Straightening Out the IBP Equations

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Introduction

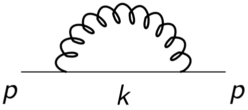
- The LHC is very demanding
  - Rapidly improving experimental precision!
  - NLO, NNLO or even N^3LO computation is necessary

- To have higher order computation of cross sections:
  - **Amplitude computation**
  - IR treatment (phase-space integration)

- To compute the needed amplitude, one needs two things:
  - **Integration-by-Parts (IBP) reduction**
  - Numerical/abstract evaluation of the master integrals

Amplitude computation

- To demonstrate how amplitudes can be computed, we use a simple example: one-loop bubble.



- QCD Feynman rules give:

$$\int d^d k \frac{(2-d)\not{k} + dm}{(k^2 - m^2)(p-k)^2}$$

  - Dimensional regularisation: $d = 4 - 2\epsilon$
  - Contains complicated numerator structure
  - Directly doable at one loop but very hard to evaluate with more loops

Feynman integrals

- The conventional technique is to decompose the matrix element into **Feynman Integrals**:

$$I_{\nu_1,\nu_2} = \int d^d k \frac{1}{(k^2 - m^2)^{\nu_1} (p-k)^{2\nu_2}}$$

  - The same propagators but raised to arbitrary power
- By partial fractioning, the matrix element becomes

$$\left[\frac{(2-d)\not{p}}{2m^2} (I_{0,1} - I_{1,0}) + dmI_{1,1} \right]$$

  - For complicated cases, the decomposition can contain hundreds or even thousands of integrals.
  - Inefficient to compute each one of them

IBP reduction

- There exist relations between Feynman integrals: **Integration-by-Parts** (IBP) identities.
- The IBP identities connect all integrals to a much smaller set of integrals: **master integrals**.
- In this example,

$$\frac{(2-d)\not{p}}{2m^2}(I_{0,1}-I_{1,0})+dmI_{1,1}\to\left(-\frac{(2-d)\not{p}}{2m^2}+\frac{d(d-2)}{2(d-3)m}\right)I_{1,0}$$

$$\{I_{0,1},I_{1,0},I_{1,1}\}\to\{I_{1,0}\}$$

- Once the IBP reduction is done, we are left with just the computation of the master integrals.

IBP equations

Here are the IBP identities for the one-loop bubble

$$\begin{aligned}0 = &-\nu_2\ I_{\nu_1-1,\nu_2+1}\\ &+(d-2\nu_1-\nu_2)\ I_{\nu_1,\nu_2}\\ &-2m^2\ \nu_1 I_{\nu_1+1,\nu_2}\end{aligned}$$

$$\begin{aligned}0 = &-\nu_2\ I_{\nu_1-1,\nu_2+1}\\ &-(\nu_1-\nu_2)\ I_{\nu_1,\nu_2}\\ &+\nu_1\ I_{\nu_1+1,\nu_2-1}\\ &-2m^2\nu_1\ I_{\nu_1+1,\nu_2}\end{aligned}$$

- Two independent equations
- Multi-variable recurrence relations
- Linear, homogenous and with rational coefficients
- Finite number of master integrals: one in this case.
- Highly tangled: the integrals in each equation contain shifts on multiple indices.

Solution of IBP equations

- Best known approach is proposed by Laporta (Laporta, 2000)
  - Substitute the abstract indices with explicit integer values
  - Produce an infinite system of equations, e.g.

$$\ldots, -I_{0,2} + I_{2,0} - 2m^2 I_{2,1} = 0, \ldots$$

  - Use Gauss elimination to solve a relevant part of it
- Limitation for more complicated problems:
  - Time-consuming: Months or even years for supercomputer with RAM consumption of several TB!
  - Huge size of simultaneous linear system of equations
- Numeric solution only:
  - It solves $I_{1,-1}$, $I_{1,-2}$ but not I_{ν_1,ν_2} for arbitrary $\nu_{1,2}$

Diagonalisation of IBP equations

- Diagonal form: changes one index at a time, keeping all other indices fixed (unshifted)

$$I_{\nu_1,\nu_2-1} = \frac{2m^2(d-\nu_1-2\nu_2)(1+d-\nu_1-2\nu_2)}{(2+d-2\nu_1-2\nu_2)(d-\nu_1-\nu_2)} I_{\nu_1,\nu_2}$$

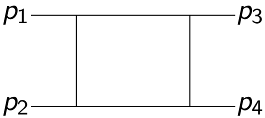
$$I_{\nu_1-1,1} = \frac{2m^2(d-\nu_1-2)(\nu_1-1)}{(d-2\nu_1)(d-1-\nu_1)} I_{\nu_1,1}$$

- Efficient solution: no simultaneous solution of multiple equations!
- One can even solve them for abstract ν_1, ν_2:

$$I_{\nu_1,\nu_2} = \frac{\Gamma(1-d+\nu_1+\nu_2)\Gamma\left(-\frac{d}{2}+\nu_1+\nu_2\right)}{\Gamma\left(2-\frac{d}{2}\right)\Gamma(\nu_1)\Gamma(1-d+\nu_1+2\nu_2)} \times (d-3)(-1)^{\nu_1}(m^2)^{2-\nu_1-\nu_2} I_{1,1}$$

What about multi-master cases?

- What about system with multiple masters?
    - Higher order?
    - Anything alternative?
- Let's consider the following example:



$$I_{\nu_1,\nu_2,\nu_3,\nu_4} = \int d^d k \frac{1}{D_1^{\nu_1} D_2^{\nu_2} D_3^{\nu_3} D_4^{\nu_4}}$$

- **Three** masters: two bubbles and a box

Matrix diagonal form

- One can derive diagonal equations with higher (third) order, but we found it better to do it in an alternative way (more in the backups).
- Consider recurrence in a **different object** but kept **first order**

$$\boldsymbol{V}_{\nu_1-1,\nu_2,\nu_3,\nu_4} = \boldsymbol{W}^{(4)}(\nu_1,\nu_2,\nu_3,\nu_4)\boldsymbol{V}_{\nu_1,\nu_2,\nu_3,\nu_4}$$

$$\begin{pmatrix} I_{\nu_1-1,\nu_2,\nu_3,\nu_4} \\ I_{\nu_1-2,\nu_2,\nu_3,\nu_4} \\ I_{\nu_1-1,\nu_2-1,\nu_3,\nu_4} \end{pmatrix} = \begin{pmatrix} W_{11}^{(4)} & W_{12}^{(4)} & W_{13}^{(4)} \\ W_{21}^{(4)} & W_{22}^{(4)} & W_{23}^{(4)} \\ W_{31}^{(4)} & W_{32}^{(4)} & W_{33}^{(4)} \end{pmatrix} \begin{pmatrix} I_{\nu_1,\nu_2,\nu_3,\nu_4} \\ I_{\nu_1-1,\nu_2,\nu_3,\nu_4} \\ I_{\nu_1,\nu_2-1,\nu_3,\nu_4} \end{pmatrix}$$

- Each line of these vector equations still contains four integrals: "third order" comes back as $\dim(\boldsymbol{V})$.

Matrix diagonal form

- Such equation can be derived for all four indices:

$$\begin{aligned}
\boldsymbol{V}_{\nu_1-1,\nu_2,\nu_3,\nu_4} &= \boldsymbol{W}^{(4)}(\nu_1,\nu_2,\nu_3,\nu_4)\boldsymbol{V}_{\nu_1,\nu_2,\nu_3,\nu_4} \\
\boldsymbol{V}_{1,\nu_2-1,\nu_3,\nu_4} &= \boldsymbol{W}^{(3)}(\nu_2,\nu_3,\nu_4)\boldsymbol{V}_{1,\nu_2,\nu_3,\nu_4} \\
\boldsymbol{V}_{1,1,\nu_3-1,\nu_4} &= \boldsymbol{W}^{(2)}(\nu_3,\nu_4)\boldsymbol{V}_{1,1,\nu_3,\nu_4} \\
\boldsymbol{V}_{1,1,1,\nu_4-1} &= \boldsymbol{W}^{(1)}(\nu_4)\boldsymbol{V}_{1,1,1,\nu_4}
\end{aligned}$$

- The explicit solution of $\boldsymbol{V}_{\nu_1,\nu_2,\nu_3,\nu_4}$ can then be achieved by iterative matrix multiplication

$$\boldsymbol{V}_{\nu_1\nu_2,\nu_3,\nu_4} = \left(\prod_{l=1}^{\nu_1+1}\boldsymbol{W}^{(\mathbf{4})}(l,\nu_2,\nu_3,\nu_4)\right)\left(\prod_{k=1}^{\nu_2+1}\boldsymbol{W}^{(\mathbf{3})}(k,\nu_3,\nu_4)\right)\times \left(\prod_{n=1}^{\nu_3+1}\boldsymbol{W}^{(\mathbf{2})}(n,\nu_4)\right)\left(\prod_{m=1}^{\nu_4+1}\boldsymbol{W}^{(\mathbf{1})}(m)\right)\boldsymbol{V}_{1,1,1,1}$$

Triangular form

- Advantages of the diagonal form:
  - Fast solution to I_{-100}, if you want
  - Analytic continuation to non-integer values
- Disadvantages of diagonal form:
  - Large rational function in $\boldsymbol{W}$
  - **Note:** a better choice of $\boldsymbol{V}$ may significantly simplify $\boldsymbol{W}$
- To improve computational efficiency: **triangular form**
  - No diagonalisation any more
  - No simultaneous solution of multiple equations
  - Much simpler rational functions in the equations

Triangular form

- Efficient reduction in complicated problems:
  - two-loop five-point planar (more benchmarks in the paper)
    - Five years ago, the very same task took more than a year
  - Benchmarked against `NeatIBP` (syzygy) (Wu et al., 2025) using the program `Kira` (Lange, Usovitsch and Wu, 2025)

Fermat	Finite field	
triang.	syzygy	triang.
3.4 h	11.3 h	8.0 h

- Timing breakdown for the Finite field approach

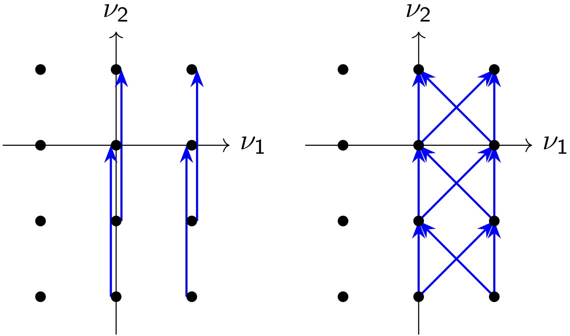
Step per probe	syzygy	triang.
Substitution	2.1%	92.7%
Solution	97.9%	7.3%

Conclusions

- We proposed an automated algorithm that achieved diagonalisation of the IBP system (**never achieved before**)
  - Potential pathway to find analytical solution to the IBP identities (ongoing)
  - Systematic approach to investigate properties of functions (more in the backups)
- As a by-product, we presented the triangular approach
  - Efficiency-oriented
  - A factor of 2 improvement in efficiency compared to the state-of-the-art algorithm (subject to additional improvements)
  - More state-of-the-art computation...

Backup: Matrix diagonal form

- Difference between higher-order diagonal (left) and matrix diagonal (right)



- It is realised with higher-order equation, one will end up with more "master integrals" than necessary (Mitov, 2005, Mitov and Moch, 2006)

Backup: Something to do with Gauss

- Contiguous relations of ${}_2F_1$-hypergeometric function

$$(c-a-1)F(a,b,c)+aF(a+1,b,c)-(c-1)F(a,b,c-1)=0,\ldots$$

- Through standard IBP pipeline, one can find
  - Finite solution
  - Two boundary conditions
    $F(1,1,1)=\frac{1}{1-z}$ $\quad F(1,1,2)=-\frac{\log(1-z)}{z}$
- Same matrix diagonal form can be derived

$$\begin{pmatrix} F(a+2,b,c) \\ F(a+1,b,c) \end{pmatrix} = \begin{pmatrix} \frac{a+bz-cz}{a(1-z)} & \frac{(a-c)(b-c)\,z}{ac(1-z)} \\ \frac{c}{a} & \frac{a-c}{a} \end{pmatrix} \begin{pmatrix} F(a+1,b,c) \\ F(a,b,c) \end{pmatrix}$$

  - Vanishing second column at $a=c$:
  - decoupling from the $\log(1-z)$ (textbook results)