Spectral Networks

& their role in 2d conformal field theory

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Background

Definition

Let C be a genus g Riemann surface with $n \geq 1$ punctures, and $\iota : \Sigma \hookrightarrow T^*C$ be a branched double cover,

$$\pi : \Sigma \xrightarrow{2:1} C.$$

For a curve $\gamma \subset C$, let γ_i denote the lift of γ to the ith sheet of Σ.

Definition

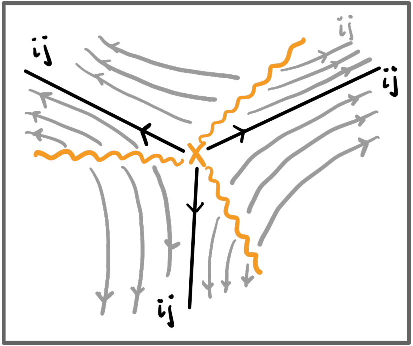
For a choice of phase θ, an $ij-$trajectory is a curve $w \subset C$ such that

$$e^{-i\theta} \int_{w_{ij}} \iota^* \lambda \in \mathbb{R}_{\geq 0}$$

where $w_{ij} := w_i - w_j$ and $\lambda \in \Omega^1(T^*C)$ is the canonical 1-form.

Definition

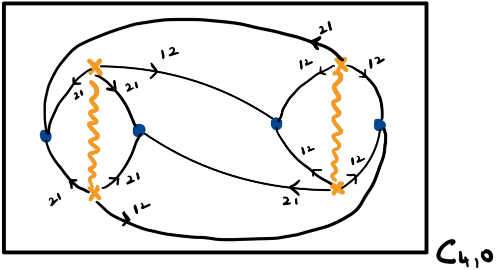
This defines a foliation of C (known as the horizontal foliation or WKB foliation). At the branch points of Σ, the foliation has a three-pronged structure.



These are known as the **critical** $ij-$**trajectories**. We orient all the critical $ij-$trajectories away from the branch point.

Definition

The collection of all the oriented critical ij trajectories, the branch points, and the ij labels is known as a (rank 2) **WKB spectral network** $\mathcal{W}_\theta(\Sigma)$.



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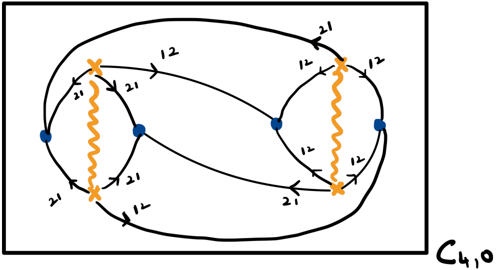


Working definition

A (rank 2) spectral network $\mathcal{W}$ is a directed graph on C with a trivalent vertex at each branch point of Σ, and all the edges terminate on either punctures or branch points.

The Physics

The 4d story

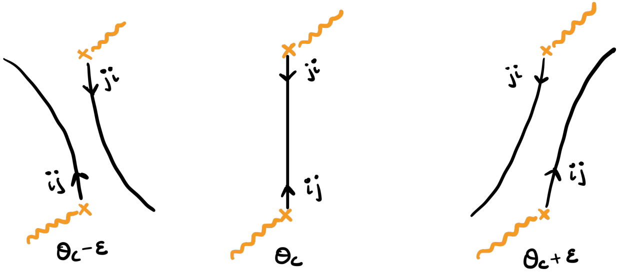
The surface C is the moduli space of (canonical) UV surface defects in a 4d class S theory, or equivalently, the **UV curve** of the class S theory.

The surface Σ is the moduli space of IR surface defects in the 4d theory, or equivalently, the **Seiberg-Witten curve** of the class S theory.

A spectral network $\mathcal{W}_\theta$ at phase θ is the locus of all the surface defects $z \in C$ that admit a BPS soliton at phase θ. The analytical condition from before is the BPS equation. [GMN13], [BHM25]

The 4d story

At certain phases, the topology of the network might degenerate. Such degenerate topologies encode the 4d BPS states.



This follows from the string/M-theory construction of class S theories.

The 2d story

The AGT correspondence states that 4d $\mathcal{N} = 2$ $SU(2)$ gauge theories are dual to 2d Liouville conformal field theory on C. [AGT10]

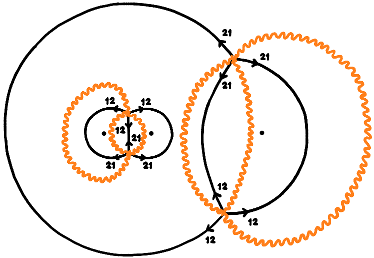
4d instanton partition functions
$\updownarrow$
2d Liouville conformal blocks

These conformal blocks admit an integral representation for a suitable choice of contours on C. [DF84]

Punchline: spectral networks can supply the required contours for these integral representations. [HN24], [HM*]

The 2d story

In particular, the integral representations $\mathcal{Z}_{\mathcal{W}_{\mathrm{FN}}}$ defined using Fenchel-Nielsen networks reproduce the *standard* Liouville conformal blocks. [HM*]



For a Fenchel-Nielsen network, the complement $C \backslash \mathcal{W}_{\mathrm{FN}}$ is a union of punctured discs. Note that none of the critical $ij-$trajectories of a Fenchel-Neilsen network terminate on a puncture.

Outlook

Related ideas

- We may obtain the algebra of loop operators in the theory using $q-$nonabelianisation. [NY20]
- We may extend this to higher-rank spectral networks and higher-rank conformal field theories.
- We would like to extend this to 3d Chern-Simons theory.

Thank you!

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