boxed{70(0)}$$

$\mathcal{N} = 8$ gravity multiplet

Our asymmetric orbifold

$$(\pm 2) + 4(\pm 3/2) + 6(\pm 1) + 4(\pm 1/2) + \boxed{2(0)}$$

axidilaton

$\mathcal{N} = 4$ gravity multiplet

If we set $v = 0$ we enhance SUSY and recover the toroidal compactification.

Computed the spectrum up to order $(q\bar{q})$

Organised all the states into 4d $\mathcal{N} = 4$ multiplets

Found 3 different spectra corresponding to the following shift vectors

v_1

v_2, v_3

v_4, v_5, v_6

3 ISLANDS!

New points in the moduli space, inaccessible to standard string compactifications!

Summary and Outlook

- An archipelago of 3 T^6/Z_9 islands (Helicity Supertraces)

  [Dabholkar, Harvey; 98], [Mizoguchi; 01], [Baykara et al; 25]

- Ongoing work: landscape of heterotic 4d islands using Niemeier lattices

  [Aldazabal et al; 25]

- Goal: classify the landscape of type IIB 4d islands.

- S-dual of these islands?

- Connection to $AdS_5 \times S_5$ via the AdS/CFT correspondence?

THANK YOU

Extra Slides

$$Z(g^3, g^{3k}) = \left\{ \frac{e^{9i\pi k v_f^2}}{\eta^4} \left(\sum_{p\in V} - \sum_{p\in Sp} \right) q^{\frac{1}{2}(p+3v_f)^2} e^{2i\pi p\cdot 3kv_f} \right\} \left\{ \frac{\mathcal{F}_3\, e^{2i\pi k E_3}}{\eta^6 \bar{\eta}^6} \prod_{a=1}^{3} \frac{\eta^3}{\widehat{\vartheta}\begin{bmatrix} \frac{1}{2} - \widetilde{3t_a} \\ \frac{1}{2} - k\widetilde{3t_a} \end{bmatrix}} \sum_{P\in I*} \bar{q}^{\frac{1}{2}(P+3v)^2_R}\, e^{i\pi k (P+3v)^2} \right\}$$

$$Z(g^3, g^{-1}) = e^{i\pi\Delta_1} \left\{ \frac{1}{\eta^4} \left(\sum_{p\in V} - \sum_{p\in Sp} \right) q^{\frac{1}{2}(p+3v_f)^2} e^{-2i\pi p.\cdot v_f} \right\} \left\{ \frac{\sqrt{3}\,\mathcal{F}_1}{\eta^6 \bar{\eta}^6} \prod_{a=1}^{3} \frac{\eta^3}{\widehat{\vartheta}\begin{bmatrix} \frac{1}{2} - \widetilde{3t_a} \\ \frac{1}{2} + t_a \end{bmatrix}} \sum_{P\in I} \bar{q}^{\frac{1}{2}(P+3v)^2_R}\, e^{-2i\pi P\cdot v} \right\}$$

$$Z(g^3, g^{-2}) = e^{i\pi\Delta_2} \left\{ \frac{1}{\eta^4} \left(\sum_{p\in V} - \sum_{p\in Sp} \right) q^{\frac{1}{2}(p+3v_f)^2} e^{-2i\pi p.\cdot 2v_f} \right\} \left\{ \frac{\sqrt{3}\,\mathcal{F}_1}{\eta^6 \bar{\eta}^6} \prod_{a=1}^{3} \frac{\eta^3}{\widehat{\vartheta}\begin{bmatrix} \frac{1}{2} - \widetilde{3t_a} \\ \frac{1}{2} + \widetilde{2t_a} \end{bmatrix}} \sum_{P\in I} \bar{q}^{\frac{1}{2}(P+3v)^2_R}\, e^{-2i\pi P\cdot 2v} \right\}$$

3

$$Z(g^3, g^{3k}) = \left\{ \frac{e^{9i\pi k v_f^2}}{\eta^4} \left(\sum_{p\in V} - \sum_{p\in Sp} \right) q^{\frac{1}{2}(p+3v_f)^2} e^{2i\pi p\cdot 3kv_f} \right\} \left\{ \frac{\mathcal{F}_3 e^{2i\pi k E_3}}{\eta^6 \bar{\eta}^6} \prod_{a=1}^{3} \frac{\eta^3}{\hat{\vartheta}\begin{bmatrix} \frac{1}{2} - \widetilde{3t_a} \\ \frac{1}{2} - k\widetilde{3t_a} \end{bmatrix}} \sum_{P\in I^*} \bar{q}^{\frac{1}{2}(P+3v)_R^2} e^{i\pi k (P+3v)^2} \right\}$$

g^0 action

i

$$Z(g^3, g^{-1}) = e^{i\pi\Delta_1} \left\{ \frac{1}{\eta^4} \left(\sum_{p\in V} - \sum_{p\in Sp} \right) q^{\frac{1}{2}(p+3v_f)^2} e^{-2i\pi p.\cdot v_f} \right\} \left\{ \frac{\sqrt{3}\,\mathcal{F}_1}{\eta^6 \bar{\eta}^6} \prod_{a=1}^{3} \frac{\eta^3}{\hat{\vartheta}\begin{bmatrix} \frac{1}{2} - \widetilde{3t_a} \\ \frac{1}{2} + t_a \end{bmatrix}} \sum_{P\in I} \bar{q}^{\frac{1}{2}(P+3v)_R^2} e^{-2i\pi P\cdot v} \right\}$$

g^2 action

$-i$

$$Z(g^3, g^{-2}) = e^{i\pi\Delta_2} \left\{ \frac{1}{\eta^4} \left(\sum_{p\in V} - \sum_{p\in Sp} \right) q^{\frac{1}{2}(p+3v_f)^2} e^{-2i\pi p.\cdot 2v_f} \right\} \left\{ \frac{\sqrt{3}\,\mathcal{F}_1}{\eta^6 \bar{\eta}^6} \prod_{a=1}^{3} \frac{\eta^3}{\hat{\vartheta}\begin{bmatrix} \frac{1}{2} - \widetilde{3t_a} \\ \frac{1}{2} + \widetilde{2t_a} \end{bmatrix}} \sum_{P\in I} \bar{q}^{\frac{1}{2}(P+3v)_R^2} e^{-2i\pi P\cdot 2v} \right\}$$

g^1 action

Operator interpretation: $Z(g^k, g^l)$ entails the action of the operator g^l on each fixed point. So that the overall factor can be interpreted as the trace.
We choose the following actions

$g^0 : (1,1,1)$ $\quad$ $g^2 : (1, e^{2\pi i/3}, e^{2\pi i/3})$ $\quad$ $g^1 : (1, e^{-2\pi i/3}, e^{-2\pi i/3})$

$g^1 g^2 = g^0$ Z_3 structure