Aspects of Duality

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Outline

1. In this talk, I will discuss (some) aspects of Duality. I will explain what this is, the idea and why it is useful in Physics.

2. Towards the end I will briefly comment on some of my recent contributions– that belong to a technical talk.

3. The plan is to first set the ground and explain which problems are suitable to be tackled using dualities. Give some examples of dualities. Focus on AdS/CFT duality. Then, discuss one recent developments for field theories displaying confinement. Finish with closing remarks and some 'epistemological' comments that this area of research brings.

4. The ideal member of the audience is a generic theoretical physicist. I only assume you know (some) Field Theory and Gravity. Please, stop me with questions if I say things that are unclear or unknown or unclear.

Theories written in terms of a Lagrangian, (Hamiltonian).

Lagrangians written in terms of particles of spin: $0, \frac{\hbar}{2}, \hbar, \frac{3\hbar}{2}, 2\hbar$.

$$L \sim -\frac{1}{2}(\partial_\mu \phi)^2 - V(\phi), \qquad \underline{\phi} : \text{has spin zero}$$

$$L \sim \bar{\psi}(i\gamma^\mu \partial_\mu + m)\psi - V(\bar{\psi}\psi), \quad \underline{\psi} : \text{has spin } \frac{\hbar}{2},$$

$$L \sim -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \qquad \underline{A_\mu} : \text{has spin } \hbar,$$

$$L \sim -\frac{1}{2}\bar{\Psi}_\mu(\epsilon^{\mu\nu ab}\gamma_5\gamma_a\partial_b - im\gamma^{[\mu,\nu]})\Psi_\nu, \ \underline{\Psi_\mu} : \text{has spin } \frac{3}{2}\hbar, \ \text{(not yet!)}$$

$$L \sim \sqrt{g}R, \qquad \underline{g_{\mu\nu}} : \text{has spin } 2\hbar.$$

The symmetries we impose play a vital role in determining the possible interactions.

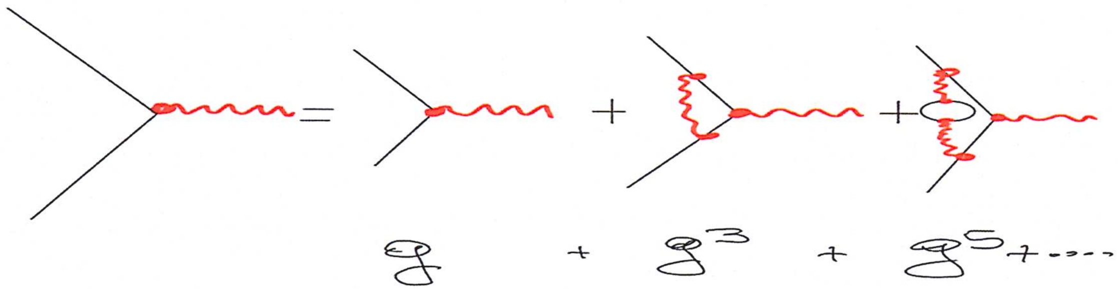
$$L \sim \sqrt{g}\left[R - \Lambda - \frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}(D_\mu\phi)^2 - \bar{\psi}(i\gamma^\mu D_\mu + m)\psi - V(\psi,\phi)\right],$$

The Standard Model of Particle Physics (with gravity) is an example of this. Generically, we have QFTs!

How to calculate observables and compare with experiments?
We use the 'Potential Energy' part of the Lagrangian. We read the interactions between particles weighted by the 'coupling constant'.

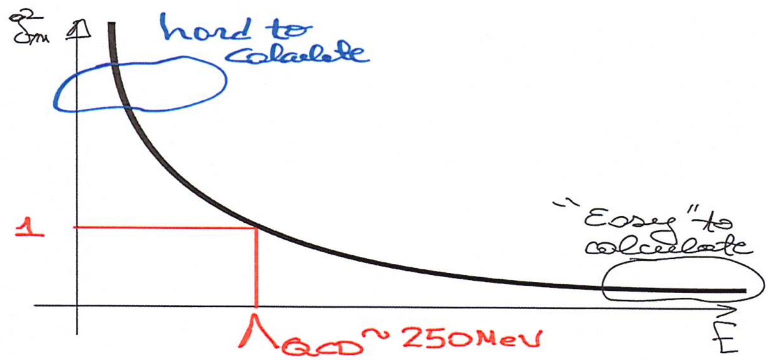
$$L_{int} \sim g_e \bar{\psi} \gamma^{\mu} \psi A_{\mu}.$$

If $g_e < 1$, perform perturbations. Analog to the Taylor series.



What if g_e is **not** smaller than one? **Any physical interest**?

Example: Chromodynamics, the theory of quarks and gluons.



If $g_e > 1$ the usual ways of calculating do not apply. Give the theory/model to a computer. Calculate the Partition Function, from which observables are derived.

This is a a well developed and beautiful area of Theoretical Physics called "**Lattice Field Theory**". The problem with the Lattice is that is **not very flexible**. Hard to use for generic QFTs.

Duality: idea 'scattered' over branches of Theoretical Physics.
Suppose that we have a Lagrangian, or a Hamiltonian, consisting of a sum of two parts, one–H_0– solvable. The other, a perturbation

$$H = H_0 + g H_1,$$

if the parameter g is small, a given physical quantity is expressed in perturbation theory (Taylor series) as,

$$\mathcal{A} = \mathcal{A}_0 + g\mathcal{A}_1 + g^2\mathcal{A}_2 + ...$$

If the parameter g is not small, we appeal to a duality

$$H = H_0 + g H_1 \leftrightarrow \tilde{H} = \tilde{H}_0 + \tilde{g}\tilde{H}_1.$$

If g is small, we perturb in terms of the original degrees of freedom x. If $\tilde{g}$ is small, perturb in terms of the dual degrees of freedom $\tilde{x}$.
The map between x and $\tilde{x}$, is nonlinear. Usually $g \sim \frac{1}{\tilde{g}}$.

Let us see an old example

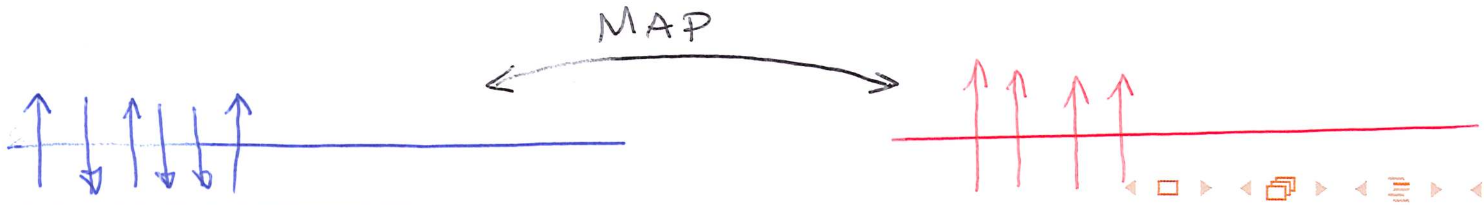
Kramers and Wannier (1940): a system of spins $\frac{\hbar}{2}$ at temperature T_1 is **equivalent** to another system of spins $\frac{\hbar}{2}$ at temperature $T_2 \sim \frac{1}{T_1}$. This is a Cold to Hot duality!

$$S_{Ising} = -\beta \sum_{links} \sigma_i \sigma_j, \quad Z[\beta] = \sum_{[\sigma]} e^{-S_{Ising}}.$$

Kramers and Wannier showed that this $Z[\beta]$ was equivalent to a $Z[\tilde{\beta}]$, with

$$Z[\tilde{\beta}] = \sum_{[s_i]} e^{-\tilde{S}_{Ising}}, \quad \tilde{S}_{Ising} = \tilde{\beta} \sum_{[dual]} s_i s_j, \quad e^{-2\beta} = \tanh[\tilde{\beta}].$$

The relation between σ_i and s_i is involved and non-linear. The parameters β and $\tilde{\beta}$ are in inverse relation.



Coleman and Mandelstam (1974).

A special two-dimensional theory in terms of a boson (ϕ) can be exactly written in terms of a two-dimensional fermion (ψ). A bosonic system, in terms of the real scalar degree of freedom ϕ and a fermionic one in terms of the Dirac field ψ.

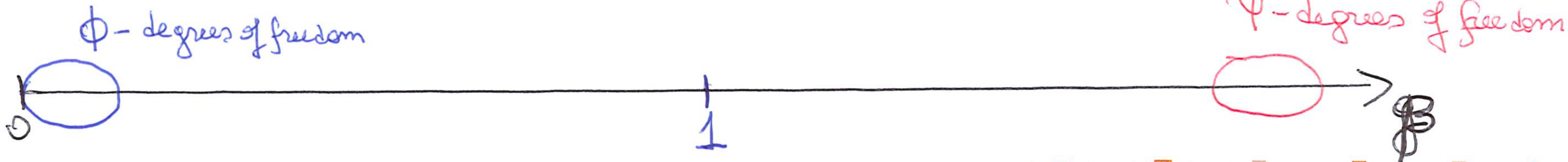
$$L_{fermion} = \int d^2x \bar{\psi}(i\gamma^\mu \partial_\mu - m_f)\psi - \frac{g}{2}(\bar{\psi}\gamma_\mu\psi)^2,$$

$$L_{boson} = \int d^2x - \frac{1}{2}(\partial_\mu\phi)^2 + m_b \cos(\beta\phi).$$

These systems are equivalent if we identify

$$\beta^2 = \frac{4\pi}{1+\frac{g}{\pi}}, \quad \cos\phi \to m_f\bar{\psi}\psi, \quad \epsilon_{\mu\nu}\partial_\nu\phi \to \bar{\psi}\gamma^\mu\psi.$$

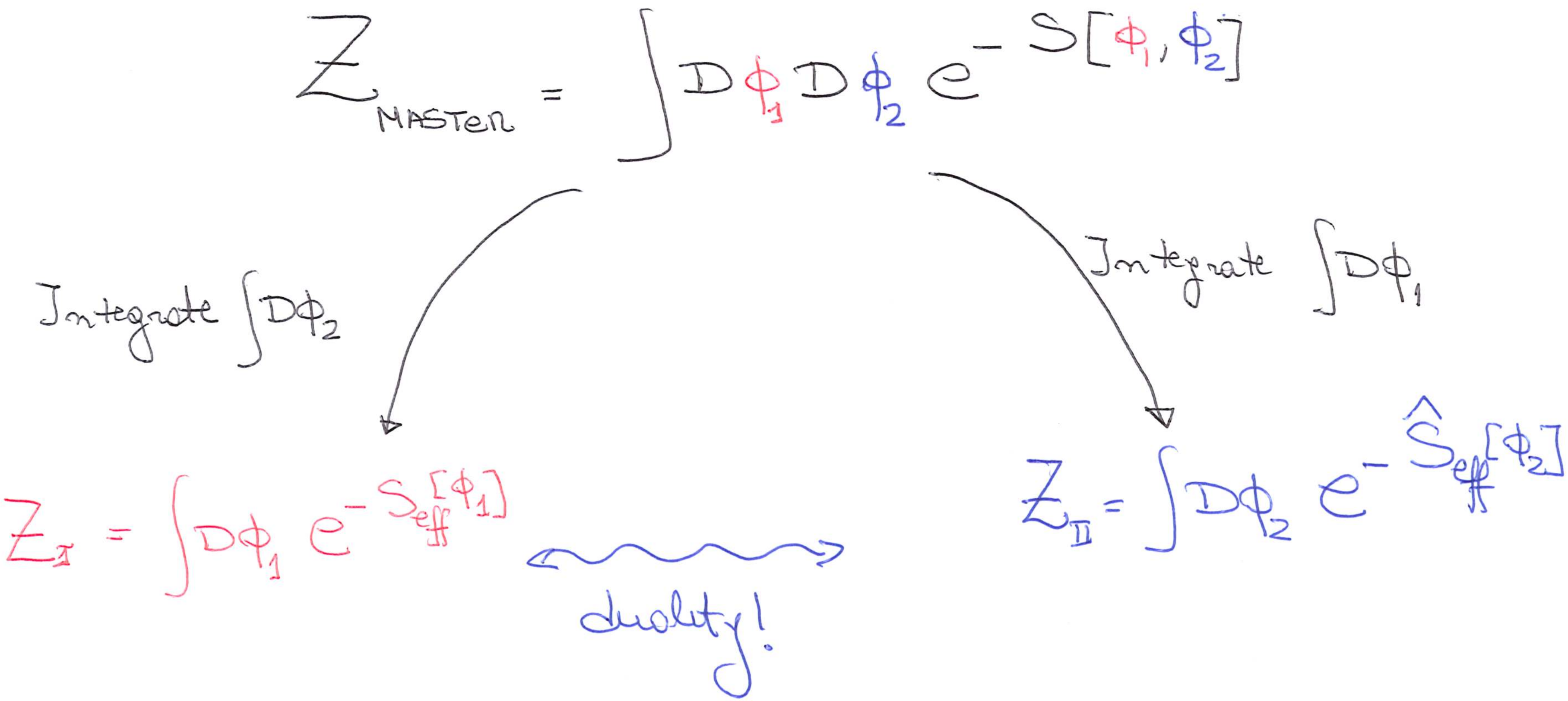
$$\psi_\pm = e^{-\frac{2i\pi}{\beta}\int_{-\infty}^{x}\partial_t\phi(x')dx' \pm \frac{i\beta}{2}\phi(x)}.$$



There are many other scattered examples of dualities!

Some can be proven, others are conjectured, and pass consistency checks.

One possible way of proving dualities



This proof relies on the possibility of performing the integrals! This is typically possible for quadratic Actions.

Examples are: Particle-Vortex duality in Condensed matter.

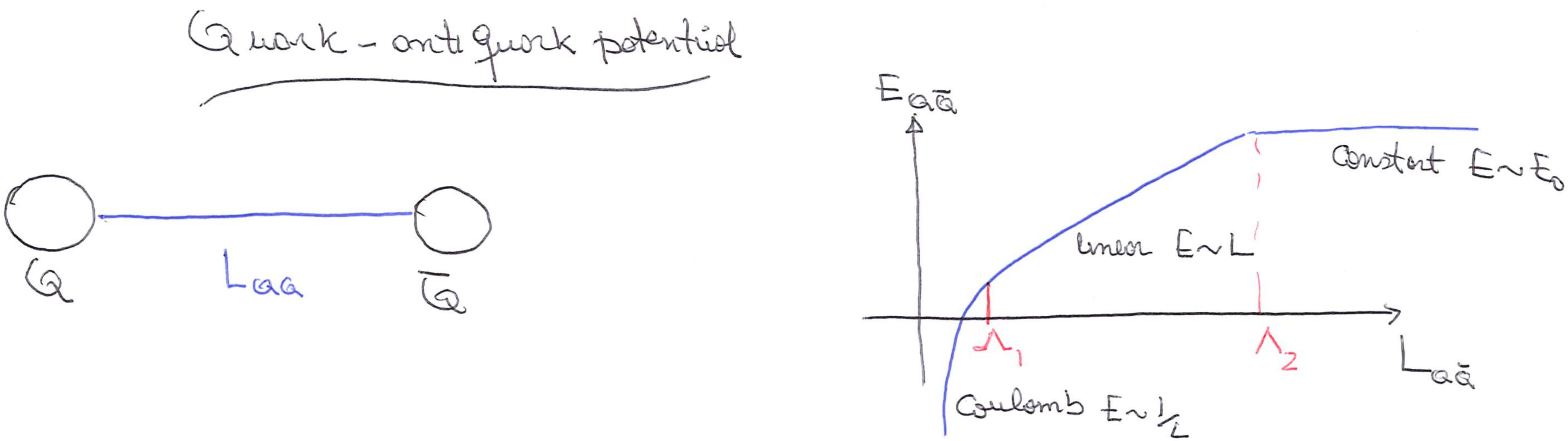
T-duality and its non-Abelian version for string theory in certain backgrounds.

Other dualities are quite difficult to prove. They involve strongly interacting systems on which some enhanced symmetries exist.

Conformality and SUSY.

What is Conformal symmetry?

Force of attraction between a quark and an anti-quark separated by a distance $L_{q\bar{q}}$. These systems are **scale dependent**.



If the observables do not show a 'scale' dependence!

These go under the name of '**conformal field theories**'. Various examples, many refer to the behaviour of systems close to a phase transition. The presence of this symmetry largely constrains the dynamics.

A Relativistic Quantum Field theory has Poincare symmetry

$$x^\mu = \Lambda^\mu_\nu x^\nu + a^\mu.$$

We add to this scale transformations

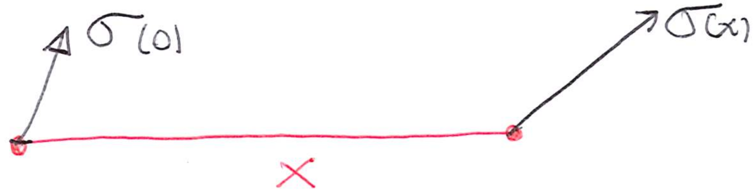
$$x^\mu \to \lambda x^\mu, \quad g_{\mu\nu} \to \lambda^{-2} g_{\mu\nu}, \quad x^\mu \to f(x^\mu) x^\mu, \qquad g_{\mu\nu} \to h(x^\mu) g_{\mu\nu}.$$

This has an impact on correlation functions (observables)

Consider a system of spins.

We ask how "in tune" a spin at $\vec{x} = 0$ is with another at $\vec{x}$.

$$< [\sigma(\vec{x}), \sigma(0)] > \sim \frac{e^{-\frac{|\vec{x}|}{\xi}}}{|\vec{x}|^{\Delta}},$$



ξ is the correlation length, as $\xi \to \infty$, we reach a CFT.

What is Supersymmetry?

A very 'odd' symmetry between bosons and fermions

Somewhat, Supersymmetry is the 'square root' of translation symmetry! Let us see some details.

To the Poincare symmetries, add a new symmetry, with Noether charges $Q_\alpha, \bar{Q}_{\dot{\alpha}}$

$$[P_\mu, Q_\alpha] = 0, \qquad [M_{\mu\nu}, Q_\alpha] = i\,(\sigma_{\mu\nu})_{\alpha\beta}\, Q^\beta, \quad (\alpha \leftrightarrow \dot{\alpha})$$

$$[Q_\alpha, Q_{\dot{\beta}}]_+ = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu.$$

$\boxed{\text{SUSY}}\ \boxed{\text{SUSY}}\ |\psi\rangle \sim \partial_\mu |\psi\rangle$

SUSY has an interesting representation theory!

Typical Lagrangians are,

$$L = (\partial\phi)^2 + i\bar{\psi}\partial\psi + |f|^2 - (\partial_\phi W)\, f - \frac{1}{2}\,(\partial_\phi^2 W)\,\bar{\psi}\psi,$$

$$\delta\phi = \bar{\epsilon}\psi, \quad \delta\psi = (i\sigma^\mu\partial_\mu\phi + f)\epsilon, \quad \delta f = i\bar{\epsilon}\sigma^\mu\partial_\mu\psi.$$

$$L = -\frac{1}{4}(F_{\mu\nu})^2 - i\bar{\lambda}\gamma^\mu D_\mu\lambda - \frac{D^2}{2}.$$

$$\delta A_\mu = i\bar{\epsilon}\sigma_\mu\lambda, \quad \delta\lambda = \sigma^{\mu\nu}F_{\mu\nu}\epsilon, \quad \delta D = \bar{\epsilon}\sigma^\mu\partial_\mu\lambda.$$

The interactions of these systems are strongly constrained!

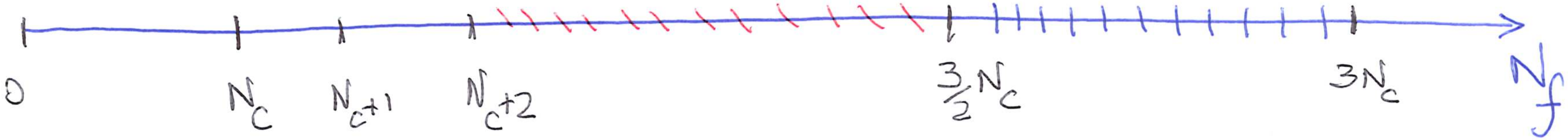
The constraint that symmetries impose on the dynamics (kinematics, interactions, etc) help us to propose dualities. In the same way, the symmetries help us test these dualities (in the absence of a proof.)

Propose other dualities (hard to prove mathematically).

Seiberg duality (1994)

$\mathcal{N}=1$ SQCD $\quad \mathcal{L} = -\frac{1}{4}F_{\mu\nu}^{2} + i\bar{\lambda}\not{D}\lambda + |D_\mu q|^2 + i\bar{\Psi}\not{D}\Psi - V(q,\tilde{q},\Psi,\tilde{\Psi},\lambda)$

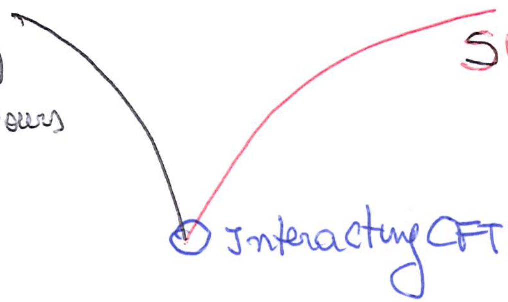
Symmetry { gauge group $SU(N_c)$
global group $SU(N_f)\times SU(N_f)\times U(1)_B \times U(1)_R$



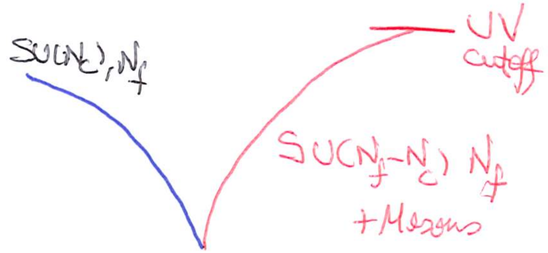
\\\ free electric phase

|||| free magnetic phase / Coulomb phase

for $N_f \in [\frac{3}{2}N_c, 3N_c]$ { $SU(N_c)$ N_f flavours



; for $N_f \in [N_c+2, \frac{3}{2}N_c]$



The 1990's brought an explosion in the number of dualities.

The generic characteristics of a duality are:

- It relates two different systems, with different degrees of freedom.
- One system is weakly coupled (perturbative, semiclassical), the dual system is generically strongly coupled (non-perturbative, highly quantum mechanical).
- Same (global) symmetries on both dual sides.

One of these dualities is particularly useful and far reaching. It was presented in 1997, by Juan Maldacena.

Today, it goes under the names of "Maldacena Conjecture" (as it is not mathematically proven!) or AdS/CFT duality.

What systems does the Maldacena Duality relate?

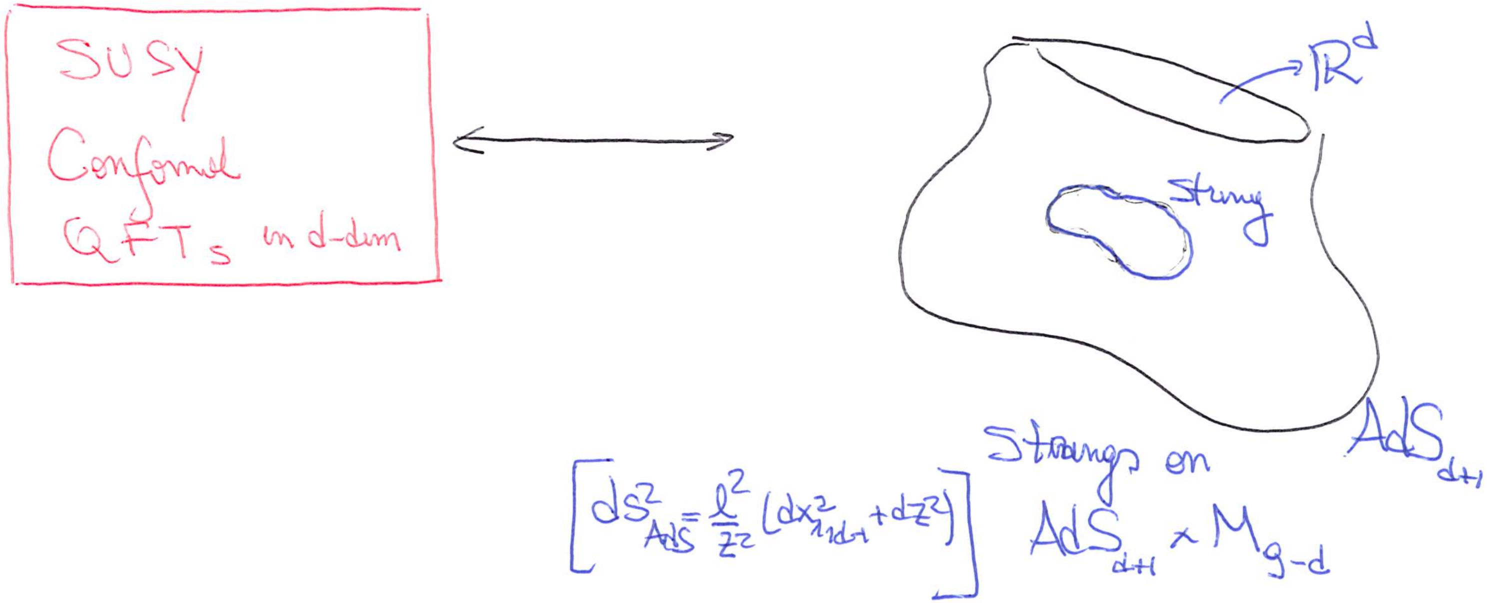
Conformal and supersymmetric quantum field theories

with

String Theory on Anti de Sitter space.

We commented what "Conformal" and "Supersymmetric" mean. I leave for another occasion to discuss what "Anti de Sitter" is. For this talk, just think about it as a particular space-time on which String theories like to propagate.

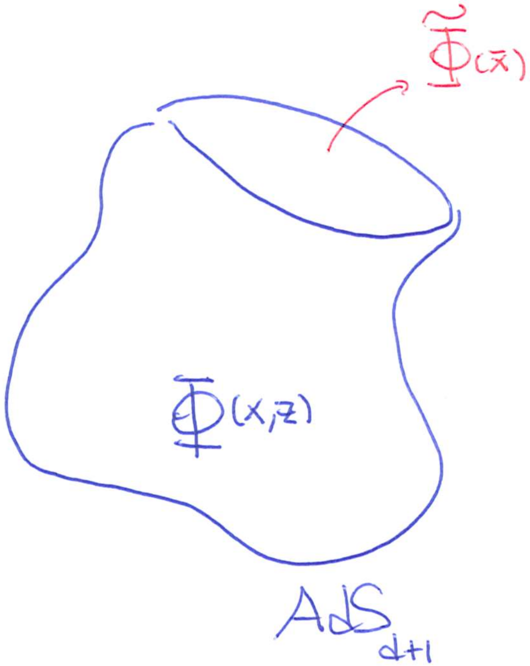
Conformality and supersymmetry → constrain the dynamics.



In slightly more precise terms, Maldacena's conjecture reads,

$$Z_{grav}[\tilde{\Phi}|_{\partial AdS}] = \int D\Phi_{string}\, e^{-S_{string}(\Phi)} = Z_{CFT}[\tilde{\Phi}].$$

Gubser-Klebanov-Polyakov, Witten (1998).

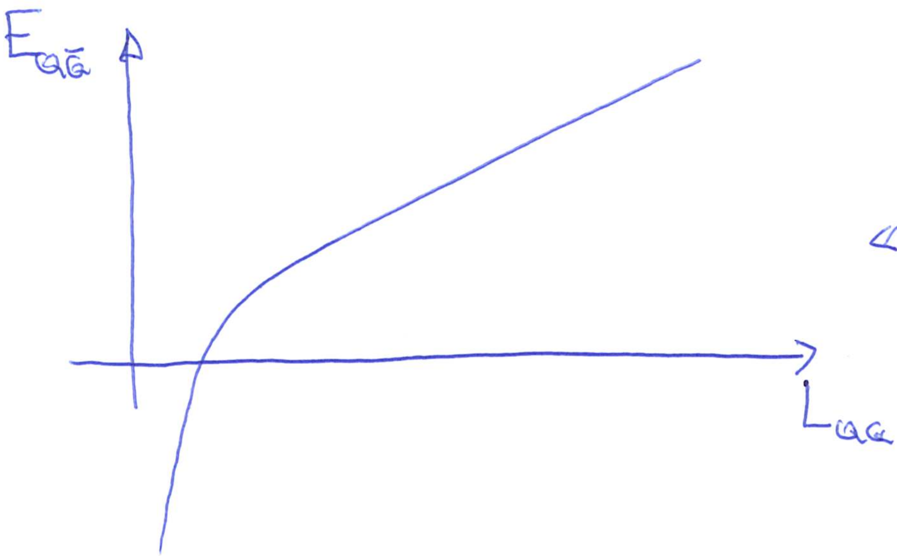


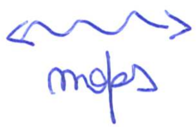
$Z_{grav} \simeq e^{-S_{gravity}[\Phi]} \longrightarrow Z_{CFT}$ at strong coupling!

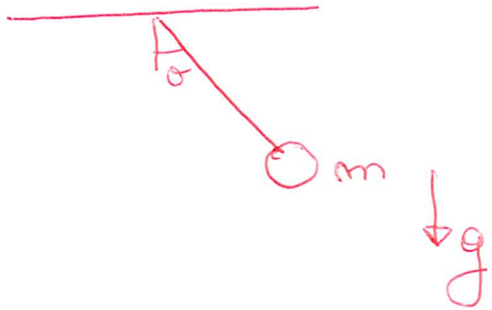
Why is this interesting?

We are mapping a very hard problem in QFT to a semiclassical calculation in a gravitational system. Allows to calculate observables in the QFT at strong coupling (highly quantum mechanical regime), using semiclassical gravitational degrees of freedom.

Achieves a geometrisation of the hard QFT problem.







More Technical. Lately, I worked on problems of this kind

How to geometrise the phenomenon of (for example) confinement?

Lots of experience gathered in the last 25 years, different models, different non-perturbative phenomena geometrised!

We learn geometry-based picture for elusive QFT-phenomena: Integrability, different vacua, dualities, etc.

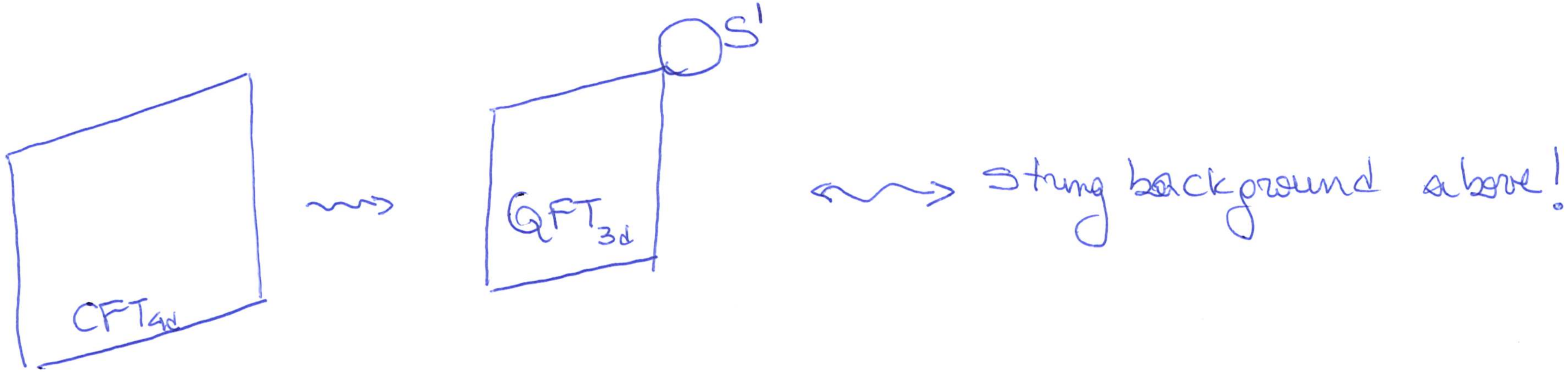
Consider a deformation of $AdS_5 \times S^5$ that two years ago was discovered by Andrés Anabalón and Simon Ross (Durham!)

$$ds_{10}^2 = \frac{\ell^2}{z^2}\left[-dt^2 + dx_1^2 + dx_2^2 + f(z)d\phi^2 + \frac{dz^2}{f(z)}\right] + \ell^2 d\tilde{\Omega}_5^2,$$

$$d\tilde{\Omega}_5^2 = d\theta^2 + \sin^2\theta d\psi^2 + \sin^2\theta \sin^2\psi\,(d\varphi_1 - A_1)^2 +$$
$$\sin^2\theta\cos^2\psi\,(d\varphi_2 - A_1)^2 + \cos^2\theta\,(d\varphi_3 - A_1)^2.$$

$$A_1 = A_\phi d\phi, \quad A_\phi = Q\left(z^2 - z_*^2\right), \qquad f(z) = 1 - \mu\ell^2 z^4 - (Q\ell\, z)^6.$$

What is going on here? Actually, it is not so complicated!



What did we recently learn using this string background?

Various lessons regarding the non-perturbative dynamics of this system and similar ones

- How confinement works here, when does screening takes place.
- Spectrum (non-perturbatively) calculated: glueballs, mesons.
- Breaking of certain global symmetries geometrised.
- Universality between many different QFTs. Some observables are 'the same' in different QFTs
- A quantity describing degrees of freedom, monotonicity.

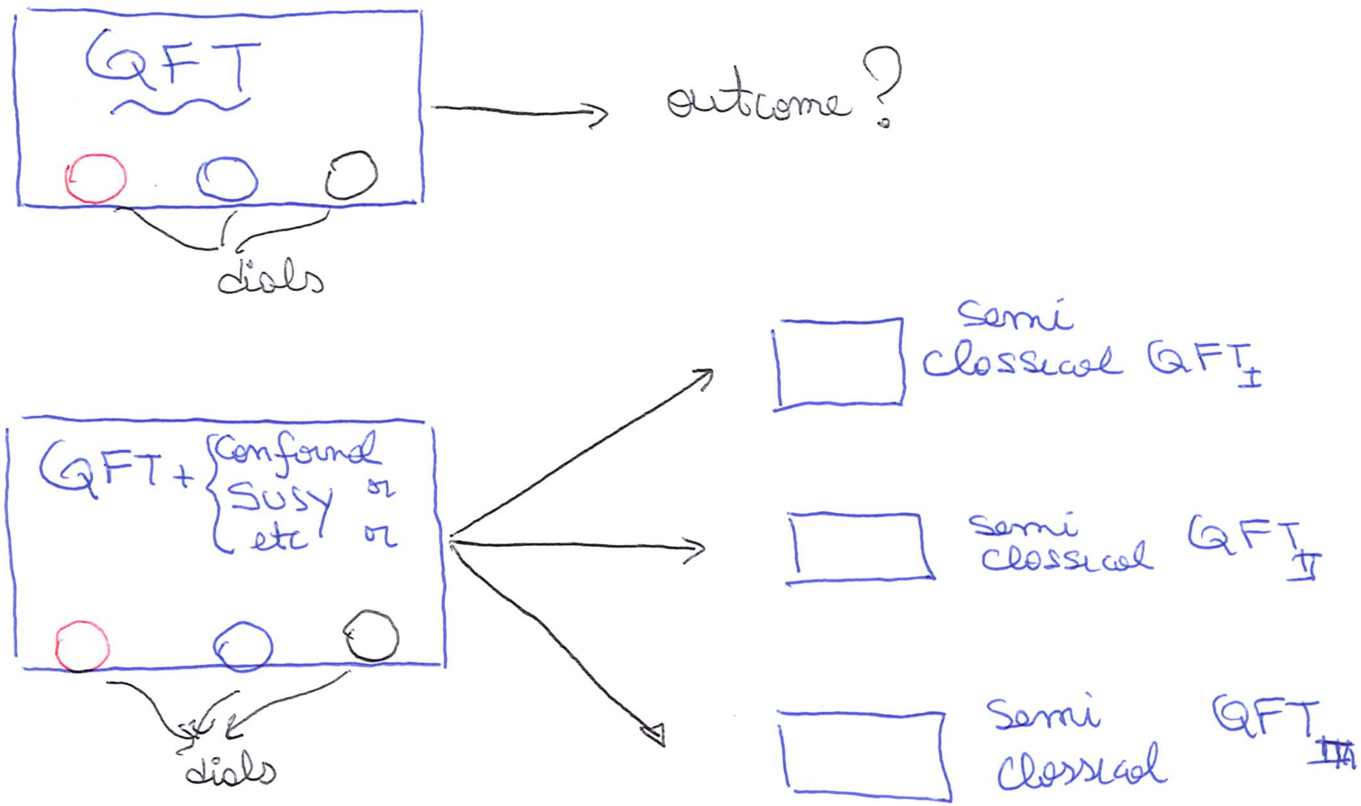
General lessons from holography.

- **A first lesson** is that a trustable background in (super) gravity defines a QFT. A theory of Strings propagating on a trustable background would be a suitable way to define a QFT, 'liberating' ourselves from the Lagrangian or Hamiltonian vision.
- **A second lesson** comes from situations where the symmetries of the problem are not that strict, like rotating and charged black holes, and map them to certain systems in condensed matter. Under certain circumstances black holes and systems of condensed matter encode the same information!
- **A third lesson**: Various phenomena in QFT, like RG-flows, symmetry breakings, confinement, global anomalies, etc, have a beautiful and surprising representation in a dual gravity theory.
- **A fourth lesson:** A suitable QFT 'defines' holographically a theory of quantum gravity!

A more generic lesson from dualities

A QFT is a very complicated system, with various 'dials' (interactions, couplings, other constants, etc). Generically very hard to understand behaviour for any value of the 'dials'.

Adding extra symmetries, we have control over various 'semi-classical limits' of the system.



To close this talk: About the idea of adding symmetries to make the problem tractable.

We solve the 'enriched by symmetries' problem, we should ask:

- Does the solution capture the essence of the original problem?
- Are we solving a qualitatively different problem?

Two Mathematics problems, to illustrate this.

First problem : calculate $\mathcal{I} = \int_{-\infty}^{\infty} e^{-x^2} dx$.

$$\mathcal{I}^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right) \times \left(\int_{-\infty}^{\infty} e^{-y^2} dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2} dxdy =$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} r\ e^{-r^2} dr = \pi \to \mathcal{I} = \sqrt{\pi}.$$

Second Problem: given (x, y, z, n) four **whole numbers**. Find a solution to $x^n + y^n = z^n$ for $n \geq 3$. Fermat (1637)→Wiles (1995).

Let x or y or z, or n to be Real numbers, infinite simple solutions: $z = (x^y + y^n)^{\frac{1}{n}}$

But the problem looses the interest!