# Localisation = Gauge Fixing with a Twist: Refinements in the Batalin-Vilkovisky Formalism

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# Overview and Results

## Supersymmetric localisation

A powerful principle by which supersymmetric path integrals reduce to (often finite-dimensional) integrals over the supersymmetric configurations of the theory.

## BRST/BV quantisation

**BRST:** A formalism for gauge fixing and quantising gauge theories, through cohomological methods, by introducing ghost fields and a BRST differential.
**BV:** A powerful extension of the BRST procedure which allows one to systematically deal with on-shell closing symmetry and higher gauge symmetries, by introducing so-called antifields and the BV differential.

## What we've been up to...

Enhancing the power of the localisation principle by putting it on the same footing as gauge symmetries, and treating it using the BV formalism.

# Localisation in Mathematics

- Early results in the mathematics literature, e.g. Duistermaat-Heckmann formula (1982) and Berline-Vergne-Atiyah-Bott formula (1984).
- Consider a manifold $M$ with a $U(1)$-action generated by $K \in \Gamma(\mathsf{T}M)$, and which has isolated fixed points $(\dim \mathcal{Z}(K) = 0)$.
- On such a manifold we consider the equivariant differential

$$\mathcal{Q} = \mathrm{d} + \iota_K \,, \qquad \mathcal{Q}^2 = \mathcal{L}_K \,.$$

- **Berline-Vergne-Atiyah-Bott formula:** Integrals over $\mathcal{Q}$-closed polyforms $\alpha \in \ker \mathcal{Q} \subset \Omega^\bullet(M)$ simplify to sums over the zeroes $\mathcal{Z}(K)$ of $K$,

$$\int_M \alpha = \sum_{p \in \mathcal{Z}(X)} \frac{\alpha_d(p)}{\mathsf{Pf}\left(\frac{1}{2\pi}\mathcal{L}_K(p)\right)} \,, \qquad \mathcal{Q}\alpha = 0 \,.$$

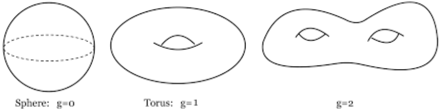
# Localisation in Quantum Mechanics

- **Witten (1982):** Proposes to use localisation in quantum mechanics
- This leads to supersymmetric proofs of index theorems:

$$\begin{array}{ccc} \text{Index of Elliptic Complex} & = & \displaystyle\int_M (\text{Characteristic Class}) \\ \| \ \textbf{localisation} & & \| \ \textbf{localisation} \\ \mathrm{tr}_{\mathcal{H}}\left[(-1)^{\mathsf{F}} e^{-\beta H_{\mathsf{SUSY}}}\right] & = & \displaystyle\int_{\mathcal{C}^\infty(S^1(\beta),\Pi\mathsf{T}M)} \mathrm{D}x\mathrm{D}\psi\ e^{-S_{\mathsf{SUSY}}[x,\psi]} \end{array}$$

- Example: **Gauss-Bonnet theorem** on Riemann surfaces,

$$2 - 2(\text{genus}) = \frac{1}{4\pi}\int \mathrm{d}^2x\sqrt{g}R$$



# Localisation in Quantum Field Theory

- **Beasley & Witten (2005):** Non-Abelian localisation of Chern-Simons theory on Seifert spaces to **flat connection moduli space**
  $\rightarrow$ application in topological QFT
- **Pestun (2007):** Supersymmetric localisation of $d = 4$, $\mathcal{N} = 2$ super Yang-Mills theory with matter to **instanton moduli space**
  $\rightarrow$ application in supersymmetric QFT
- **Arvanitakis & Kanakaris (2024):** Results towards localising $d = 3$, $\mathcal{N} = 0$ Yang-Mills theory to its phase space $\rightarrow$ non-supersymmetric non-topological QFT

However there are some complications:

- SUSY representations in field theory generically only close **on-shell**, i.e.

$$[\mathcal{Q}_\alpha, \mathcal{Q}_\beta] = {\gamma_{\alpha\beta}}^\mu \mathcal{B}_\mu + \text{field equations}$$

- Problem: Supersymmetric localisation requires **off-shell representations**
  - $\geq 4$ supercharges: Off-shell reps require infinite towers of auxiliary fields
  - $\leq 9$ supercharges: Sacrifice Lorentz invariance for off-shell rep with finite auxiliaries (**Berkovitz construction**)

# Idea behind Localisation

Consider a supermanifold $\mathcal{M}$ with a $\mathcal{U} := U(\mathsf{b}|1)$-action, generated by odd vector field $\mathcal{Q}$ and $\mathcal{B} := \frac{1}{2}[\mathcal{Q}, \mathcal{Q}]$. Consider

$$Z = \int_{\mathcal{M}} \mathrm{d}\mu \; f \,,$$

which is $\mathcal{Q}$-invariant $\mathcal{Q}f = 0 = \mathrm{div}_{\mathrm{d}\mu}\, \mathcal{Q} = 0$. Integrating out orbits we find

$$Z = \int_{\mathcal{M}/\mathcal{U}} \mathrm{d}\bar{\mu} \;\; \mathsf{Vol}(\mathsf{Orb}) f \,,$$

where for $p \notin \mathcal{Z}(\mathcal{Q})$ we find

$$\mathsf{Vol}(\mathsf{Orb}(p)) \propto \int \mathrm{d}(\text{fermion}) \cdot 1 = 0$$

$\Rightarrow$ **Contributions localise to the zeroes $\mathcal{Z}(\mathcal{Q})$ of $\mathcal{Q}$!**

More concretely, we are free to deform $Z$ via a localising fermion $\Psi_{\mathsf{loc}} \in \ker \mathcal{B} \subset \mathcal{C}^\infty_{\mathsf{odd}}(\mathcal{M})$

$$Z(t) = \int \mathrm{d}\mu \; e^{-t^2 \mathcal{Q}\Psi_{\mathsf{loc}}} f = Z(0)$$

If we choose $\Psi_{\mathsf{loc}}$ s.t. $(\mathcal{Q}\Psi_{\mathsf{loc}})_\circ \geq 0$ we find in the limit $t \to \infty$ that

$$Z = \int_{\mathcal{Q}\Psi_{\mathsf{loc}}=0} \mathrm{d}\bar{\mu} \; \frac{f}{\sqrt{\mathrm{Hess}\, \mathcal{Q}\Psi_{\mathsf{loc}}}}$$

when $f = \mathcal{O}e^{-S} \Rightarrow S(t) = S + t^2 \mathcal{Q}\Psi_{\mathsf{loc}}$

# BRST Quantisation of Gauge Theory

Problem: Gauge symmetries give rise to a number of complications:

- **Classical physics & perturbation theory:** Time evolution & propagators are ill defined due to the badly degenerate kinetic operator.
- **Path integral:** Badly divergent due to the fact that the volume $\mathsf{Vol}(\mathcal{G}) = \infty$ of the gauge group $\mathcal{G}$ $\Rightarrow$ Gauge orbits with infinite volume!
- Ad hoc $\mathfrak{F} \to \mathfrak{F}/\mathcal{G}$ breaks manifest symmetries and locality, and is badly singular.

Solution: **BRST procedure:** Cancel out $\mathsf{Vol}(\mathcal{G})$ by introducing **ghost fields** $\mathcal{C}$,

$$\langle \mathcal{O} \rangle = \int_{\mathfrak{F}} \frac{\mathrm{D}\phi}{\mathsf{Vol}(\mathcal{G})}\, \mathcal{O} e^{-S} = \int_{\mathfrak{F}_{\mathsf{BRST}}} \mathrm{D}\phi \mathrm{D}\mathcal{C}\, \mathcal{O} e^{-S - Q_{\mathsf{BRST}} \Psi_{\mathsf{gf}}}$$

- Local gauge symmetry $\to$ global BRST symmetry which encodes observables in its cohomology $H(Q_{\mathsf{BRST}})$
- $Q_{\mathsf{BRST}} \Psi_{\mathsf{gf}}$ absorbs the factor $\mathsf{Vol}(\mathcal{G})^{-1}$ by integrating out ghosts $\int \mathrm{D}\mathcal{C}$, and **fixes the gauge**. Choice of **gauge-fixing fermion** $\Psi_{\mathsf{gf}}$ **= gauge choice**.

# BRST Quantisation vs Localisation

Localisation and BRST quantisation share many similar features:

| | |
|---|---|
| supermanifold $\mathcal{M}$ | $\mathbb{N}$-graded supermanifold $\mathfrak{F}_{\mathsf{BRST}}$ |
| supersymmetry $\mathcal{Q} \in \Gamma_{\mathsf{odd}}(\mathsf{T}\mathcal{M})$ | BRST differential $Q_{\mathsf{BRST}} \in \Gamma_{(+1,\mathsf{even})}(\mathsf{T}\mathfrak{F}_{\mathsf{BRST}})$ |
| BPS observables $H(\mathcal{Q}\vert \ker \mathcal{B})$ | gauge observables $H^0(Q_{\mathsf{BRST}})$ |
| localising fermion $\Psi_{\mathsf{loc}} \in \mathcal{C}^\infty_{\mathsf{odd}}(\mathcal{M})$ | gauge-fixing fermion $\Psi_{\mathsf{gf}} \in \mathcal{C}^\infty_{(-1,\mathsf{even})}(\mathfrak{F}_{\mathsf{BRST}})$ |

Question: Can these be treated on a similar footing? **Yes, they can!**

- Extend BRST space to **gBRST space** $\mathfrak{F}_{\mathsf{gBRST}} := \mathbb{C}^{\mathsf{b}|1}[1]^\times \times \mathfrak{F}_{\mathsf{BRST}}$ by introducing (resp. even, odd) **global ghosts** $\varepsilon, \xi$ associated respectively to $\mathcal{Q}, \mathcal{B}$
- Extend the BRST differential $Q_{\mathsf{BRST}}$ to the **gBRST differential**

$$Q_{\mathsf{gBRST}} = Q_{\mathsf{BRST}} + \varepsilon\mathcal{Q} + \xi\mathcal{B} - \tfrac{1}{2}\varepsilon^2 \frac{\partial}{\partial \xi}$$

- Gauge-BPS observables get encapsulated in the cohomology as

$$H^0(Q_{\mathsf{gBRST}}) = H^0\big(\mathcal{Q}\big| \ker(\mathcal{B}|H^0(Q_{\mathsf{BRST}}))\big)$$

# BRST Quantisation vs Localisation

- The gauge-fixing fermion $\Psi_{\mathsf{gf}}$ and localising fermion $\Psi_{\mathsf{loc}}$ get combined into a single fermion

$$\Psi(t) = \Psi_{\mathsf{gf}} + \frac{t^2}{\varepsilon}\Psi_{\mathsf{loc}}\,, \qquad S \to S(t) = S + Q_{\mathsf{gBRST}}\Psi(t)$$

- We don't integrate over global ghosts, however global ghost dependence of the path integral drops out!

$$\int \mathrm{D}\phi\mathrm{D}\mathcal{C} : \mathcal{C}^\infty(\mathfrak{F}_{\mathsf{gBRST}}) \to \mathbb{R}[\![\varepsilon, \tfrac{1}{\varepsilon}, \xi]\!] \supset \mathbb{R}$$

- Localisation in the presence of gauge symmetries yields

$$\langle \mathcal{O}_{\mathsf{BPS}} \rangle = \int_{\mathfrak{F}_{\mathsf{BRST}}} \mathrm{D}\phi\mathrm{D}\mathcal{C}\ \mathcal{O}_{\mathsf{BPS}} e^{-S - Q_{\mathsf{BRST}}\Psi_{\mathsf{gf}}} = \int_{\mathfrak{F}_{\mathsf{loc}}} \mathrm{d}\phi_0\ \frac{\mathcal{O}_{\mathsf{BPS}} e^{-S}}{\sqrt{\mathrm{Hess}(\mathcal{Q}\Psi_{\mathsf{loc}} + Q_{\mathsf{BRST}}\Psi_{\mathsf{gf}})}}$$

# BV formalism

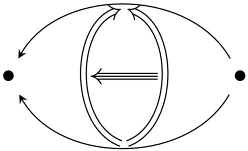
## BV formalism

A formal treatment of infinitesimal (global or gauge) symmetries by **extending the BRST formalism** to deal with **higher, open symmetry structures.**

Introduce a **chain complex** of **fields** $\phi^i$**, (higher) ghosts** $c^{(k)}$ and their **antifield counterparts** $\phi_i^*$, $c^*_{(k)}$:

$$\underbrace{\cdots \to c^*_{(2)} \xrightarrow{\text{higher Noether}} c^*_{(1)} \xrightarrow{\text{Noether}} \phi_i^* \xrightarrow{\text{field eqns}}}_{\text{deals with field equations}} \underbrace{\phi^i \xrightarrow{\text{gauge}} c^{(1)} \xrightarrow{\text{higher gauge}} c^{(2)} \to \cdots}_{\text{describes symmetry transformations}}$$

- BRST space $\to$ BV space $(\mathfrak{F}_{\mathsf{BV}}, Q_{\mathsf{BV}}, (-,-))$
- gauge-fixing $=$ fixing antifields to $\Phi^* = \frac{\delta}{\delta\Phi}\Psi_{\mathsf{gf}}$
- BV action $S_{\mathsf{BV}}$ s.t. $Q_{\mathsf{BV}} = (S_{\mathsf{BV}}, -)$ & $S^{\mathsf{gf}}_{\mathsf{BV}} = S + Q_{\mathsf{BRST}}\Psi_{\mathsf{gf}} + \cdots$
- **Master equation:** $Q^2_{\mathsf{BV}} = 0 \Leftrightarrow (S_{\mathsf{BV}}, S_{\mathsf{BV}}) = 0$



# Ongoing work: BV refined localisation

So why should we care about the BV formalism?

→ the presence of on-shell supersymmetry representations is a hindering factor for supersymmetric localisation

→ the BV formalism is precisely the kind of tool we need to circumvent this issue altogether

Some work towards getting around this using the BV formalism:

- **Losev & Lysov (2023, 2024):** BV treatment of supersymmetry on finite-dimensional spaces
- **Cattaneo & Jiang (2025):** Reinterpreting proofs of localisation theorems through the BV formalism
- **Arvanitakis, Borsten, Kanakaris & Kim (2025):** Applying the BV formalism to localise supersymmetric gauge theory

<u>Outlook:</u> General localisation theorems? Higher global symmetries used for localisation?

# Thank You!