Meson-Glueball Spectroscopy from $Sp(4)$ Lattice Gauge Theories with mixed representation fermions

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Prelude

Glueballs

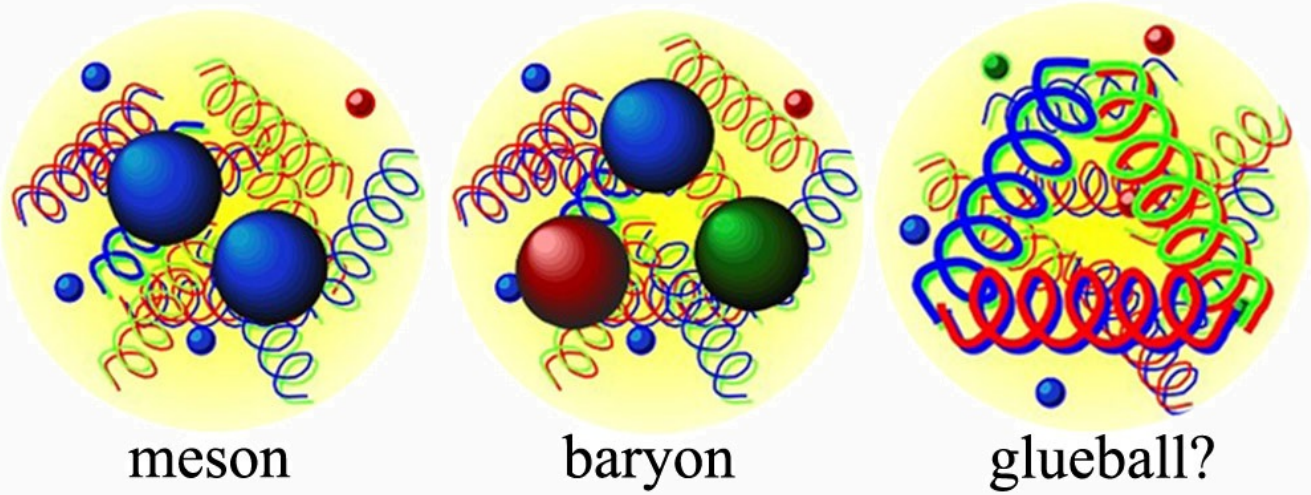

Figure 1: Combining quarks and gluons according to their *colour* to create *colourless* states. Image from Brookhaven National Laboratory `bnl.gov`.

Recent interest in glueballs

Confirmation of the spin numbers of the $X(2370)$ resonance:

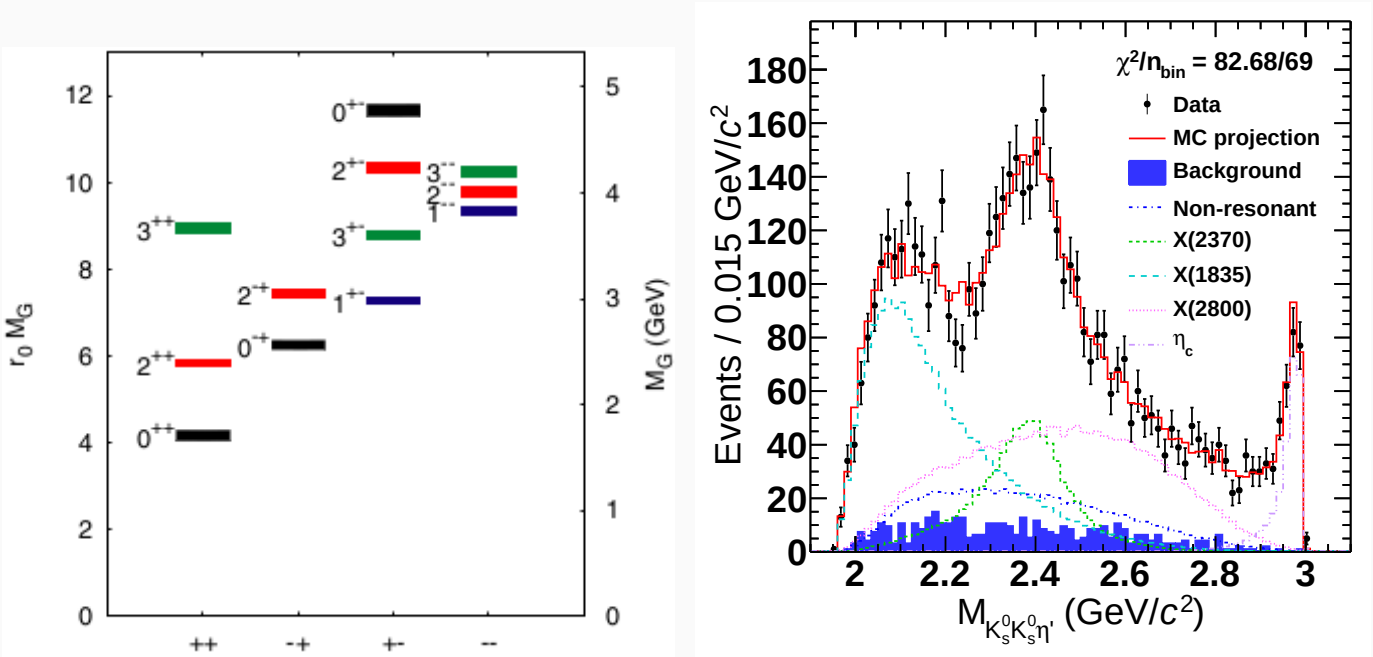

Figure 2: Left: Lattice Studies of Glueballs in Yang-Mills (Y. Chen et al in hep-lat/0510074). Right: Invariant Mass Histogram from the $J/\psi \longrightarrow \gamma K_S^0 K_S^0 \eta'$ BESIII events confirming a 0^{-+} resonance at 2.37 GeV (from BESIII Collaboration in 2312.05324).

Dynamical Fermions

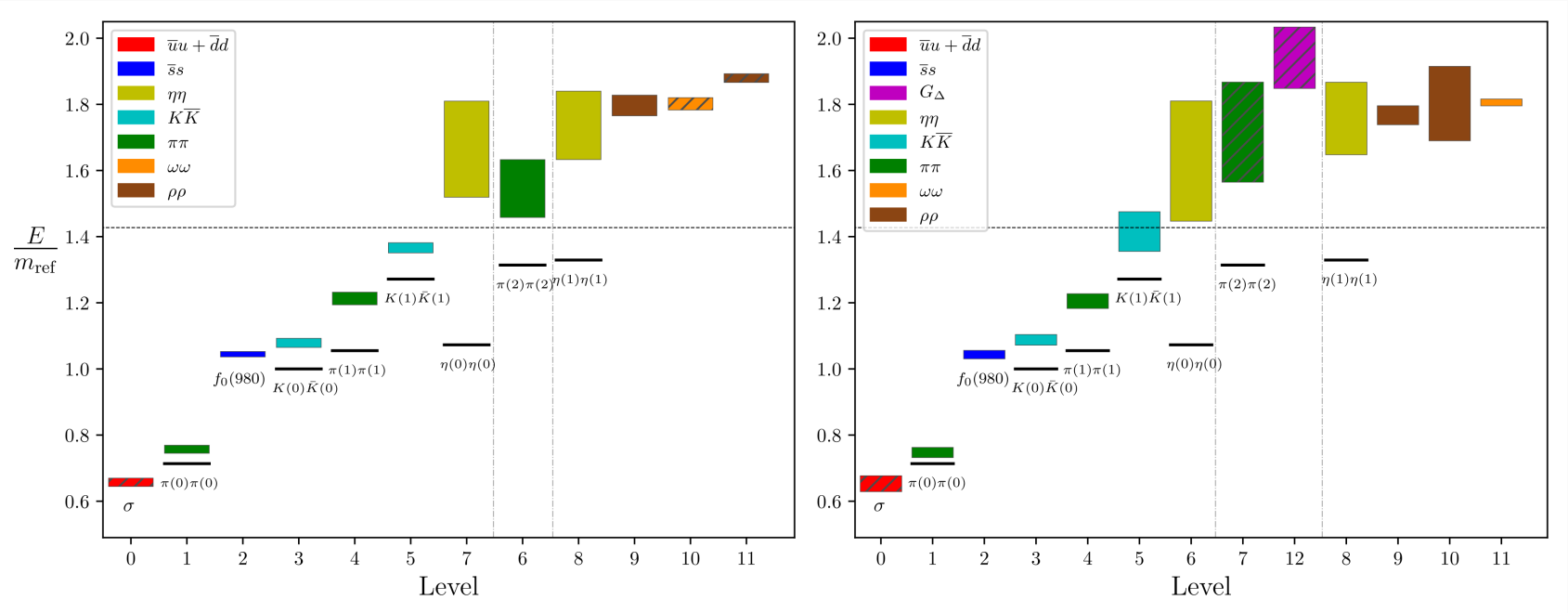

Figure 3: Flavour singlet scalar spectroscopy from dynamical fermion ensembles at $m_\pi \approx 390$ MeV. Pink is the new state obtained with the introduction of a glueball operator. From R. Brett et al. in 1909.07306.

Glueball studies with dynamical fermions is hard (decay, mixing)!

$Sp(2N)$ theories

Our theory:

$$\mathcal{L}_{Sp(2N)} = -\frac{1}{2}\mathrm{Tr}\, F^{\mu\nu}F_{\mu\nu} + \sum_{I=1}^{N_{\mathrm{fun}}} \bar{Q}_I(i\gamma^\mu D_\mu - m_{\mathrm{fun}})Q_I + \sum_{j=1}^{N_{\mathrm{as}}} \bar{\psi}_j(i\gamma^\mu D_\mu - m_{\mathrm{as}})\psi_j$$

Group definition:

$$Sp(2N) = \{U \in M_{2N\times 2N}(\mathbb{C}) : U^T\Omega U = U\}, \quad \Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$$

Transformation laws:

$$Q_I \longrightarrow Q_I' = UQ_I, \quad \psi_i \longrightarrow \psi_i' = U^{(\mathrm{as})}\psi_i, \quad F_{\mu\nu} \longrightarrow U^{(adj)}F_{\mu\nu}(U^{(adj)})^\dagger$$

Why $Sp(2N)$ theories?

- Composite Higgs Models;
- Strongly interacting Dark Matter Models;
- Large-N_c behaviour of $SU(N)$ theories;
- Finite temperature phase transitions;
- Gravitational Wave detection;
- **?????** - *To be unlocked*.

TELOS Collaboration '23: arXiv.2304.01070.

Our objective

- Measure Glueball spectrum in $Sp(4)$;
- Measure flavour-singlet Mesonic spectrum in the same theory;
- Study possible mixing effects and decay patterns.

Lattice Simulation

Lattice Action

Observables:

$$\langle O \rangle = \int \mathcal{D}U \; O[U] \exp(-S[U]) \longrightarrow \langle O \rangle \approx \frac{1}{N_{\mathrm{conf}}} \sum_{\{U\} \sim e^{-S}}^{N_{\mathrm{conf}}} O[U]$$

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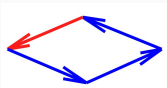
Wilson gauge action for S_{gauge}:

$$S_{\mathrm{gauge}} = \beta \sum_{x} \sum_{\mu<\nu} \Bigg(\Bigg)$$



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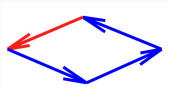

Wilson fermions for $S_{\mathrm{ferm}}^{(R)}$:

$$S_{\mathrm{ferm}}^{(R)} = a^4 \sum_{j=1}^{N_R} \sum_{x} \overline{\psi}^j(x) D^{(R)} \left[U; m^{(R)} \right) \psi^j(x)$$

Ensembles

This presentation:

Name	$\beta = 4N/g^2$	am_0^{as}	am_0^{f}	N_t	N_s	N_{conf}	$\langle P \rangle$
M3	6.5	-1.01	-0.71	96	20	2500	0.585156(13)

In the pipeline:

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M3	6.5	-1.01	-0.71	96	20	4500	0.585156(13)
M4	6.5	-1.01	-0.70	64	20	5500	0.584228(12)

Lattice Spectroscopy

Lattice Spectroscopy

Spectroscopy on the lattice is obtained from the spectral decomposition of 2-point correlators. Define zero-momentum operators:

$$O(t) \equiv O(\mathbf{p} = 0, t) = \sum_{\mathbf{x}} e^{(-i\mathbf{0}\cdot\mathbf{x})} \; O(x, t)$$

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Correlators defined as:

$$C(t) = \langle \Omega | O(t) O(0) | \Omega \rangle \stackrel{!}{=} \sum_{n} |c_n|^2 e^{-E_n t},$$

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Momentum zero operators implies $E_n = m_n$. Can extract an *effective mass*:

$$m_{\text{eff}}(t) = \ln\left(\frac{C(t)}{C(t+1)}\right) \xrightarrow{t \longrightarrow \infty} m_{\text{eff}}(t) \sim m_0$$

Operators

Pseudoscalar Meson operators:

$$O_{\mathrm{as}}(x) = \frac{1}{\sqrt{N_{\mathrm{as}}}} \sum_{k=1}^{N_{\mathrm{as}}} \bar{\Psi}^k(x)\gamma_5\Psi^k(x)\,, \quad O_{\mathrm{fun}}(x) = \frac{1}{\sqrt{N_{\mathrm{f}}}} \sum_{l=1}^{N_{\mathrm{f}}} \bar{Q}^l(x)\gamma_5 Q^l(x)\,.$$

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Glueball operators:

$$O_G^{0^{\pm+}}(L_x) = \sum_R R(L_x) \pm P(R(L_x)) + h.c = \mathrm{Re}\left(\sum_R R(L_x) \pm P(R(L_x))\right)$$

TELOS Collaboration '24: 2405.05765

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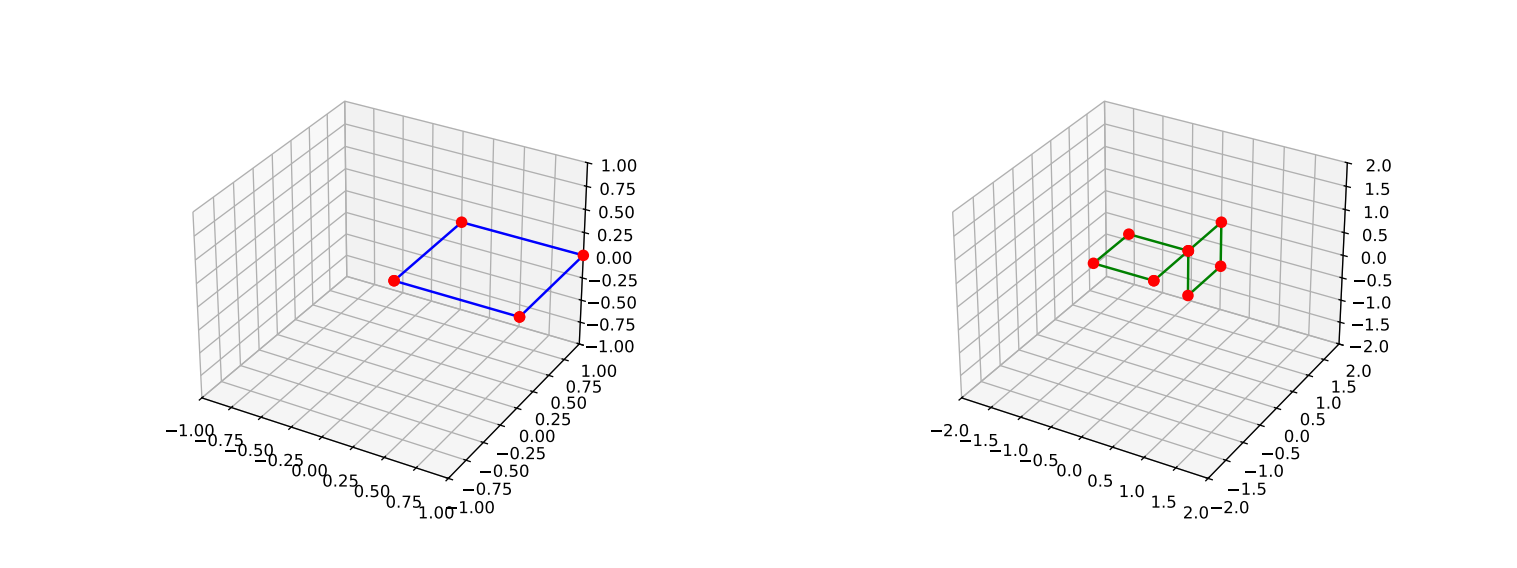

Figure 4: Left: The plaquette loop projected to $J^{PC} = 0^{++}$. Right: the broken chair projected to $J^{PC} = 0^{-+}$. Blue lines: Take the trace and sum. Green lines: Take the trace and subtract.

GEVP: Combining the operators

In theory, it is possible to write:

$$\hat{O} = \sum_{i=1}^{\infty} c_i \hat{O}_i, \quad \text{so that} \quad \langle \hat{O}(t_0) O(t_1) \rangle = |c_0|^2 e^{-m_0(t_0 - t_1)}$$

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Variational problem to find c_i:

$$\tilde{m}_0 = \min_{\{c_i\}} \{m_{\mathrm{eff}}(t)\} = \min_{\{c_i\}} \left\{ \frac{D(t)}{D(t+1)} \right\}, \quad D(t) = \sum_{ij} c_i c_j \langle O_i O_j \rangle (t)$$

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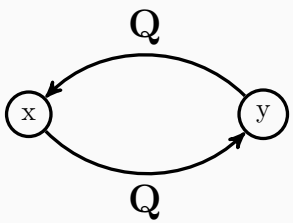
$$\implies C(t_0) v(t_0, t_1) = \lambda(t_0, t_1) C(t_1) v(t_0, t_1) \implies \hat{O}^\dagger \approx \sum_{i=1}^{N_{\text{ops}}} v_i(t_0, t_1) O_i^\dagger$$

where $C(t)$ is the correlation matrix:

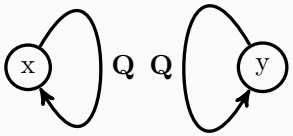
$$C_{ij}(t) = \langle O_i O_j \rangle (t) = \begin{bmatrix} C_{G-G} & C_{G-\text{fun}} & C_{G-\text{as}} \\ C_{\text{fun}-G} & C_{\text{fun}-\text{fun}} & C_{\text{fun}-\text{as}} \\ C_{\text{as}-G} & C_{\text{as}-\text{fun}} & C_{\text{as}-\text{as}} \end{bmatrix}$$

Correlators

$$C_{\mathrm{fun-fun}}(x,y) \equiv \langle \bar{O}_{\mathrm{fun}}(x) O_{\mathrm{fun}}(y) \rangle = \pm$$

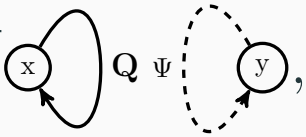


$$+ N_{\mathrm{f}}$$



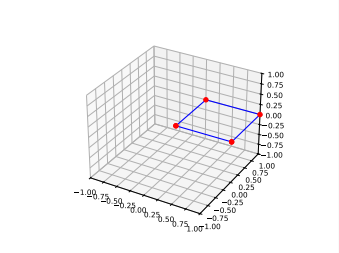
$$,$$

$$C_{\mathrm{fun-as}}(x,y) \equiv \langle \bar{O}_{\mathrm{as}}(x) O_{\mathrm{fun}}(y) \rangle = +\sqrt{N_{\mathrm{as}} N_{\mathrm{f}}}$$

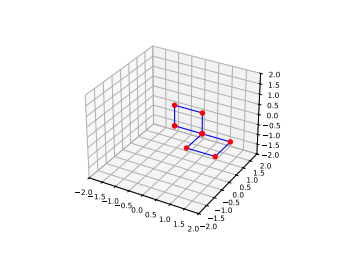


$$,$$

$$C_{G-G}(x,y) \equiv \langle \mathrm{Tr} \sum$$



$$(x) \cdot \mathrm{Tr} \sum$$



$$(y) \rangle$$

Correlators

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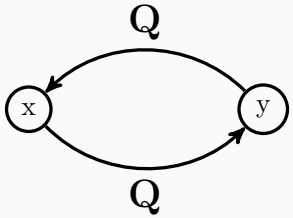

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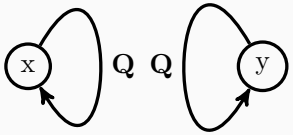

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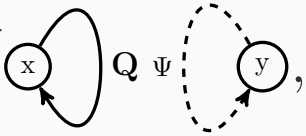

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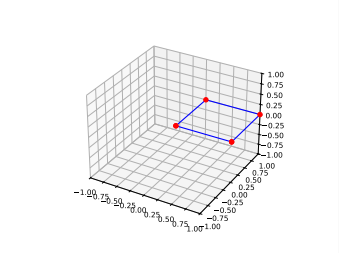

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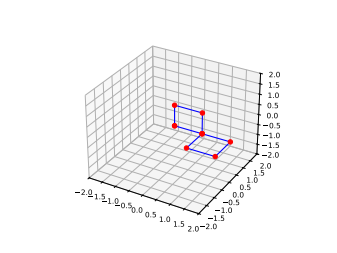
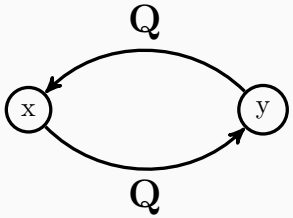


$$(y) \rangle$$

E.g, first diagram is:



$$= \mathrm{Tr}\left(\Gamma\ D^{-1}_{\mathrm{fun}}(x;y)\ \Gamma\ D^{-1}_{\mathrm{fun}}(y;x)\right), \quad \Gamma = \mathbb{I}, \gamma_5$$

TELOS Collaboration '24: 2405.05765

Results

Pseudoscalar Channel

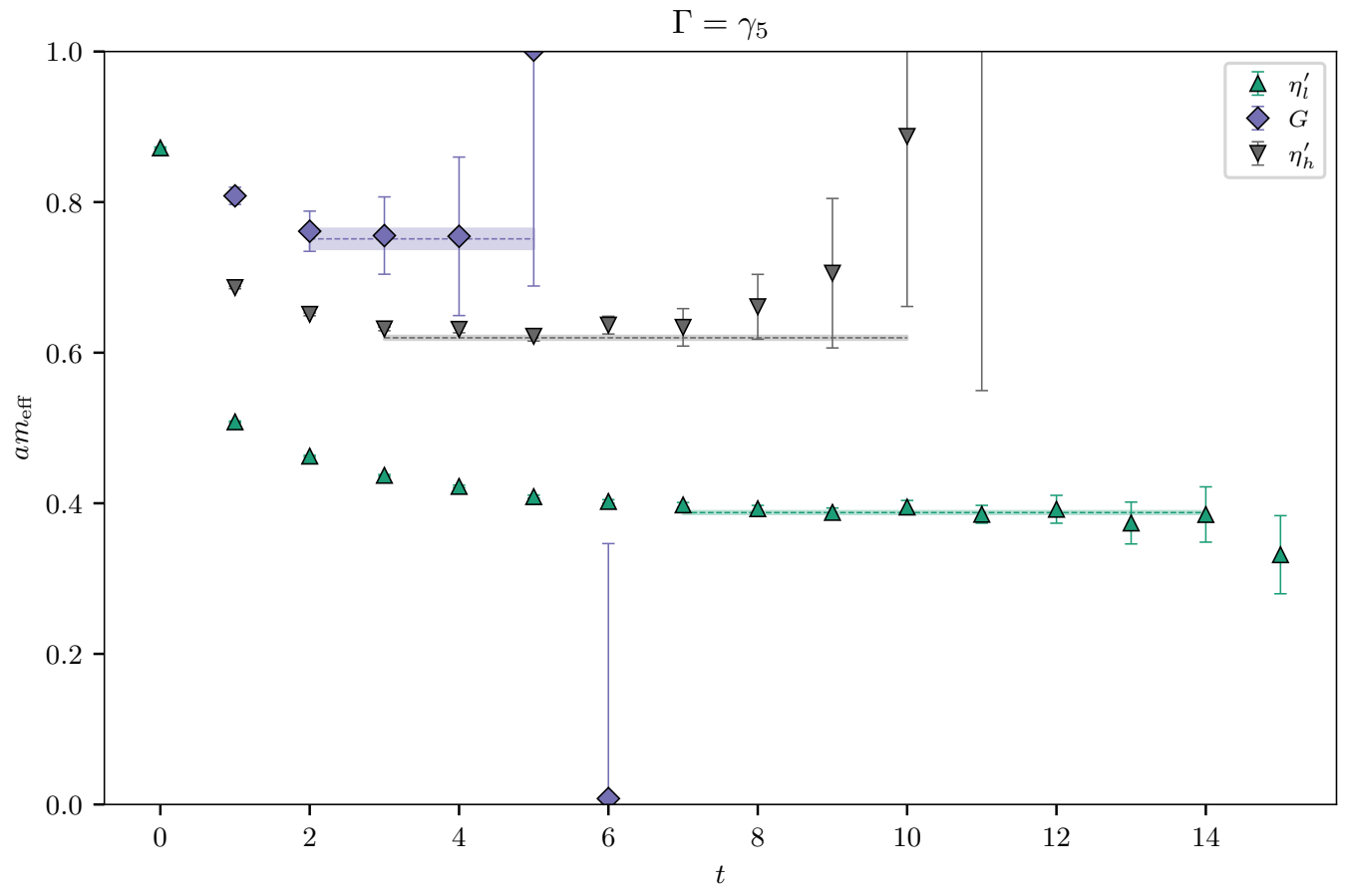


Figure 5: Effective masses and their respective exponential fits in the pseudoscalar channel.

Scalar Channel

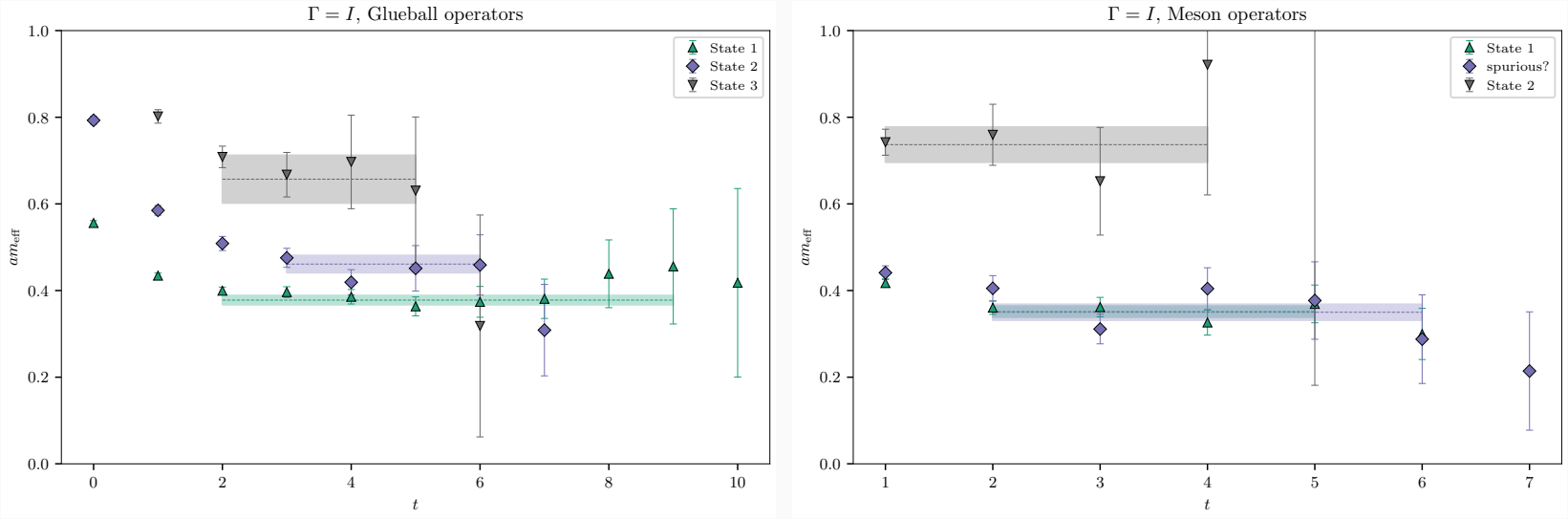


Figure 6: Effective masses and their respective exponential fits in the scalar channel.

Mass fits on both channels

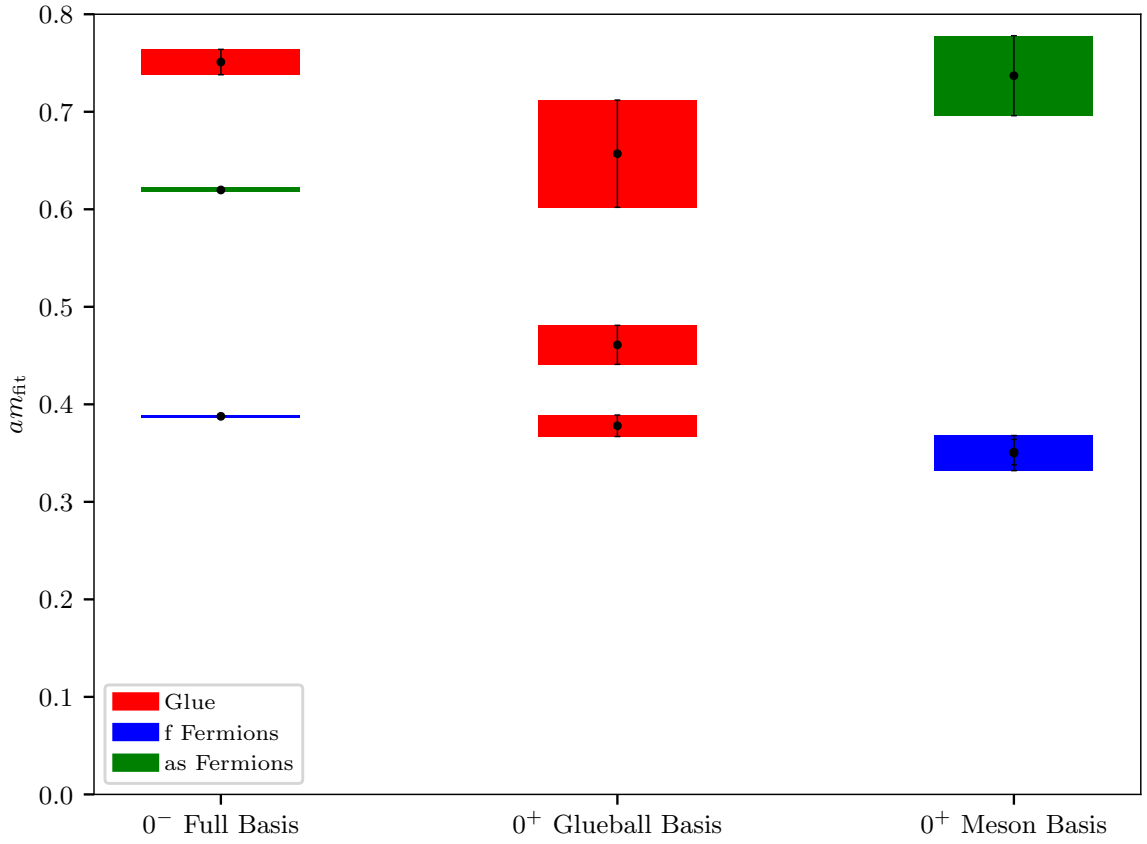


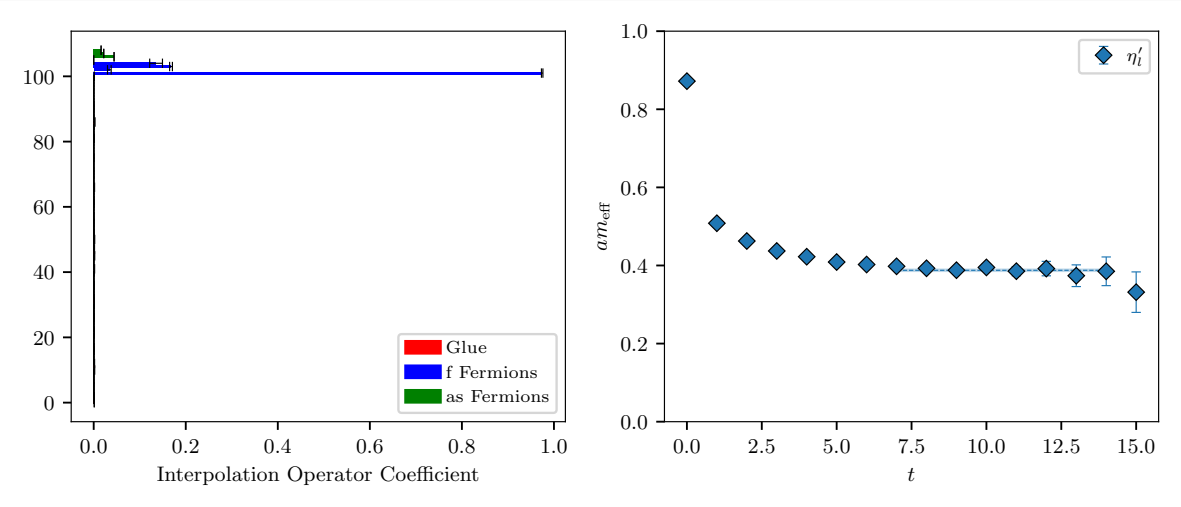
Figure 7: Comparison of masses obtained from exponential fits

Interpolating operators

From the GEVP:

$$\hat{O}^\dagger \approx \sum_{i=1}^{N_{\mathrm{ops}}} v_i(t_0, t_1) O_i^\dagger$$

In the pseudoscalar channel:



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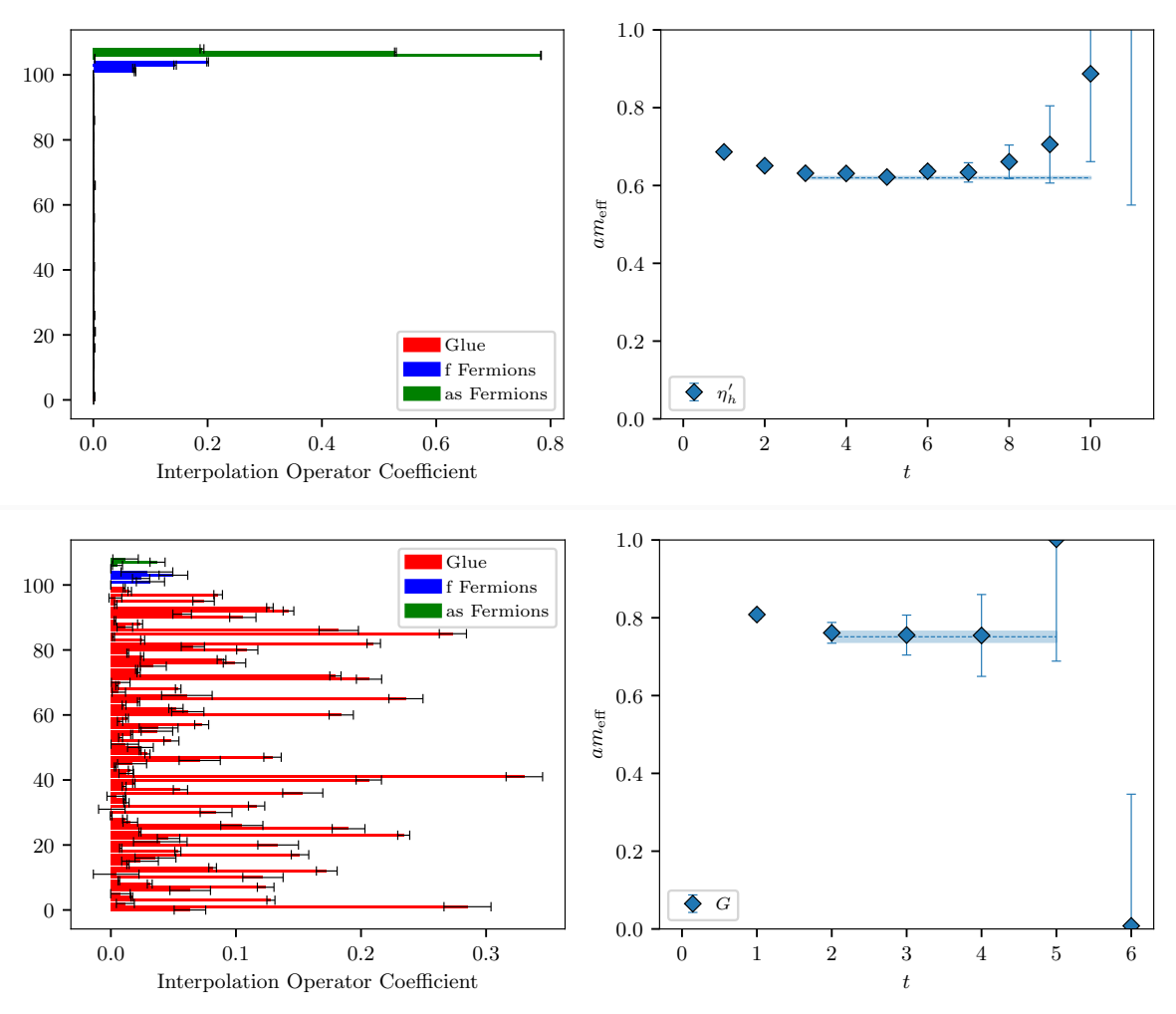

Conclusion

Takeaway points

- Despite its existence not being excluded from first principles, Glueballs remain experimentally undetected;
- Very hard to identify and study on Lattices with dynamical fermions;
- $Sp(4)$ gauge theory provides a QCD-like theory with a near-threshold stable glueball state.
- Good starting place to tune parameters and study glueball decays.

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Backup

Lattice Regularization

- *Bare* QFT $\longrightarrow$ Regularization $\longrightarrow$ Renormalization $\longrightarrow$ Renormalized QFT $\longrightarrow$ Amplitudes;
- Lattice Regularization $\longrightarrow$ non-perturbative!
  - Space-time discretization:

$$x^\mu \in \mathbb{R} \longrightarrow x_E^\mu = a(n_0, n_2, n_2, n_3), n_\mu \in \mathbb{N}$$

  - Wick rotation:

$$x^\mu = (x^0, \mathbf{x}) \longrightarrow x_E^\mu = (ix^0, \mathbf{x})$$
$$x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu \longrightarrow x_E^\mu x_{E,\mu} = (x^0)^2 + |\mathbf{x}|^2 \quad \text{Euclidean!}$$

Glueballs

- QCD is the theory of strong interactions:

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{2}\mathrm{Tr}\ F^{\mu\nu}F_{\mu\nu} + \sum_f \bar{\psi}_f(i\gamma^\mu D_\mu - m_f)\psi_f$$

- $SU(3)_c$ gauge transformations:

$$\psi_f \longrightarrow \psi'_f = U\psi_f, \quad U \in SU(3)$$
$$A_\mu \longrightarrow A'_\mu = UA_\mu U^{-1} + iU\partial_\mu U^{-1}$$

- Observed colour confinement:

$$|\mathrm{phys}\rangle = \mathcal{O}^\dagger|\Omega\rangle, \quad \mathcal{O} \longrightarrow \mathcal{O}' = U\mathcal{O}U^{-1} = \mathcal{O}$$

Composite Higgs Models and $Sp(2N)$ theories

Re-write

$$Q^{J\,a} \equiv \begin{pmatrix} q^{J\,a} \\ \Omega^{ab}(-\tilde{C}q^{J+2\,*})_b \end{pmatrix}, \qquad \Psi^{j\,ab} \equiv \begin{pmatrix} \psi^{j\,ab} \\ \Omega^{ac}\Omega^{bd}(-\tilde{C}\psi^{j+3\,*})_{cd} \end{pmatrix},$$

Get:

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\, V_{\mu\nu}V^{\mu\nu} +$$

$$+\frac{1}{2}\sum_{M=1}^{4}\left(i\,(q^M)^\dagger{}_a\bar{\sigma}^\mu\,(D_\mu q^M)^a\right) - \frac{1}{2}m^{(f)}\sum_{M,N=1}^{4}\tilde{\Omega}_{MN}\left(q^{M\,a\,T}\Omega_{ab}\tilde{C}q^{N\,b}\right) +$$

$$+\frac{1}{2}\sum_{m=1}^{6}\left(i\,(\psi^m)^\dagger{}_{ab}\bar{\sigma}^\mu\,(D_\mu\psi^m)^{ab}\right) - \frac{1}{2}m^{(as)}\sum_{m,n=1}^{6}\omega_{mn}\left(\psi^{m\,ab\,\mathrm{T}}\Omega_{ac}\Omega_{bd}\tilde{C}\psi^{n\,cd}\right)$$

Approximate $SU(4)\times SU(6)$ flavour symmetry. Breaking patterns:

- $SU(6) \longrightarrow SU(6)/SO(6) \rightsquigarrow$ 20 pseudo Nambu-Goldstone bosons;
- $SU(4) \longrightarrow SU(4)/Sp(4) \rightsquigarrow$ 5 pNBG's (Higgs doublet + scalar field).

Higgs Naturalness/Fine-Tuning/Hierarchy problem

Hierarchy Problem:

$$\mathcal{L}_{\rm Higgs} = c\Lambda_{\rm SM}^2 H^\dagger H, \quad m_{\rm Higgs} = 125\ \text{GeV},\ \Lambda_{\rm SM} \approx 10^{15}\ \text{GeV} \implies c \sim 10^{-28} \ll$$

Radiative corrections:

$$m_{\rm Higgs} = m_{\rm Higgs,\,bare} + \delta m_{\rm Higgs,\,SM} + \delta m_{\rm Higgs,\,BSM}$$

$$\delta m_{\rm Higgs,\,SM} \sim \frac{3y_t^2}{8\pi^2}\Lambda_{\rm SM}^2$$

Fine-tuning!

$$\Delta \geq \frac{\delta m_{\rm Higgs,\,SM}}{m_{\rm Higgs}} \approx \left(\frac{\Lambda_{\rm SM}}{450\ \text{GeV}}\right)^2 \sim 10^{24}\ \text{GeV}$$

24 digit cancelation!

Lattice Observables

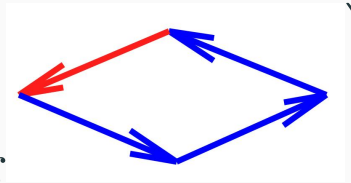
Need to discretize action. Using links:

$$U_\mu(x) = \exp\left(i \int_x^{x+a\hat{\mu}} ds^\mu A_\mu(s)\right) \approx \exp\left(iaA_\mu(x)\right) \in Sp(4)$$

The action comes as:

$$S = S_{\text{gauge}} + S^f_{\text{ferm}} + S^{as}_{\text{ferm}}$$

$$S_{\text{gauge}} = \beta \sum_x \sum_{\mu<\nu} \left(\text{Re Tr} \right.$$



$$\left. \right)$$

$$S^{(R)}_{\text{ferm}} = a^4 \sum_{j=1}^{N_R} \sum_x \overline{\psi}^j(x) D^{(R)}\left[U; m^{(R)}\right) \psi^j(x)$$

Lattice Setup

Measurement of observables is done in the Path Integral Formalism:

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}\phi \; O[\phi] \exp(-S[\phi]), \quad Z = \int \mathcal{D}\phi \exp(-S[\phi])$$

Even with lattice discretization, hard to evaluate in most cases. Sample via Monte-Carlo:

$$\langle O \rangle \approx \frac{1}{N_{\mathrm{conf}}} \sum_{\{U\}}^{N_{\mathrm{conf}}} O[U], \quad \{U\}_{i=1,\ldots,N_{\mathrm{conf}}} \sim \exp(-S[U])$$

In practice, Markov Chain Monte Carlo:

$$U_0 \longrightarrow U_1 \longrightarrow U_2 \longrightarrow \cdots \longrightarrow U_{N_{\mathrm{conf}}} \quad P_{\mathrm{accept}} = \min\left(1, e^{-\Delta S}\right)$$