Localisation = Gauge Fixing with a Twist: Refinements in the Batalin-Vilkovisky Formalism

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Dimitri Kanakaris Decavel
University of Hertfordshire

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Overview and Results

Supersymmetric localisation

A powerful principle by which supersymmetric path integrals reduce to (often finite-dimensional) integrals over the supersymmetric configurations of the theory.

BRST/BV quantisation

BRST: A formalism for gauge fixing and quantising gauge theories, through cohomological methods, by introducing ghost fields and a BRST differential.
BV: A powerful extension of the BRST procedure which allows one to systematically deal with on-shell closing symmetry and higher gauge symmetries, by introducing so-called antifields and the BV differential.

What we've been up to...

Enhancing the power of the localisation principle by putting it on the same footing as gauge symmetries, and treating it using the BV formalism.

Localisation in Mathematics

- Early results in the mathematics literature, e.g. Duistermaat-Heckmann formula (1982) and Berline-Vergne-Atiyah-Bott formula (1984).
- Consider a manifold M with a $U(1)$-action generated by $K \in \Gamma(\mathsf{T}M)$, and which has isolated fixed points ($\dim \mathcal{Z}(K) = 0$).
- On such a manifold we consider the equivariant differential

$$\mathcal{Q} = \mathrm{d} + \iota_K \,, \qquad \qquad \mathcal{Q}^2 = \mathcal{L}_K \,.$$

- **Berline-Vergne-Atiyah-Bott formula:** Integrals over $\mathcal{Q}$-closed polyforms $\alpha \in \ker \mathcal{Q} \subset \Omega^\bullet(M)$ simplify to sums over the zeroes $\mathcal{Z}(K)$ of K,

$$\int_M \alpha = \sum_{p \in \mathcal{Z}(X)} \frac{\alpha_d(p)}{\mathsf{Pf}\left(\frac{1}{2\pi}\mathcal{L}_K(p)\right)} \,, \qquad \qquad \mathcal{Q}\alpha = 0\,.$$

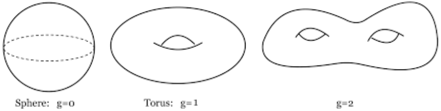
Localisation in Quantum Mechanics

- **Witten (1982):** Proposes to use localisation in quantum mechanics
- This leads to supersymmetric proofs of index theorems:

$$\begin{array}{ccc} \text{Index of Elliptic Complex} & = & \displaystyle\int_M (\text{Characteristic Class}) \\ \Big\| \ \textbf{localisation} & & \Big\| \ \textbf{localisation} \\ \mathrm{tr}_{\mathcal{H}}\left[(-1)^{\mathsf{F}} e^{-\beta H_{\mathsf{SUSY}}}\right] & = & \displaystyle\int_{\mathcal{C}^\infty(S^1(\beta),\Pi \mathsf{T} M)} \mathrm{D}x\mathrm{D}\psi\ e^{-S_{\mathsf{SUSY}}[x,\psi]} \end{array}$$

- Example: **Gauss-Bonnet theorem** on Riemann surfaces,

$$2 - 2(\text{genus}) = \frac{1}{4\pi}\int \mathrm{d}^2x\sqrt{g}R$$



Localisation in Quantum Field Theory

- **Beasley & Witten (2005):** Non-Abelian localisation of Chern-Simons theory on Seifert spaces to **flat connection moduli space**
  $\rightarrow$ application in topological QFT
- **Pestun (2007):** Supersymmetric localisation of $d = 4$, $\mathcal{N} = 2$ super Yang-Mills theory with matter to **instanton moduli space**
  $\rightarrow$ application in supersymmetric QFT
- **Arvanitakis & Kanakaris (2024):** Results towards localising $d = 3$, $\mathcal{N} = 0$ Yang-Mills theory to its phase space

<u>However</u> there are some complications:

- SUSY representations in field theory generically only close **on-shell**, i.e.

$$[\mathcal{Q}_\alpha, \mathcal{Q}_\beta] = \gamma_{\alpha\beta}{}^{\mu}\mathcal{B}_\mu + \text{field equations}$$

- <u>Problem:</u> Supersymmetric localisation requires **off-shell representations**
  - ≥ 4 supercharges: Off-shell reps require infinite towers of auxiliary fields
  - ≤ 9 supercharges: Sacrifice Lorentz invariance for off-shell rep with finite auxiliaries (**Berkovitz construction**)

Idea behind Localisation

Consider a supermanifold $\mathcal{M}$ with a $\mathcal{U} := U(\mathrm{b}|1)$-action, generated by odd vector field $\mathcal{Q}$ and $\mathcal{B} := \frac{1}{2}[\mathcal{Q}, \mathcal{Q}]$. Consider

$$Z = \int_{\mathcal{M}} \mathrm{d}\mu \; f \,,$$

which is $\mathcal{Q}$-invariant $\mathcal{Q}f = 0 = \mathrm{div}_{\mathrm{d}\mu}\, \mathcal{Q} = 0$. Integrating out orbits we find

$$Z = \int_{\mathcal{M}/\mathcal{U}} \mathrm{d}\bar{\mu} \;\; \mathsf{Vol}(\mathsf{Orb}) f \,,$$

where for $p \notin \mathcal{Z}(\mathcal{Q})$ we find

$$\mathsf{Vol}(\mathsf{Orb}(p)) \propto \int \mathrm{d}(\mathsf{fermion}) \cdot 1 = 0$$

$\Rightarrow$ **Contributions localise to the zeroes $\mathcal{Z}(\mathcal{Q})$ of $\mathcal{Q}$!**

More concretely, we are free to deform Z via a localising fermion $\Psi_{\mathsf{loc}} \in \ker \mathcal{B} \subset \mathcal{C}^{\infty}_{\mathsf{odd}}(\mathcal{M})$

$$Z(t) = \int \mathrm{d}\mu \; e^{-t^2 \mathcal{Q}\Psi_{\mathsf{loc}}} f = Z(0)$$

If we choose Ψ_{loc} s.t. $(\mathcal{Q}\Psi_{\mathsf{loc}})_\circ \geq 0$ we find in the limit $t \to \infty$ that

$$Z = \int_{\mathcal{Q}\Psi_{\mathsf{loc}}=0} \mathrm{d}\bar{\mu} \; \frac{f}{\sqrt{\mathrm{Hess}\, \mathcal{Q}\Psi_{\mathsf{loc}}}}$$

when $f = \mathcal{O}e^{-S} \Rightarrow S(t) = S + t^2 \mathcal{Q}\Psi_{\mathsf{loc}}$

BRST Quantisation of Gauge Theory

Problem: Gauge symmetries give rise to a number of complications:

- **Classical physics & perturbation theory:** Time evolution & propagators are ill defined due to the badly divergent kinetic operator.
- **Path integral:** Badly divergent due to the fact that the volume $\mathsf{Vol}(\mathcal{G}) = \infty$ of the gauge group $\mathcal{G}$ $\Rightarrow$ Gauge orbits with infinite volume!
- Ad hoc $\mathfrak{F} \to \mathfrak{F}/\mathcal{G}$ breaks manifest symmetries and locality, and is badly singular.

Solution: **BRST procedure:** Cancel out $\mathsf{Vol}(\mathcal{G})$ by introducing **ghost fields** $\mathcal{C}$,

$$\langle \mathcal{O} \rangle = \int_{\mathfrak{F}} \frac{\mathrm{D}\phi}{\mathsf{Vol}(\mathcal{G})}\ \mathcal{O} e^{-S} = \int_{\mathfrak{F}_{\mathsf{BRST}}} \mathrm{D}\phi \mathrm{D}\mathcal{C}\ \mathcal{O} e^{-S-Q_{\mathsf{BRST}}\Psi_{\mathsf{gf}}}$$

- Local gauge symmetry $\to$ global BRST symmetry which encodes observables in its cohomology $H(Q_{\mathsf{BRST}})$
- $Q_{\mathsf{BRST}}\Psi_{\mathsf{gf}}$ absorbs the factor $\mathsf{Vol}(\mathcal{G})^{-1}$ by integrating out ghosts $\int \mathrm{D}\mathcal{C}$, and **fixes the gauge**. Choice of **gauge-fixing fermion** Ψ_{gf} **= gauge choice**.

BRST Quantisation vs Localisation

Localisation and BRST quantisation share many similar features:

supermanifold $\mathcal{M}$	$\mathbb{N}$-graded supermanifold $\mathfrak{F}_{\mathsf{BRST}}$
supersymmetry $\mathcal{Q} \in \Gamma_{\mathsf{odd}}(\mathsf{T}\mathcal{M})$	BRST differential $Q_{\mathsf{BRST}} \in \Gamma_{(+1,\mathsf{even})}(\mathsf{T}\mathfrak{F}_{\mathsf{BRST}})$
BPS observables $H(\mathcal{Q}\vert \ker \mathcal{B})$	gauge observables $H^0(Q_{\mathsf{BRST}})$
localising fermion $\Psi_{\mathsf{loc}} \in \mathcal{C}^\infty_{\mathsf{odd}}(\mathcal{M})$	gauge-fixing fermion $\Psi_{\mathsf{gf}} \in \mathcal{C}^\infty_{(-1,\mathsf{even})}(\mathfrak{F}_{\mathsf{BRST}})$

Question: Can these be treated on a similar footing? **Yes, they can!**

- Extend BRST space to **gBRST space** $\mathfrak{F}_{\mathsf{gBRST}} := \mathbb{C}^{\mathsf{b}|1}[1]^\times \times \mathfrak{F}_{\mathsf{BRST}}$ by introducing (resp. even, odd) **global ghosts** ε, ξ associated respectively to $\mathcal{Q}, \mathcal{B}$
- Extend the BRST differential Q_{BRST} to the **gBRST differential**

$$Q_{\mathsf{gBRST}} = Q_{\mathsf{BRST}} + \varepsilon\mathcal{Q} + \xi\mathcal{B} - \tfrac{1}{2}\varepsilon^2 \frac{\partial}{\partial\xi}$$

- Gauge-BPS observables get encapsulated in the cohomology as

$$H^0(Q_{\mathsf{gBRST}}) = H^0\big(\mathcal{Q}\big| \ker(\mathcal{B}|H^0(Q_{\mathsf{BRST}}))\big)$$

BRST Quantisation vs Localisation

- The gauge-fixing fermion Ψ_{gf} and localising fermion Ψ_{loc} get combined into a single fermion

$$\Psi(t) = \Psi_{\mathsf{gf}} + \frac{t^2}{\varepsilon}\Psi_{\mathsf{loc}}\,, \qquad\qquad S \to S(t) = S + Q_{\mathsf{gBRST}}\Psi(t)$$

- We don't integrate over global ghosts, however global ghost dependence of the path integral drops out!

$$\int \mathrm{D}\phi\mathrm{D}\mathcal{C} : \mathcal{C}^{\infty}(\mathfrak{F}_{\mathsf{gBRST}}) \to \mathbb{R}[\varepsilon, \tfrac{1}{\varepsilon}, \xi] \supset \mathbb{R}$$

- Localisation in the presence of gauge symmetries yields

$$\langle \mathcal{O}_{\mathsf{BPS}} \rangle = \int_{\mathfrak{F}_{\mathsf{BRST}}} \mathrm{D}\phi\mathrm{D}\mathcal{C}\ \mathcal{O}_{\mathsf{BPS}} e^{-S - Q_{\mathsf{BRST}}\Psi_{\mathsf{gf}}} = \int_{\mathfrak{F}_{\mathsf{loc}}} \mathrm{d}\phi_0\ \frac{\mathcal{O}_{\mathsf{BPS}} e^{-S}}{\sqrt{\mathrm{Hess}(\mathcal{Q}\Psi_{\mathsf{loc}} + Q_{\mathsf{BRST}}\Psi_{\mathsf{gf}})}}$$

BV formalism

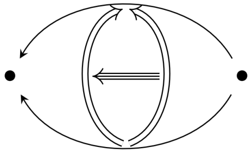
BV formalism

A formal treatment of infinitesimal (global or gauge) symmetries by **extending the BRST formalism** to deal with **higher, open symmetry structures.**

Introduce a **chain complex** of **fields** ϕ^i**, (higher) ghosts** $c^{(k)}$ and their **antifield counterparts** ϕ_i^*, $c_{(k)}^*$:

$$\underbrace{\cdots \to c^*_{(2)} \xrightarrow{\text{higher Noether}} c^*_{(1)} \xrightarrow{\text{Noether}} \phi^*_i \xrightarrow{\text{field eqns}}}_{\text{deals with field equations}} \underbrace{\phi^i \xrightarrow{\text{gauge}} c^{(1)} \xrightarrow{\text{higher gauge}} c^{(2)} \to \cdots}_{\text{describes symmetry transformations}}$$

- BRST space $\to$ BV space $\left(\mathfrak{F}_{\mathsf{BV}}, Q_{\mathsf{BV}}, (-,-)\right)$
- gauge-fixing $=$ fixing antifields to $\Phi^* = \frac{\delta}{\delta\Phi}\Psi$
- BV action S_{BV} s.t. $Q_{\mathsf{BV}} = (S_{\mathsf{BV}}, -)$ & $S^{\mathsf{gf}}_{\mathsf{BV}} = S + Q_{\mathsf{BRST}}\Psi_{\mathsf{gf}} + \cdots$
- **Master equation:** $Q^2_{\mathsf{BV}} = 0 \Leftrightarrow (S_{\mathsf{BV}}, S_{\mathsf{BV}}) = 0$



Ongoing work: BV refined localisation

So why should we care about the BV formalism?

→ the presence of on-shell supersymmetry representations is a hindering factor for supersymmetric localisation

→ the BV formalism is precisely the kind of tool we need to circumvent this issue altogether

Some work towards getting around this using the BV formalism:

- **Losev & Lysov (2023, 2024):** BV treatment of supersymmetry on finite-dimensional spaces
- **Cattaneo & Jiang (2025):** Reinterpreting proofs of localisation theorems through the BV formalism
- **Arvanitakis, Borsten, Kanakaris & Kim (2025):** Applying the BV formalism to localise supersymmetric gauge theory

Outlook: General localisation theorems? Higher global symmetries used for localisation?

Thank You!