Meson-Glueball Spectroscopy from $Sp(4)$ Lattice Gauge Theories with mixed representation fermions

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University of Plymouth - Centre for Mathematical Sciences

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Prelude

Glueballs

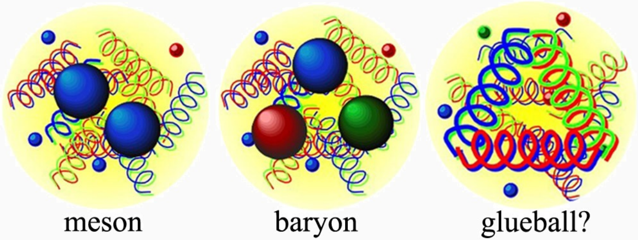

Figure 1: Combining quarks and gluons according to their *colour* to create *colourless* states. Image from Brookhaven National Laboratory `bnl.gov`.

Confirmation of the spin numbers of the $X(2370)$ resonance:

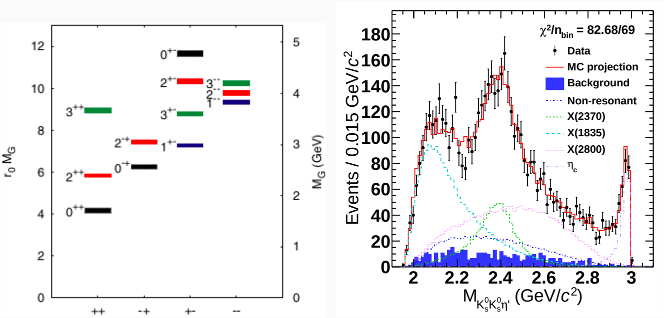

Figure 2: Left: Lattice Studies of Glueballs in Yang-Mills (Y. Chen et al in hep-lat/0510074). Right: Invariant Mass Histogram from the $J/\psi \longrightarrow \gamma K_S^0 K_S^0 \eta'$ BESIII events confirming a 0^{-+} resonance at 2.37 GeV (from BESIII Collaboration in 2312.05324).

Dynamical Fermions

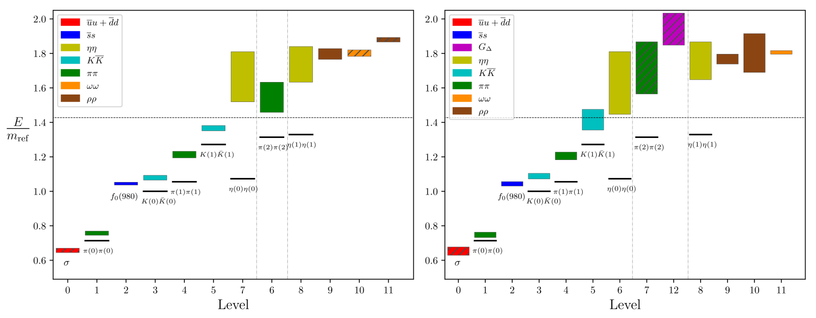

Figure 3: Flavour singlet scalar spectroscopy from dynamical fermion ensembles at $m_\pi \approx 390$ MeV. Pink is the new state obtained with the introduction of a glueball operator. From R. Brett et al. in 1909.07306.

Glueball studies with dynamical fermions is hard (decay, mixing)!

$Sp(2N)$ theories

Our theory:

$$\mathcal{L}_{Sp(2N)} = -\frac{1}{2}\mathrm{Tr}\, F^{\mu\nu}F_{\mu\nu} + \sum_{I=1}^{N_{\mathrm{fun}}} \bar{Q}_I (i\gamma^\mu D_\mu - m_{\mathrm{fun}}) Q_I + \sum_{j=1}^{N_{\mathrm{as}}} \bar{\psi}_j (i\gamma^\mu D_\mu - m_{\mathrm{as}}) \psi_j$$

Group definition:

$$Sp(2N) = \{U \in M_{2N\times 2N}(\mathbb{C}) : U^T \Omega U = U\}, \quad \Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$$

Transformation laws:

$$Q_I \longrightarrow Q_I' = UQ_I, \quad \psi_i \longrightarrow \psi_i' = U^{(\mathrm{as})}\psi_i, \quad F_{\mu\nu} \longrightarrow U^{(adj)} F_{\mu\nu} (U^{(adj)})^\dagger$$

Why $Sp(2N)$ theories?

- Composite Higgs Models;
- Strongly interacting Dark Matter Models;
- Large-N_c behaviour of $SU(N)$ theories;
- Finite temperature phase transitions;
- Relic background of gravitational waves;
- **?????** - *To be unlocked*.

TELOS Collaboration '23: arXiv.2304.01070.

Our objective

- Measure Glueball spectrum in $Sp(4)$;
- Measure flavour-singlet Mesonic spectrum in the same theory;
- Study possible mixing effects and decay patterns.

Lattice Simulation

Lattice Discretization

- Space-time discretization, lattice spacing a;

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$$t = x^0 \longrightarrow \tau = ix^0$$

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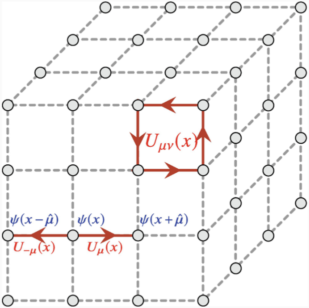

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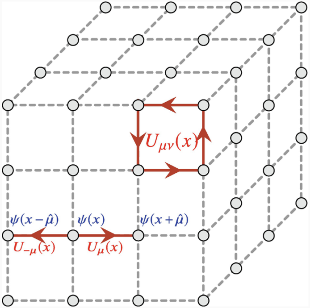

- Observables:

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}U \; O[U, \psi, \bar{\psi}] \exp(-S)$$

Image from O. Kaczmarek's *Flavored Aspects of QCD Thermodynamics from Lattice QCD*.

Lattice Action

Observables:

$$\langle O \rangle = \int \mathcal{D}U \; O[U] \exp(-S[U]) \longrightarrow \langle O \rangle \approx \frac{1}{N_{\mathrm{conf}}} \sum_{\{U\}\sim e^{-S}}^{N_{\mathrm{conf}}} O[U]$$

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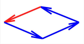
Wilson gauge action for S_{gauge}:

$$S_{\mathrm{gauge}} = \beta \sum_x \sum_{\mu<\nu} \Bigg($$

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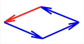
$$\Bigg)$$

Wilson fermions for $S_{\text{ferm}}^{(R)}$:

$$S_{\text{ferm}}^{(R)} = a^4 \sum_{j=1}^{N_R} \sum_{x} \overline{\psi}^{j}(x) D^{(R)} \left[U; m^{(R)} \right) \psi^{j}(x)$$

Ensembles

This presentation:

Name	$\beta = 4N/g^2$	am_0^{as}	am_0^{f}	N_t	N_s	N_{conf}	$\langle P \rangle$
M3	6.5	-1.01	-0.71	96	20	2500	0.585156(13)

In the pipeline:

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M4	6.5	-1.01	-0.70	64	20	5500	0.584228(12)

Lattice Spectroscopy

Lattice Spectroscopy

Spectroscopy on the lattice is obtained from the spectral decomposition of 2-point correlators. Define zero-momentum operators:

$$O(t) \equiv O(\mathbf{p}=0,t) = \sum_{\mathbf{x}} e^{(-i\mathbf{0}\cdot\mathbf{x})} O(x,t)$$

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Momentum zero operators implies $E_n = m_n$. Can extract an *effective mass*:

$$m_{\mathrm{eff}}(t) = \ln\left(\frac{C(t)}{C(t+1)}\right) \xrightarrow{t\longrightarrow\infty} m_{\mathrm{eff}}(t) \sim m_0$$

Operators

Pseudoscalar Meson operators:

$$O_{\mathrm{as}}(x) = \frac{1}{\sqrt{N_{\mathrm{as}}}} \sum_{k=1}^{N_{\mathrm{as}}} \bar{\Psi}^k(x)\gamma_5 \Psi^k(x)\,, \quad O_{\mathrm{fun}}(x) = \frac{1}{\sqrt{N_{\mathrm{f}}}} \sum_{l=1}^{N_{\mathrm{f}}} \bar{Q}^l(x)\gamma_5 Q^l(x)\,.$$

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Glueball operators:

$$O_G^{0^\pm}(L_x) = \sum_R R(L_x) \pm P(R(L_x))$$

TELOS Collaboration '24: 2405.05765

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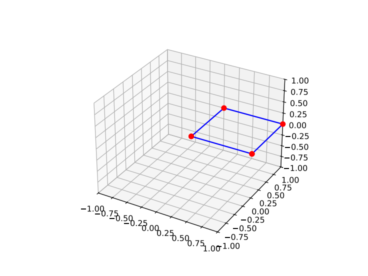

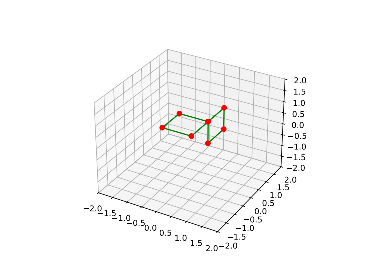

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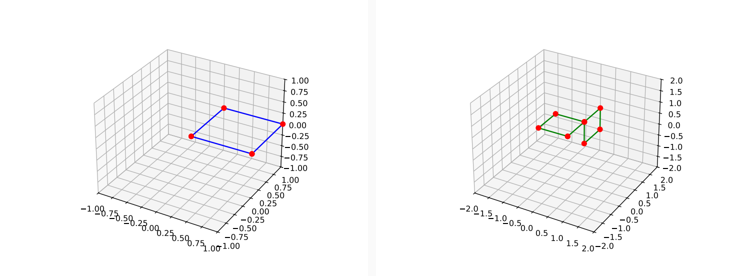

Figure 4: Left: The plaquette loop projected to $J^P = 0^+$. Right: the broken chair projected to $J^P = 0^-$. Blue lines: Take the trace and sum. Green lines: Take the trace and subtract.

GEVP: Combining the operators

In theory, it is possible to write:

$$\hat{O} = \sum_{i=1}^{\infty} c_i \hat{O}_i, \quad \text{so that} \quad \langle \hat{O}(t_0) O(t_1) \rangle = |c_0|^2 e^{-m_0(t_0 - t_1)}$$

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Variational problem to find c_i:

$$\text{min}_{\{c_i\}} \{m_{\text{eff}}(t)\} = \text{min}_{\{c_i\}} \left\{ \frac{D(t)}{D(t+1)} \right\}, \quad D(t) = \sum_{ij} c_i c_j \langle O_i O_j \rangle(t)$$

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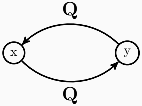
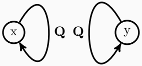
$$\implies C(t_0) v(t_0, t_1) = \lambda(t_0, t_1) C(t_1) v(t_0, t_1) \implies \hat{O}^\dagger \approx \sum_{i=1}^{N_{\mathrm{ops}}} v_i(t_0, t_1) O_i^\dagger$$

where $C(t)$ is the correlation matrix:

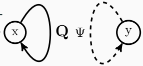
$$C_{ij}(t) = \langle O_i O_j \rangle (t) = \begin{bmatrix} C_{G-G} & C_{G-\mathrm{fun}} & C_{G-\mathrm{as}} \\ C_{\mathrm{fun}-G} & C_{\mathrm{fun}-\mathrm{fun}} & C_{\mathrm{fun}-\mathrm{as}} \\ C_{\mathrm{as}-G} & C_{\mathrm{as}-\mathrm{fun}} & C_{\mathrm{as}-\mathrm{as}} \end{bmatrix}$$

Correlators

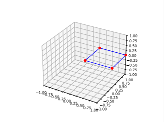
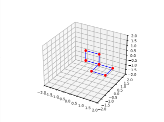
$$C_{\mathrm{fun-fun}}(x,y) \equiv \langle \bar{O}_{\mathrm{fun}}(x) O_{\mathrm{fun}}(y) \rangle = \pm$$ 

$$+ N_{\mathrm{f}}$$ 

,

$$C_{\mathrm{fun-as}}(x,y) \equiv \langle \bar{O}_{\mathrm{as}}(x) O_{\mathrm{fun}}(y) \rangle = +\sqrt{N_{\mathrm{as}} N_{\mathrm{f}}}$$ 

,

$$C_{G-G}(x,y) \equiv \langle \sum \mathrm{Tr}$$  $$(x) \cdot \sum \mathrm{Tr}$$  $$(y) \rangle$$

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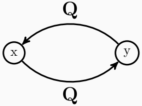
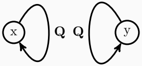
$C_{\mathrm{fun-fun}}(x, y) \equiv \langle \bar{O}_{\mathrm{fun}}(x) O_{\mathrm{fun}}(y) \rangle = \pm$ 

$+ N_{\mathrm{f}}$ 

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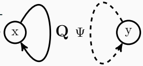
$C_{\mathrm{fun-as}}(x, y) \equiv \langle \bar{O}_{\mathrm{as}}(x) O_{\mathrm{fun}}(y) \rangle = +\sqrt{N_{\mathrm{as}} N_{\mathrm{f}}}$ 

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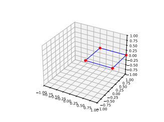
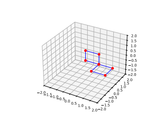
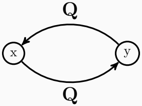
$C_{G-G}(x, y) \equiv \langle \sum \mathrm{Tr}$  $(x) \cdot \sum \mathrm{Tr}$  $(y) \rangle$

E.g, first diagram is:



$= \mathrm{Tr} \left(\Gamma \; D^{-1}_{\mathrm{fun}}(x; y) \; \Gamma \; D^{-1}_{\mathrm{fun}}(y; x) \right), \quad \Gamma = \mathbb{I}, \gamma_5$

TELOS Collaboration '24: 2405.05765

Results

Pseudoscalar Channel

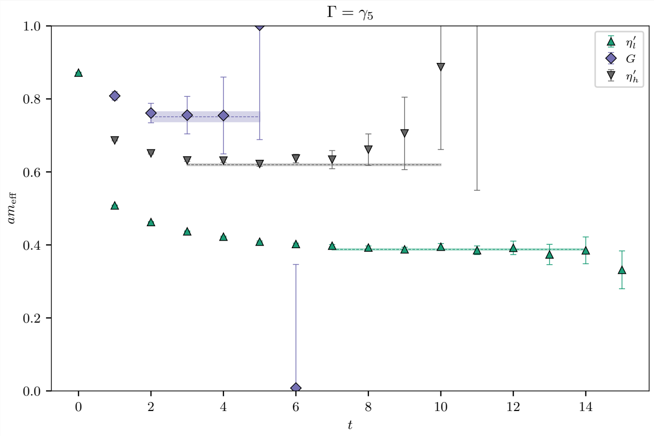


Figure 5: Effective masses $m_{\text{eff}}(t) = \ln\left(C(t)/C(t+1)\right)$ and their respective exponential fits (shaded rectangles) in the pseudoscalar channel.

Scalar Channel

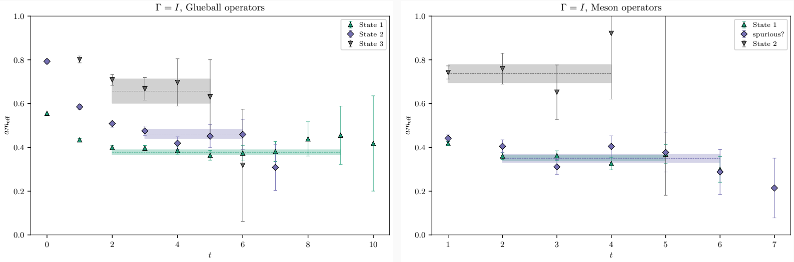


Figure 6: Effective masses $m_{\text{eff}}(t) = \ln\left(C(t)/C(t+1)\right)$ and their respective exponential fits (shaded rectangles) in the scalar channel.

Mass fits on both channels

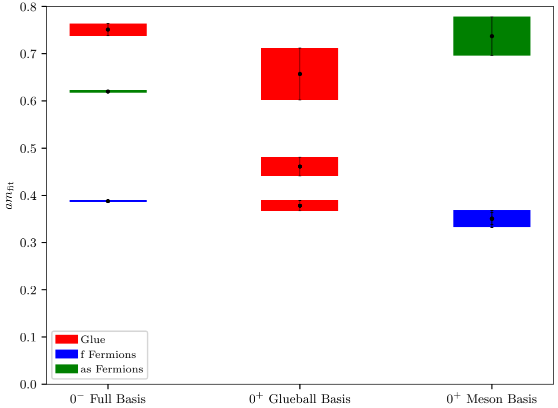


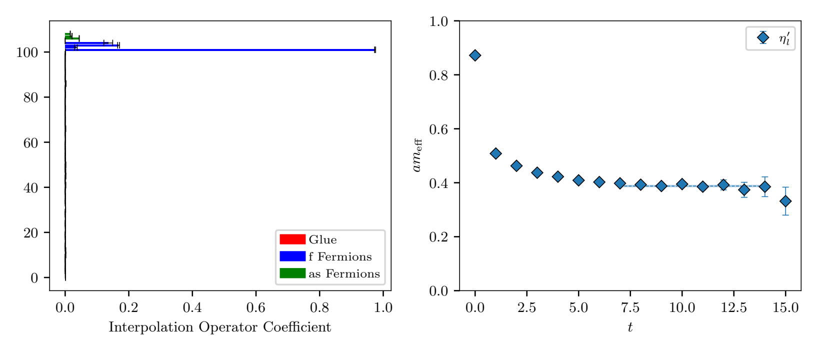
Figure 7: Comparison of masses obtained from exponential fits

Interpolating operators

From the GEVP:

$$\hat{O}^\dagger \approx \sum_{i=1}^{N_{\mathrm{ops}}} v_i(t_0, t_1) O_i^\dagger$$

In the pseudoscalar channel:



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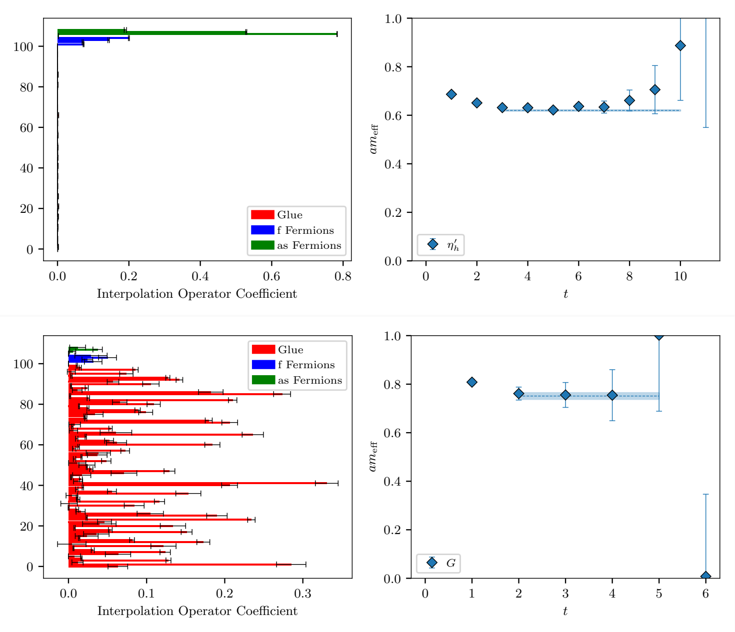

Conclusion

Takeaway points

- Despite its existence not being excluded from first principles, Glueballs remain experimentally undetected;
- Very hard to identify and study on Lattices with dynamical fermions;
- $Sp(4)$ gauge theory provides a QCD-like theory with a near-threshold stable glueball state.
- Good starting place to tune parameters and study glueball decays.

References

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Backup

Lattice Regularization

- *Bare* QFT $\longrightarrow$ Regularization $\longrightarrow$ Renormalization $\longrightarrow$ Renormalized QFT $\longrightarrow$ Amplitudes;
- Lattice Regularization $\longrightarrow$ non-perturbative!
    - Space-time discretization:

$$x^{\mu} \in \mathbb{R} \longrightarrow x_E^{\mu} = a(n_0, n_2, n_2, n_3), n_{\mu} \in \mathbb{N}$$

    - Wick rotation:

$$\begin{aligned} x^{\mu} &= (x^0, \mathbf{x}) \longrightarrow x_E^{\mu} = (ix^0, \mathbf{x}) \\ x^{\mu} x_{\mu} &= g_{\mu\nu} x^{\mu} x^{\nu} \longrightarrow x_E^{\mu} x_{E,\mu} = (x^0)^2 + |\mathbf{x}|^2 \quad \text{Euclidean!} \end{aligned}$$

Glueballs

- QCD is the theory of strong interactions:

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{2}\mathrm{Tr}\ F^{\mu\nu}F_{\mu\nu} + \sum_f \bar{\psi}_f(i\gamma^\mu D_\mu - m_f)\psi_f$$

- $SU(3)_c$ gauge transformations:

$$\begin{aligned} \psi_f &\longrightarrow \psi'_f = U\psi_f, \quad U \in SU(3) \\ A_\mu &\longrightarrow A'_\mu = UA_\mu U^{-1} + iU\partial_\mu U^{-1} \end{aligned}$$

- Observed colour confinement:

$$|\mathrm{phys}\rangle = \mathcal{O}^\dagger|\Omega\rangle, \quad \mathcal{O} \longrightarrow \mathcal{O}' = U\mathcal{O}U^{-1} = \mathcal{O}$$

Composite Higgs Models and $Sp(2N)$ theories

Re-write

$$Q^{J\,a} \equiv \left(\begin{array}{c} q^{J\,a} \\ \Omega^{ab}(-\tilde{C} q^{J+2\,*})_b \end{array} \right) , \quad \Psi^{j\,ab} \equiv \left(\begin{array}{c} \psi^{j\,ab} \\ \Omega^{ac}\Omega^{bd}(-\tilde{C}\psi^{j+3\,*})_{cd} \end{array} \right) ,$$

Get:

$$\mathcal{L} = -\frac{1}{2}\mathrm{Tr}\, V_{\mu\nu} V^{\mu\nu} +$$
$$+ \frac{1}{2}\sum_{M=1}^{4} \left(i \, (q^M)^\dagger{}_a \bar{\sigma}^\mu \, (D_\mu q^M)^a \right) - \frac{1}{2} m^{(f)} \sum_{M,N=1}^{4} \tilde{\Omega}_{MN} \left(q^{M\,a\,T} \Omega_{ab} \tilde{C} q^{N\,b} \right) +$$
$$+ \frac{1}{2}\sum_{m=1}^{6} \left(i \, (\psi^m)^\dagger{}_{ab} \bar{\sigma}^\mu \, (D_\mu \psi^m)^{ab} \right) - \frac{1}{2} m^{(as)} \sum_{m,n=1}^{6} \omega_{mn} \left(\psi^{m\,ab\,\mathrm{T}} \Omega_{ac}\Omega_{bd} \tilde{C} \psi^{n\,cd} \right)$$

Approximate $SU(4) \times SU(6)$ flavour symmetry. Breaking patterns:

- $SU(6) \longrightarrow SU(6)/SO(6) \rightsquigarrow$ 20 pseudo Nambu-Goldstone bosons;
- $SU(4) \longrightarrow SU(4)/Sp(4) \rightsquigarrow$ 5 pNBG's (Higgs doublet + scalar field).

Higgs Naturalness/Fine-Tuning/Hierarchy problem

Hierarchy Problem:

$$\mathcal{L}_{\mathrm{Higgs}} = c\Lambda_{\mathrm{SM}}^2 H^\dagger H, \quad m_{\mathrm{Higgs}} = 125 \text{ GeV}, \Lambda_{\mathrm{SM}} \approx 10^{15} \text{ GeV} \implies c \sim 10^{-28} \ll$$

Radiative corrections:

$$m_{\mathrm{Higgs}} = m_{\mathrm{Higgs,\,bare}} + \delta m_{\mathrm{Higgs,\,SM}} + \delta m_{\mathrm{Higgs,\,BSM}}$$
$$\delta m_{\mathrm{Higgs,\,SM}} \sim \frac{3y_t^2}{8\pi^2}\Lambda_{\mathrm{SM}}^2$$

Fine-tuning!

$$\Delta \geq \frac{\delta m_{\mathrm{Higgs,\,SM}}}{m_{\mathrm{Higgs}}} \approx \left(\frac{\Lambda_{\mathrm{SM}}}{450 \text{ GeV}}\right)^2 \sim 10^{24} \text{ GeV}$$

24 digit cancelation!

Lattice Observables

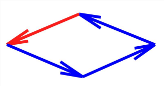
Need to discretize action. Using links:

$$U_\mu(x) = \exp\left(i\int_x^{x+a\hat{\mu}} ds^\mu A_\mu(s)\right) \approx \exp\left(iaA_\mu(x)\right) \in Sp(4)$$

The action comes as:

$$S = S_{\mathrm{gauge}} + S^{f}_{\mathrm{ferm}} + S^{as}_{\mathrm{ferm}}$$

$$S_{\mathrm{gauge}} = \beta \sum_x \sum_{\mu<\nu} \Bigg(\mathrm{Re}\,\mathrm{Tr}$$



$$\Bigg)$$

$$S^{(R)}_{\mathrm{ferm}} = a^4 \sum_{j=1}^{N_R} \sum_x \overline{\psi}^j(x) D^{(R)}\left[U; m^{(R)}\right) \psi^j(x)$$

Lattice Setup

Measurement of observables is done in the Path Integral Formalism:

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}\phi \; O[\phi] \exp(-S[\phi]), \quad Z = \int \mathcal{D}\phi \exp(-S[\phi])$$

Even with lattice discretization, hard to evaluate in most cases. Sample via Monte-Carlo:

$$\langle O \rangle \approx \frac{1}{N_{\text{conf}}} \sum_{\{U\}}^{N_{\text{conf}}} O[U], \quad \{U\}_{i=1,\dots,N_{\text{conf}}} \sim \exp(-S[U])$$

In practice, Markov Chain Monte Carlo:

$$U_0 \longrightarrow U_1 \longrightarrow U_2 \longrightarrow \cdots \longrightarrow U_{N_{\text{conf}}} \quad P_{\text{accept}} = \min\left(1, e^{-\Delta S}\right)$$