Analytical results for the C-angularity soft function at NNLO

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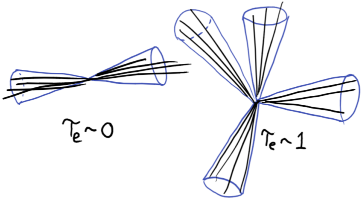
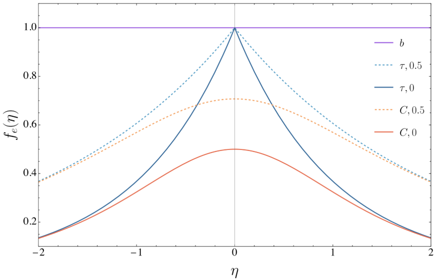
Overview

Aims

- Calculation of a soft function to NNLO in QCD
  - Piece of a cross section that describes correction due to low momentum (soft) radiation, can be combined with other pieces to get full result
  - Also gives NNLO soft anomalous dimension required for resummation
- Calculate for a new family of event shapes called "C-angularities"
  - Could be useful in extracting α_S from e^+e^- collisions
  - Includes results for C-parameter event shape
- Streamline calculation for easier N3LO calculation
  - N3LO (C-)angularity anomalous dimension last piece for N3LL resummation of (C-)angularities
  - N3LO C-parameter soft function last piece for N3LL' accuracy (Between N3LL and N4LL accuracy)

Event Shapes





- Thrust $\mathcal{T}_\tau = |p_\perp| e^{-|y|}$
  - Discontinuous derivative at $y = 0$
- C-Parameter $\mathcal{T}_C = \frac{|p_\perp|}{2\cosh(y)}$
  - Fully differentiable
- Jet broadening $\mathcal{T}_b = |p_\perp|$
- Angularities $\mathcal{T}_{\tau,a} = |p_\perp| e^{-|y|(1-a)}$
  - Varies between Thrust and Jet broadening
- C-Angularities $\mathcal{T}_{C,a} = \frac{|p_\perp|}{(2\cosh(y))^{1-a}}$
  - Varies between C-Parameter and Jet broadening

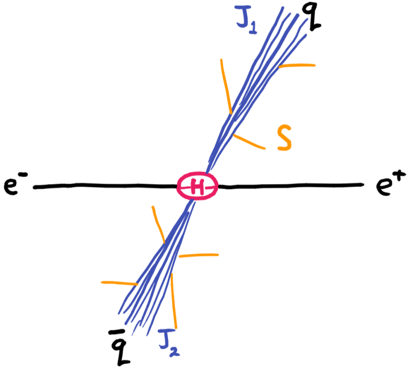
Motivation - Measuring α_s

- Event Shape distribution $\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_e}$ can be used to determine the strong coupling constant α_S
- Fit tail of dijet region to α_S and non perturbative parameter Ω_1
  - Use different values of *a* to break degeneracies between α_S and Ω_1
  - Has been done with angularities [Bell et al., 2019]
  - C-angularities - simpler calculation due to smooth behaviour
- Must re-sum large logarithms that break perturbation
  - Requires anomalous dimension

SCET Factorisation

Factorisation

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\tau_{C,a}} = H(Q,\mu) \times (S(\mathcal{T}_S,\mu) \otimes J_1(\mathcal{T}_J,\mu) \otimes J_2(\mathcal{T}_J,\mu))$$



- $S(\mathcal{T}_S,\mu)$ Soft Function
  - Soft radiation, at low energy scale $\mathcal{T}_S$
- $H(Q,\mu)$ Hard Function
  - Scattering process at high energy scale Q
- $J(\mathcal{T}_J,\mu)$ Jet Functions
  - Collinear final state radiation, at an intermediate energy scale $\mathcal{T}_J$

($B(\mathcal{T}_J,\mu)$ Beam functions)
  - Collinear initial state radiation + PDFs

Calculation Setup

- Integrate over soft amplitudes with measurement on real radiation
  - Has a complicated divergence structure when momenta become soft/collinear

Key Idea - Split soft function into two pieces

- "global" piece S^{Σ}
  - Much easier to calculate, can be extracted from kinematic limit of existing beam function results
- correction term ΔS
  - Has a much simpler divergence structure
  - can be expanded in a, coefficients can be calculated analytically

Calculation Setup - Measurement Function

Apply $\mathcal{M} = \delta(\mathcal{T}_{C,a} = \mathcal{T}_S)$ or $\theta(\mathcal{T}_{C,a} < \mathcal{T}_{\text{cut}})$ to real radiation

- Split measurement function into two pieces
- C-Angularity $\mathcal{T}_{C,a} = \sum_i \mathcal{T}_{C,a}(k_i)$
  - Correct IR safe variable
- "global" C-Angularity $\mathcal{T}^{\Sigma}_{C,a} = \mathcal{T}_{C,a}(\Sigma_i k_i)$
  - Incorrect variable (IR unsafe)...
  - ... but easier to calculate
- Splits soft function into piece using global variable plus a correction term
- $\mathcal{M}(\mathcal{T}_{\text{cut}}) = \mathcal{M}^{\Sigma}(\mathcal{T}_{\text{cut}}) + \Delta\mathcal{M}(\mathcal{T}_{\text{cut}})$, where
- $\Delta\mathcal{M}(\mathcal{T}_{\text{cut}}) = \theta(\mathcal{T}_{C,a} < \mathcal{T}_{\text{cut}}) - \theta(\mathcal{T}^{\Sigma}_{C,a} < \mathcal{T}_{\text{cut}})$

Calculation Setup - Divergences

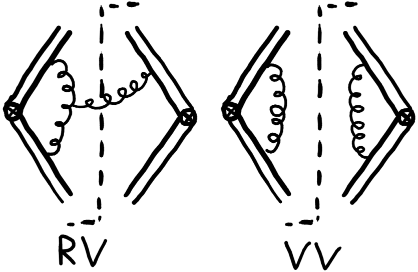
- Momenta becoming soft/collinear produce divergences - potentially complicating calculation
- S_Σ extracted from existing NNLO beam function results, so ΔS is the only complicated bit
- For most cases, $\mathcal{T}^{\Sigma}_{C,a}$ and $\mathcal{T}_{C,a}$ coincide in the IR limit $\implies$ $\Delta\mathcal{M}$ controls divergence of the amplitude

Two Exceptions!

- Each parton momentum become collinear to the opposite jet (anti-collinear)
- Total parton momenta become collinear to a jet

Calculation Setup - Amplitudes

Soft amplitudes created from two linear Wilson lines



- Amplitudes with 2 virtual loops (VV) scaleless $\implies S = 0$
- Amplitudes with 1 real emission, 1 virtual (RV): $\Delta\mathcal{M} = 0 \implies \Delta S = 0$

Calculation Setup - Amplitudes

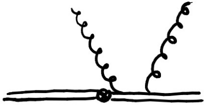


Figure 1: Uncorrelated Emission

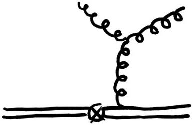


Figure 2: Correlated Emission

- Split RR diagrams into diagrams with correlated and uncorrelated emissions
- Uncorrelated emissions: emissions from different vertices, has colour factor C_R^2
    - Can be easily calculated from NLO results (non-abelian exponentiation)
- Correlated emissions: emissions from same vertex, colour factors $C_R C_A$, $C_R n_f T_F$
    - Prob. of anti-collinear emission $\to 0 \implies$ no anti-collinear divergence
    - So only one possible divergence in ΔS

Calculation Setup - Summary

- Sort amplitudes by colour factors into correlated and uncorrelated amplitudes
- $S_a^{(2,\text{uncorr.})}$ calculated from NLO result
- $S_a^{(2,\text{corr.})}$ split into $S_a^{(2,\Sigma)}$ and $\Delta S_a^{(2)}$
- $S_a^{(2,\Sigma)}$ extracted from beam function results
- Calculate $\Delta S_a^{(2)}$ for C_A and $n_f T_F$ colour factors

Calculation of $S_a^{(2,\mathrm{uncorr.})}$ and results

- Can be calculated from covolution of NLO soft function $S^{(1)}$

$$S_a^{(2,\mathrm{uncorr.})} = \frac{1}{2} S_a^{(1)} \otimes S_a^{(1)}$$

$$S_a^{(2,\mathrm{uncorr.})}(\mathcal{T}_{\mathrm{cut}}, \mu) = \frac{4C_R^2}{(1-a)^2} \left[2\log^4\left(\frac{\mathcal{T}^{\mathrm{cut}}}{\mu}\right) + \frac{(-2a^2+4a-11)\pi^2}{6} \log^2\left(\frac{\mathcal{T}^{\mathrm{cut}}}{\mu}\right) \right.$$
$$\left. + 16\zeta(3)\log\left(\frac{\mathcal{T}^{\mathrm{cut}}}{\mu}\right) + \frac{\pi^4\left(20a^4 - 80a^3 + 140a^2 - 120a + 13\right)}{1440} \right]$$

Calculation of $S_a^{(2,\Sigma)}$

- Can be calculated from existing NNLO beam function results [Baranowski, 2020]
  - Beam functions depend on the momentum fraction z
  - In the limit of $z \to 1$, the beam function produces a structure similar to the global variable soft function
  - Specifically, S^Σ can be written as

$$S^\Sigma = f(\epsilon) \int dr^+ dr^- (r^+ r^-)^{1-2\epsilon} \Theta(\mathcal{T}_{C,a}(r) < \mathcal{T}_{\text{cut}}) \tag{1}$$

where $f(\epsilon)$ is extracted from beam function results, and the integral over the total momentum $r^\pm = k_1^\pm + k_2^\pm$ is the NLO integral

$S_a^{(2,\Sigma)}$ results

$$
\begin{aligned}
S_a^{\Sigma} = C_R \Bigg\{ & C_A \Bigg[-\frac{18\pi^4\left(2a^2-4a+1\right)-201\pi^2\left(4a^2-8a+7\right)}{162(1-a)} \\
& +\frac{4\left(-1188a^3\zeta_3+3564a^2\zeta_3-3564a\zeta_3+693\zeta_3+1214\right)}{162(1-a)}\Bigg] \\
& -\frac{4n_f T_F\left[15\pi^2\left(4a^2-8a+7\right)+4\left(108a^3\zeta_3-324a^2\zeta_3+324a\zeta_3-63\zeta_3-82\right)\right]}{162(1-a)} \\
& -\frac{4\left(C_A\left(33\pi^2\left(a^2-2a+2\right)+378\zeta_3-404\right)-4\left(3\pi^2\left(a^2-2a+2\right)-28\right)n_f T_F\right)}{27(1-a)}\log\left(\frac{\mathcal{T}^{\text{cut}}}{\mu}\right) \\
& \frac{8\left(\left(3\pi^2-67\right)C_A+20n_f T_F\right)}{9(1-a)}\log^2\left(\frac{\mathcal{T}^{\text{cut}}}{\mu}\right) \\
& +\frac{16(11C_A-4n_f T_F)}{9(1-a)}\log^3\left(\frac{\mathcal{T}^{\text{cut}}}{\mu}\right)\Bigg\}
\end{aligned} \tag{2}
$$

Calculation of $\Delta S_a^{(2)}$

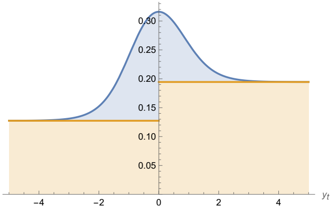


Figure 3: $C_F n_F T_F$ integrand y_t dependence. Evaluated at $\Delta y = 2$, $z = 0.5$, $a = 0.5$. Asymptotic behaviour given by orange line

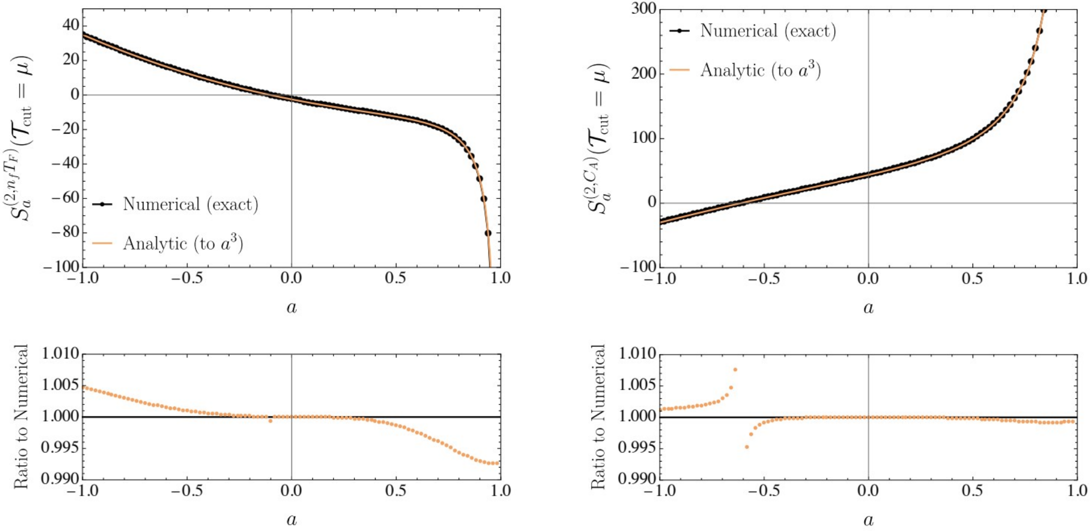
- For both C_A and $n_f T_F$ colour channels:
- Start with 8 - 2 δ functions = 6 variables
- Change variables to $\cos\Delta\phi$, T, z, y_t, Δy (plus trivial ϕ_t integral)
  - The $\cos\Delta\phi$ and T integrals are simple to integrate over
  - This leaves integrals over z, Δy and y_t
- We expect a divergence as $y_t \to \pm\infty$
  - Subtract off $y_t \to \pm\infty$ asymptote (orange line), and split integrand into regularised piece $I_{\text{reg.}}$ (blue) and divergent piece $I_{\text{div.}}$ (orange)
- Finally, expand $I_{\text{div.}}$ to $\mathcal{O}(\epsilon)$ and expand both in a to desired accuracy, and perform remaining integrals

$\Delta S_a^{(2)}$ results

$I^{c,n,m}$	$c = C_R n_f T_F$	$c = C_R C_A$
$I_{\text{div.}}^{c,0,1}$	$\frac{10}{3}$	$-\frac{41}{3} + \frac{4}{3}\pi^2 + 8\zeta_3$
$I_{\text{div.}}^{c,0,2}$	$\frac{56}{45} - \frac{12}{5}\zeta_3$	$-\frac{118}{45} + \frac{1}{3}\pi^2 - \frac{24}{5}\zeta_3$
$I_{\text{div.}}^{c,0,3}$	$\frac{11}{54} - \frac{2}{9}\zeta_3$	$-\frac{65}{108} + \frac{1}{9}\pi^2 - \frac{8}{9}\zeta_3 + \frac{1}{90}\pi^4$
$I_{\text{div.}}^{c,0,4}$	$\frac{13}{210} - \frac{2}{15}\zeta_3 + \frac{1}{7}\zeta_5$	$-\frac{19}{105} + \frac{1}{24}\pi^2 - \frac{13}{30}\zeta_3 + \frac{1}{120}\pi^4 - \frac{4}{7}\zeta_5$
$I_{\text{div.}}^{c,1,1}$	$\frac{239}{9} - \frac{32}{3}\zeta_3$	$-\frac{1793}{18} + \frac{16}{3}\pi^2 + \frac{160}{3}\zeta_3 + \frac{4}{9}\pi^4$
$I_{\text{div.}}^{c,1,2}$	$\frac{2162}{225} - \frac{2}{9}\pi^2 - \frac{212}{75}\zeta_3 - \frac{1}{9}\pi^4$	$-\frac{2741}{225} + \frac{4}{9}\pi^2 + \frac{2116}{75}\zeta_3 - \frac{26}{45}\pi^4$
$I_{\text{div.}}^{c,1,3}$	$\frac{467}{1620} - \frac{1}{27}\pi^2 + \frac{202}{135}\zeta_3 - \frac{7}{405}\pi^4$	$-\frac{4187}{3240} + \frac{559}{135}\zeta_3 - \frac{31}{405}\pi^4 + \frac{2}{9}\pi^2\zeta_3 + \zeta_5$
$I_{\text{reg.}}^{c,0,0}$	$-\frac{8}{3} + \frac{16}{3}\zeta_3$	$\frac{4}{3} - \frac{44}{3}\zeta_3 + \frac{4}{15}\pi^4$
$I_{\text{reg.}}^{c,0,1}$	$\frac{25}{6} - \frac{2}{3}\pi^2 - \frac{8}{3}\zeta_3$	$-\frac{97}{12} + \frac{1}{3}\pi^2 - \frac{34}{3}\zeta_3 - \frac{11}{45}\pi^4$
$I_{\text{reg.}}^{c,0,2}$	$\frac{76}{135} - \frac{2}{9}\pi^2 + \frac{8}{225}\pi^4$	$-\frac{1021}{270} + \frac{7}{36}\pi^2 + 3\zeta_3 + \frac{13}{300}\pi^4$
$(*)\ I_{\text{reg.}}^{c,0,3}$	$\frac{161}{1296} - \frac{1}{27}\pi^2 - \frac{11}{54}\zeta_3 + \frac{1}{162}\pi^4$	$-\frac{5885}{2592} + \frac{5}{36}\pi^2 + \frac{127}{54}\zeta_3 - \frac{59}{3240}\pi^4 - \frac{1}{9}\pi^2\zeta_3 + \frac{1}{3}\zeta_5$

$$\Delta S_a^{(2,c)}(\mathcal{T}_{\text{cut}}, \mu) = -\frac{2I_{\text{div.}}^{c,0}}{(1-a)} \log\left(\frac{\mathcal{T}_{\text{cut}}}{\mu}\right) + \frac{I_{\text{div.}}^{c,1}}{2(1-a)} + 2I_{\text{reg.}}^{c,0} + \mathcal{O}(\epsilon)$$

Full results



Comparisons to Literature Results - Anomalous dimension

- Soft anomalous dimension γ_s - describes how a soft function changes with scale

$$\mu \frac{\mathrm{d}S}{\mathrm{d}\mu} = \gamma_s \otimes S$$

- Can be used to partially check $\mathcal{T}_{C,a}$ results
- Slicing variables with the same forward rapidity limit (e.g. Thrust and C-parameter) have the same anomalous dimension
- Angularities $\mathcal{T}_{\tau,a}$ coincides with $\mathcal{T}_{C,a}$ for forward rapidities, so can use the angularities anomalous dimension to check results.
- Check against [Bauer et al., 2021], all terms agree except the $C_A\ a^2$ - Tiny disagreement (0.2 per mille, so phenomenologically irrelevant)
    - We are in contact with the authors to hopefully solve the disagreement

Conclusion - N3LO Calculation

- Method can be used for a N3LO calculation
- Global piece can be found from limit of N3LO beam function results
- At N3LO, there are:
  - RRR, RRV, RVV and VVV diagrams
  - RVV and VVV $\rightarrow 0$ for correction piece
  - RRR can be split into CCC, CCU and UUU for correlated (C) and uncorrelated (U) emissions.
    - CCU and UUU simple to calculate like the NNLO uncorrelated piece
- Counter-terms would be necessary between RRR and RRV
- The correction piece diverges like a NLO calculation, so can modify a NLO subtraction technique

Thank you

References

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