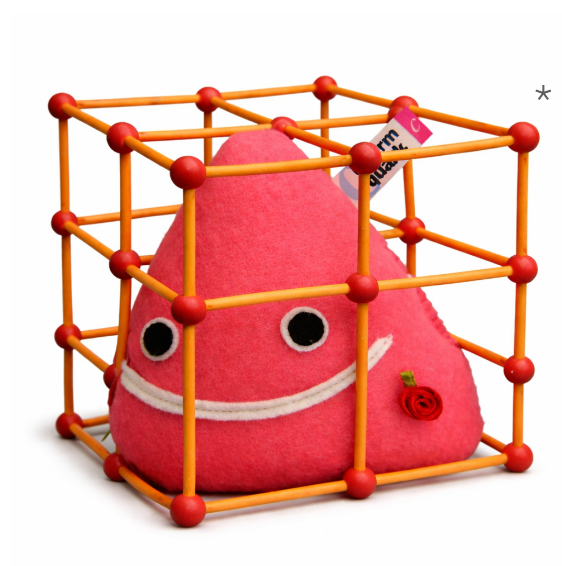
11 February 2026

Higgs Maxwell Workshop, Edinburgh

Charm quark mass from lattice QCD



Rajnandini Mukherjee & Matthew Black

THE UNIVERSITY *of* EDINBURGH

* Charm quark plushie by *The Particle Zoo*

Charm physics from lattice QCD

- Standard model parameters and indirect searches for New Physics:
  CKM matrix elements, quark masses, α_s, …
- Lattice QCD: first-principles based, systematically improvable
  Currently in an era of precision charm physics!

Charm physics from lattice QCD

- Standard model parameters and indirect searches for New Physics: CKM matrix elements, quark masses, α_s, …
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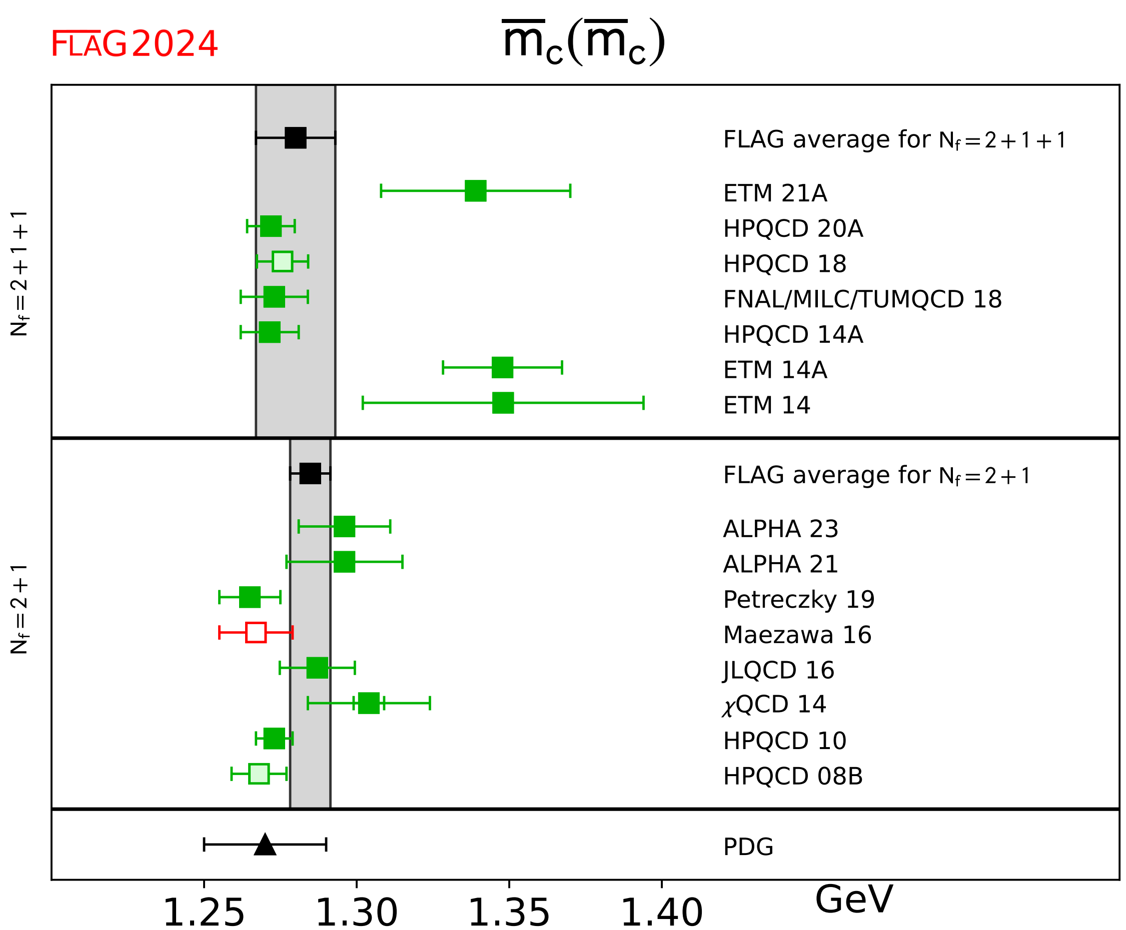
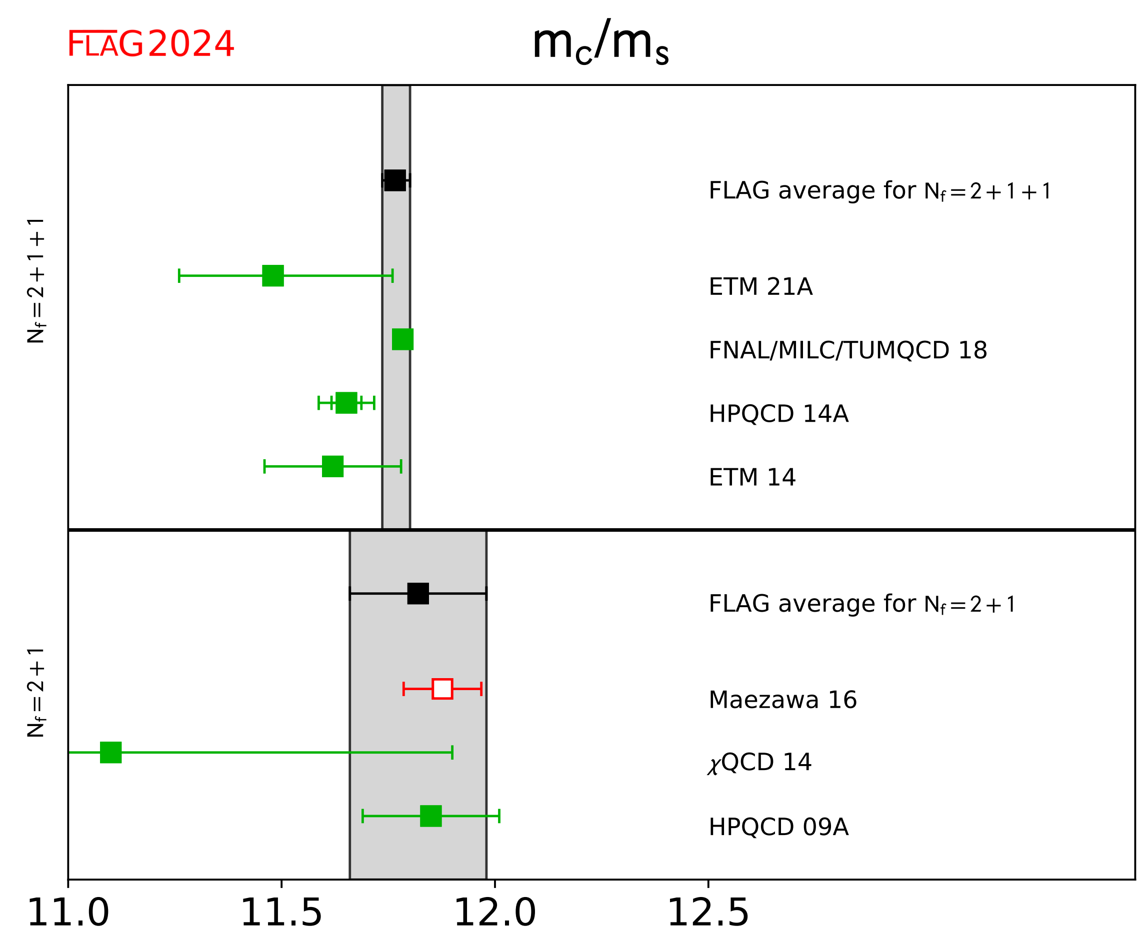
LQCD determinations: [FLAG 2024, arXiv:2411.04268]

SM parameter	exp/process	lattice QCD	precision (%)
$\lvert V_{cd} \rvert$	$D \to l\nu$	f_D	~ 0.3
	$D \to \pi l\nu$	$f_{+,0}^{D\pi}(0)$	~ 0.8
$\lvert V_{cs} \rvert$	$D_s \to l\nu$	f_{D_s}	~ 0.2
	$D \to K l\nu$	$f_+^{DK}(0)$	~ 0.4
$\lvert V_{cb} \rvert$	$B \to D^{(*)} l\nu$	$\mathcal{F}_{B\to D^{(*)}}$	~ 1.1
$m_c^{\overline{\mathrm{MS}}}$	$J/\psi,\, \eta_c$	D, D_s, η_c	~ 1.1
m_c/m_s	-	D, D_s, η_c	~ 0.3

Charm physics from lattice QCD

quark masses: $m_c^{\overline{\rm MS}}$ and m_c/m_s [FLAG 2024, arXiv:2411.04268]





$N_f = 2+1+1$: $m_c^{\overline{\rm MS}}(m_c^{\overline{\rm MS}}) = 1.280(13)$ GeV $[1.1\%]$, $m_c^{\rm RGI} = 1.528(15)_m(21)_\Lambda$ GeV $[1.7\%]$, $m_c/m_s = 11.766(30)$ $[0.3\%]$

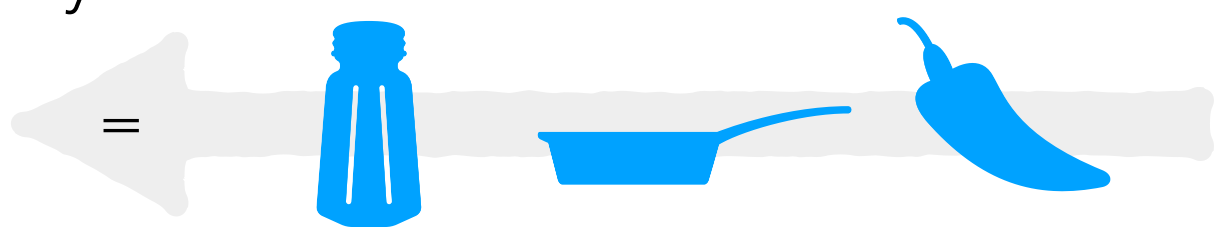
$N_f = 2+1$: $m_c^{\overline{\rm MS}}(m_c^{\overline{\rm MS}}) = 1.278(6)$ GeV $[0.5\%]$, $m_c^{\rm RGI} = 1.526(7)_m(21)_\Lambda$ GeV $[1.4\%]$, $m_c/m_s = 11.820(160)$ $[1.4\%]$

$\Lambda_{N_f=4} = 297(12)$ MeV

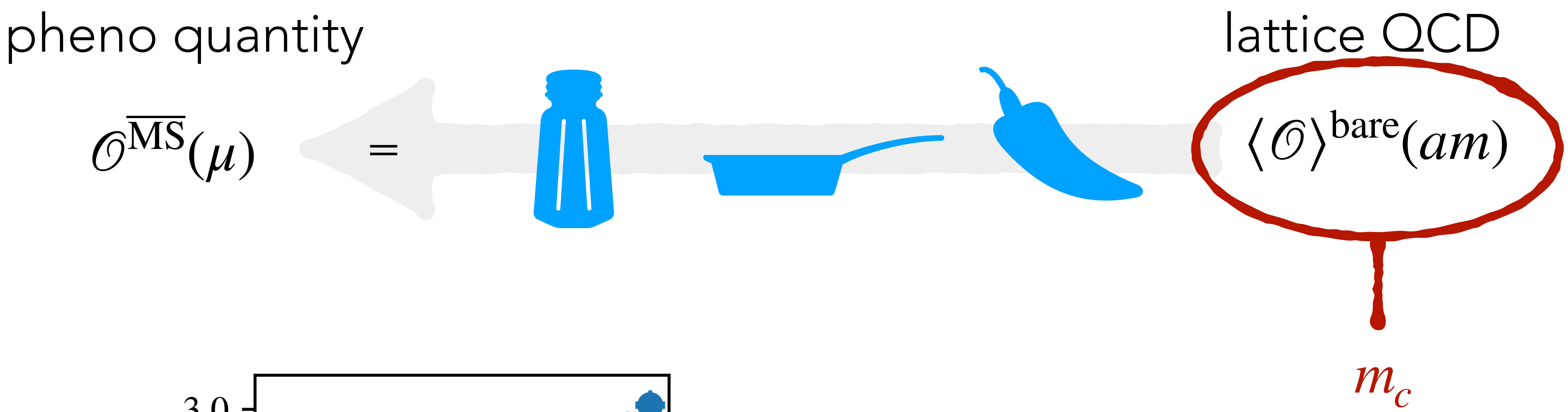
Charm mass determination

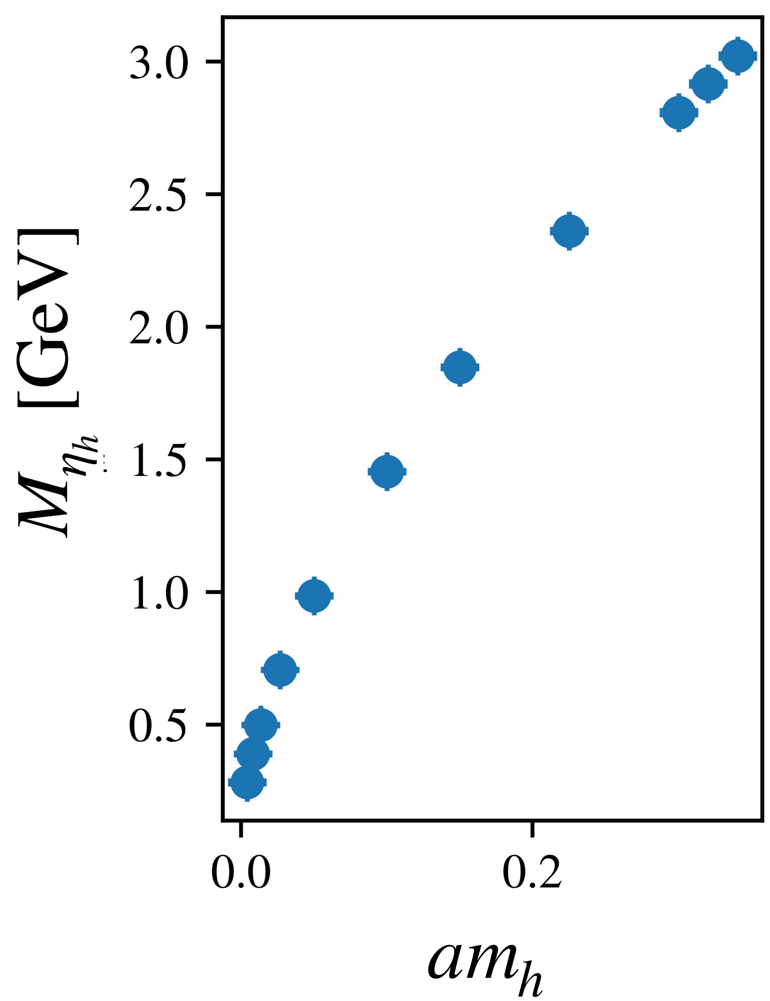
pheno quantity | lattice QCD

$\mathscr{O}^{\overline{\mathrm{MS}}}(\mu) \quad = \quad \langle \mathscr{O} \rangle^{\mathrm{bare}}(am)$



Charm mass determination





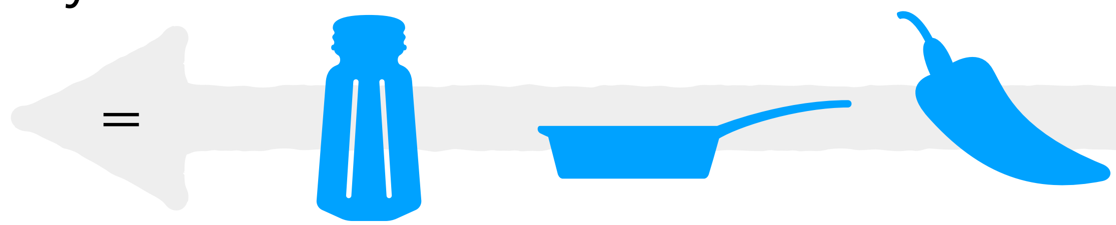
- Measure meson mass vs input bare quark mass

Charm mass determination

pheno quantity

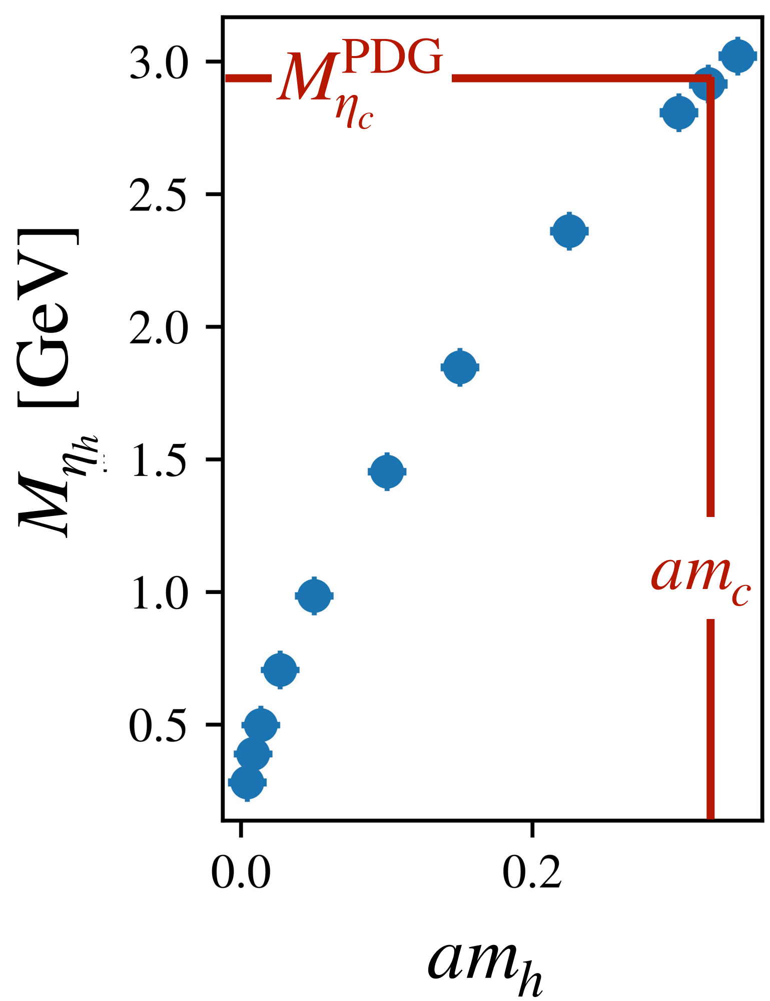
lattice QCD

$\mathcal{O}^{\overline{\mathrm{MS}}}(\mu)$ =



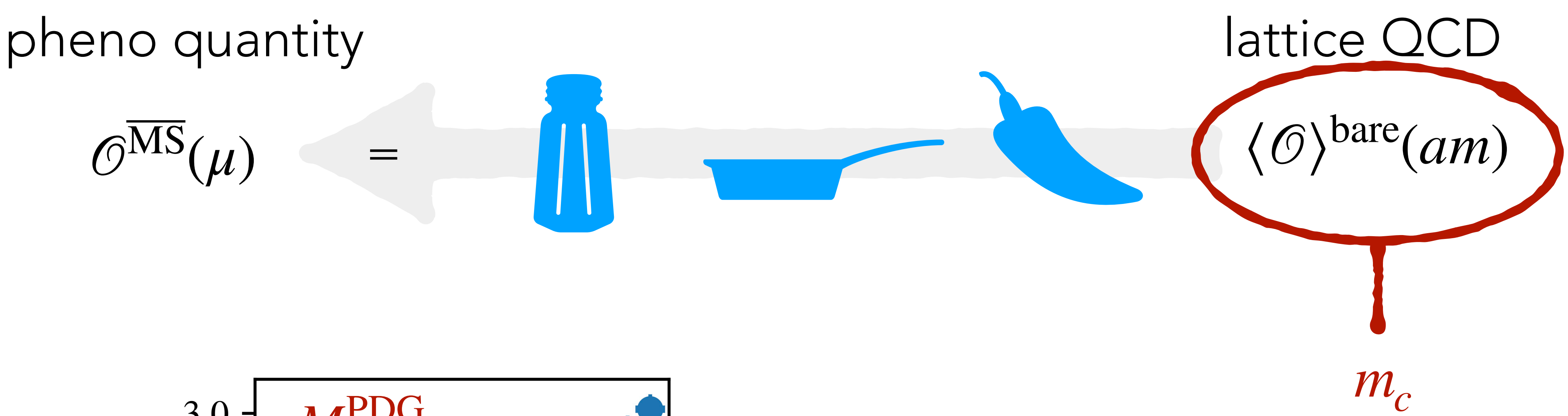
$\langle \mathcal{O} \rangle^{\mathrm{bare}}(am)$

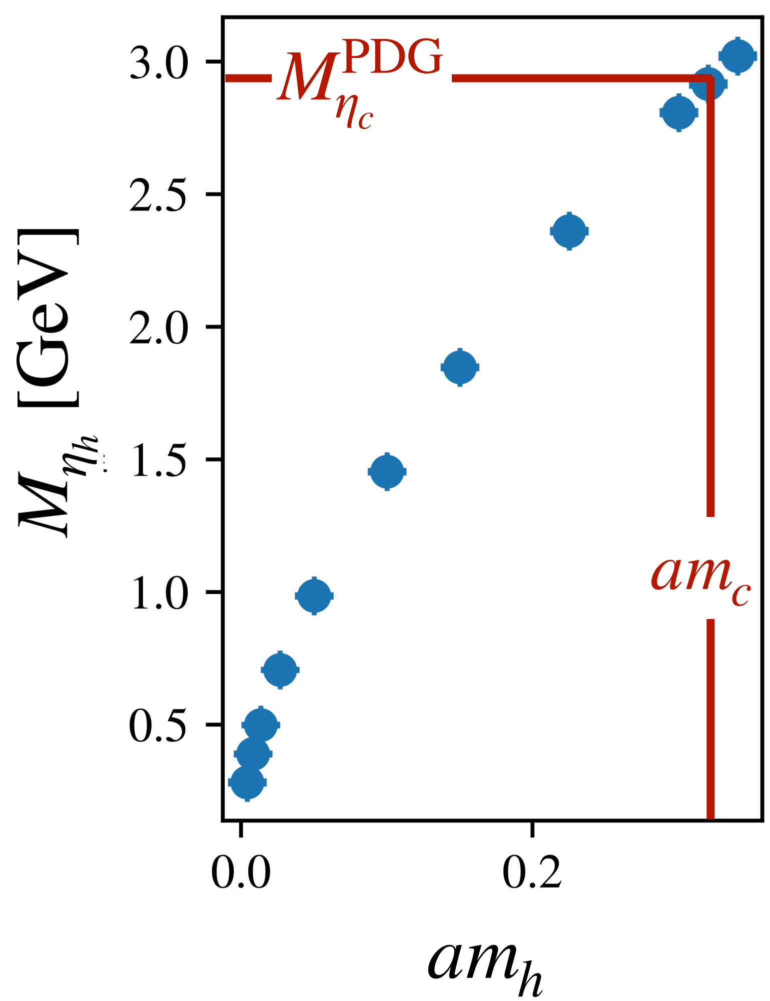
m_c



- Measure meson mass vs input bare quark mass
- Tune m_c to match exp. value of meson mass

Charm mass determination

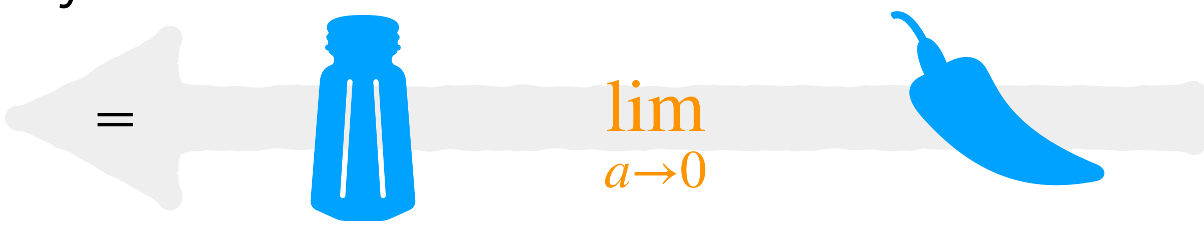




- Measure meson mass vs input bare quark mass
- Tune m_c to match exp. value of meson mass
- Challenge: simulating am_h to reach $M_{\eta_c}^{\mathrm{PDG}}$
  → fermion action, lattice spacing

Charm mass determination

pheno quantity | **continuum limit** | lattice QCD



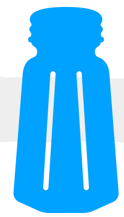
$$\mathcal{O}^{\overline{\mathrm{MS}}}(\mu) \quad = \quad \lim_{a \to 0} \quad \langle \mathcal{O} \rangle^{\mathrm{bare}}(am)$$

Charm mass determination

pheno quantity

continuum limit

lattice QCD

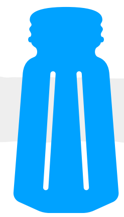


$$\mathcal{O}^{\overline{\mathrm{MS}}}(\mu) = \lim_{a\to 0} Z^{S}_{\mathcal{O}}(am, a\mu)\, \langle \mathcal{O} \rangle^{\mathrm{bare}}(am)$$

renormalisation

Charm mass determination

pheno quantity | continuum limit | lattice QCD



$$\mathcal{O}^{\overline{\mathrm{MS}}}(\mu) = \lim_{a\to 0} Z^S_{\mathcal{O}}(am, a\mu) \langle \mathcal{O} \rangle^{\mathrm{bare}}(am)$$

renormalisation

perturbative

- Lattice perturbation theory [Capitani Phys. Rept. 382 (2003) 113]

non-perturbative

- Rome-Southampton method [Martinelli et al NPB 445 (1995)]

  regularisation-independent (RI) momentum-subtraction (MOM) scheme

- Variations: RI/MOM', **RI/SMOM**, RI/IMOM…[Sturm et al PRD 80 (2009), Garron et al PRD 108 (2023)]

- Other NPR schemes: [Lüscher et al NPB 384 (1992), Sint NPB 421 (1994), Tomii et al PRD 94 (2016)] …

  Gradient flow (Matthew's talk!)

Charm mass determination

pheno quantity | continuum limit | lattice QCD

$$\mathcal{O}^{\overline{\mathrm{MS}}}(\mu) = R_{\mathcal{O}}^{\overline{\mathrm{MS}}\leftarrow S}(\mu) \lim_{a\to 0} Z_{\mathcal{O}}^{S}(am, a\mu) \langle \mathcal{O} \rangle^{\mathrm{bare}}(am)$$

matching | renormalisation

Charm mass determination

pheno quantity | continuum limit | lattice QCD

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matching | renormalisation

- Challenge: PT convergence
  → higher-loop matching

Charm mass determination

pheno quantity | continuum limit | lattice QCD

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matching | renormalisation

- Challenge: heavy quarks suffer large discretisation effects

Charm mass determination

pheno quantity

continuum limit

lattice QCD

$$\mathcal{O}^{\overline{\mathrm{MS}}}(\mu) \quad = \quad R_{\mathcal{O}}^{\overline{\mathrm{MS}}\leftarrow S}(\mu) \lim_{a\to 0} Z_{\mathcal{O}}^{S}(am, a\mu) \quad \langle \mathcal{O} \rangle^{\mathrm{bare}}(am)$$

matching

renormalisation

- Challenge: heavy quarks suffer large discretisation effects

$$\langle \mathcal{O} \rangle^{S}_{\mathrm{lat}}(am, a\mu) = Z^{S}_{\mathcal{O}}(a\mu)\, \langle \mathcal{O} \rangle^{\mathrm{bare}}_{\mathrm{lat}}(am) = \langle \mathcal{O} \rangle^{S}_{\mathrm{cont}}(m, \mu)\left[1 + \hat{\delta}\left(am, a\mu\right)\right]$$

lattice artefacts

Charm mass determination

pheno quantity | continuum limit | lattice QCD

$$\mathcal{O}^{\overline{\mathrm{MS}}}(\mu) = R_{\mathcal{O}}^{\overline{\mathrm{MS}}\leftarrow S}(\mu) \lim_{a\to 0} Z_{\mathcal{O}}^{S}(am, a\mu) \langle\mathcal{O}\rangle^{\mathrm{bare}}(am)$$

matching | renormalisation

- Challenge: heavy quarks suffer large discretisation effects

what we want	what we need	what we have
$\lim_{a\to 0} \hat{\delta} \leq O(a^2)$	$m \ll \mu \ll a^{-1}$	$m_c \approx a^{-1}$

$$\langle\mathcal{O}\rangle_{\mathrm{lat}}^{S}(am, a\mu) = Z_{\mathcal{O}}^{S}(a\mu)\,\langle\mathcal{O}\rangle_{\mathrm{lat}}^{\mathrm{bare}}(am) = \langle\mathcal{O}\rangle_{\mathrm{cont}}^{S}(m,\mu)\left[1 + \hat{\delta}\left(am, a\mu\right)\right]$$

lattice artefacts

Charm mass determination

pheno quantity | continuum limit | lattice QCD

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matching | renormalisation

- Challenge: heavy quarks suffer large discretisation effects

what we want	what we need	what we have
$\lim_{a\to 0} \hat{\delta} \leq O(a^2)$	$m \ll \mu \ll a^{-1}$	$a = 0.12$ fm $\rightsquigarrow am_c^{\mathrm{RGI}} \approx 0.9$ $a = 0.04$ fm $\rightsquigarrow am_c^{\mathrm{RGI}} \approx 0.3$

Charm mass determination

pheno quantity — continuum limit — lattice QCD

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matching — renormalisation

- Challenge: heavy quarks suffer large discretisation effects

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- Typically *massless* NPR schemes are used: $Z_{\mathcal{O}}^{S}(a\mu)$ defined for $am \ll 1$

Charm mass determination

pheno quantity | continuum limit | lattice QCD

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- Typically *massless* NPR schemes are used: $Z_{\mathcal{O}}^{S}(a\mu)$ defined for $am \ll 1$

STRATEGY: define a scheme S where $Z_{\mathcal{O}}^{S}$ absorbs $\hat{\delta}$ in $\lim a \to 0$

A massive NPR scheme for heavy quark observables

Phys.Rev.D 110 (2024)

L Del Debbio, F Erben, J Flynn, **RM**, J T Tsang

University of Southampton

THE UNIVERSITY of EDINBURGH

RI/mSMOM [Boyle et al PRD 95 (2017)]

- NPR scheme for fermion bilinears
  - ✓ Massive ward identities satisfied
  - ✓ Z-factors in continuum limit similar to $\overline{\mathrm{MS}}$
  - ✓ Valid beyond the regime $am \ll 1$

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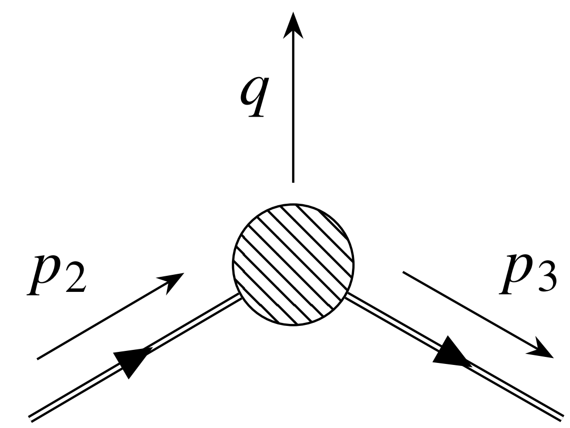

- Renormalisation conditions imposed at some finite mass

RI/SMOM: $\lim_{m_R \to 0} Z_{\mathcal{O}} \langle p_3 | \mathcal{O}^{\text{bare}} | p_2 \rangle_{q^2=\mu^2} \equiv \langle p_3 | \mathcal{O}_{\text{tree}} | p_2 \rangle,$ $\qquad Z_{\mathcal{O}} = Z_{\mathcal{O}}(a\mu)$

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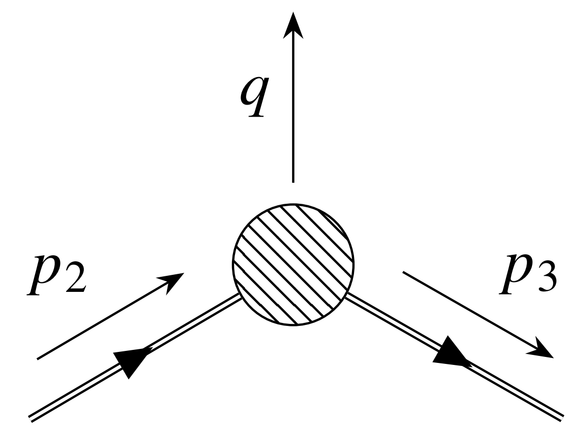

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RI/mSMOM: $\lim_{m_R \to \overline{m}_R} Z_{\mathcal{O}} \langle p_3 | \mathcal{O}^{\text{bare}} | p_2 \rangle_{q^2=\mu^2} \equiv \langle p_3 | \mathcal{O}_{\text{tree}} | p_2 \rangle,$ $\quad Z_{\mathcal{O}} = Z_{\mathcal{O}}(a\mu, a\overline{m})$

◉ Can **tune** $\overline{m}_R$ to absorb discretisation effects in $Z_{\mathcal{O}}\langle \mathcal{O} \rangle$

Numerical implementation
renormalised charm quark mass

- In RI/mSMOM scheme

$$m_{c,R}(\mu, \overline{m}_R) = \lim_{a \to 0} Z_m(a\mu, a\overline{m})\, m_c$$

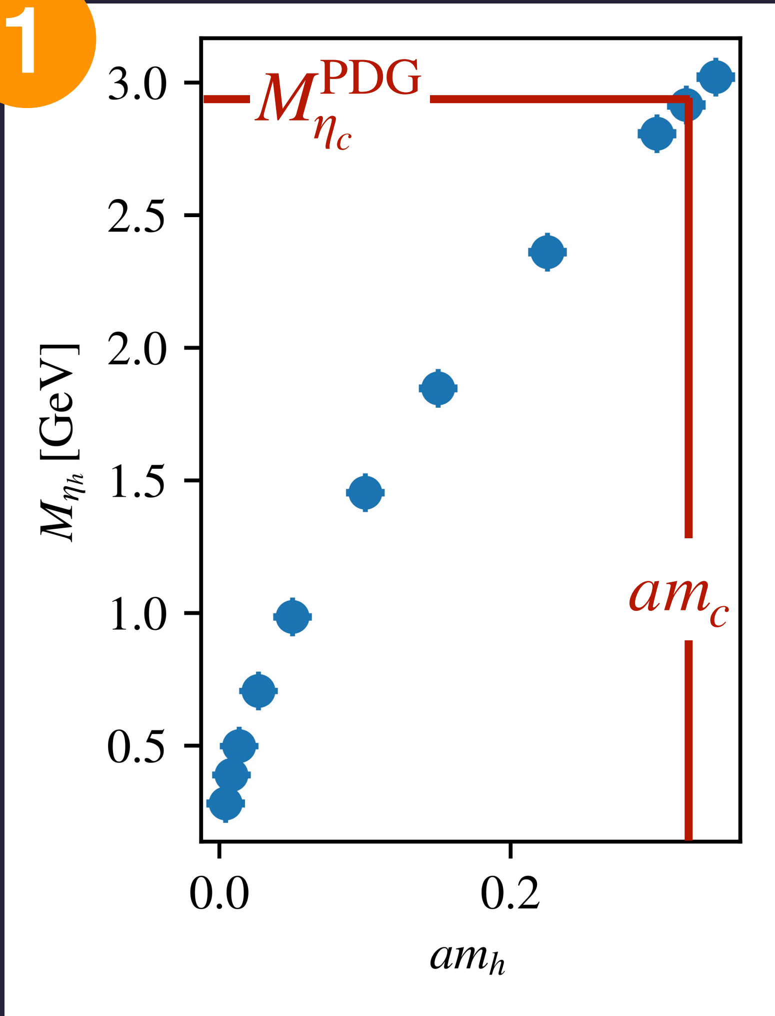
- Using 6 RBC/UKQCD domain wall fermion ensembles
  3 lattice spacings $a = 0.11 - 0.08$ fm [PRD84(2011), PRD93(2016), PRD110(2024)]

Numerical implementation
renormalised charm quark mass

- In RI/mSMOM scheme

$$m_{c,R}(\mu, \overline{m}_R) = \lim_{a \to 0} Z_m(a\mu, a\overline{m})\, m_c$$

1



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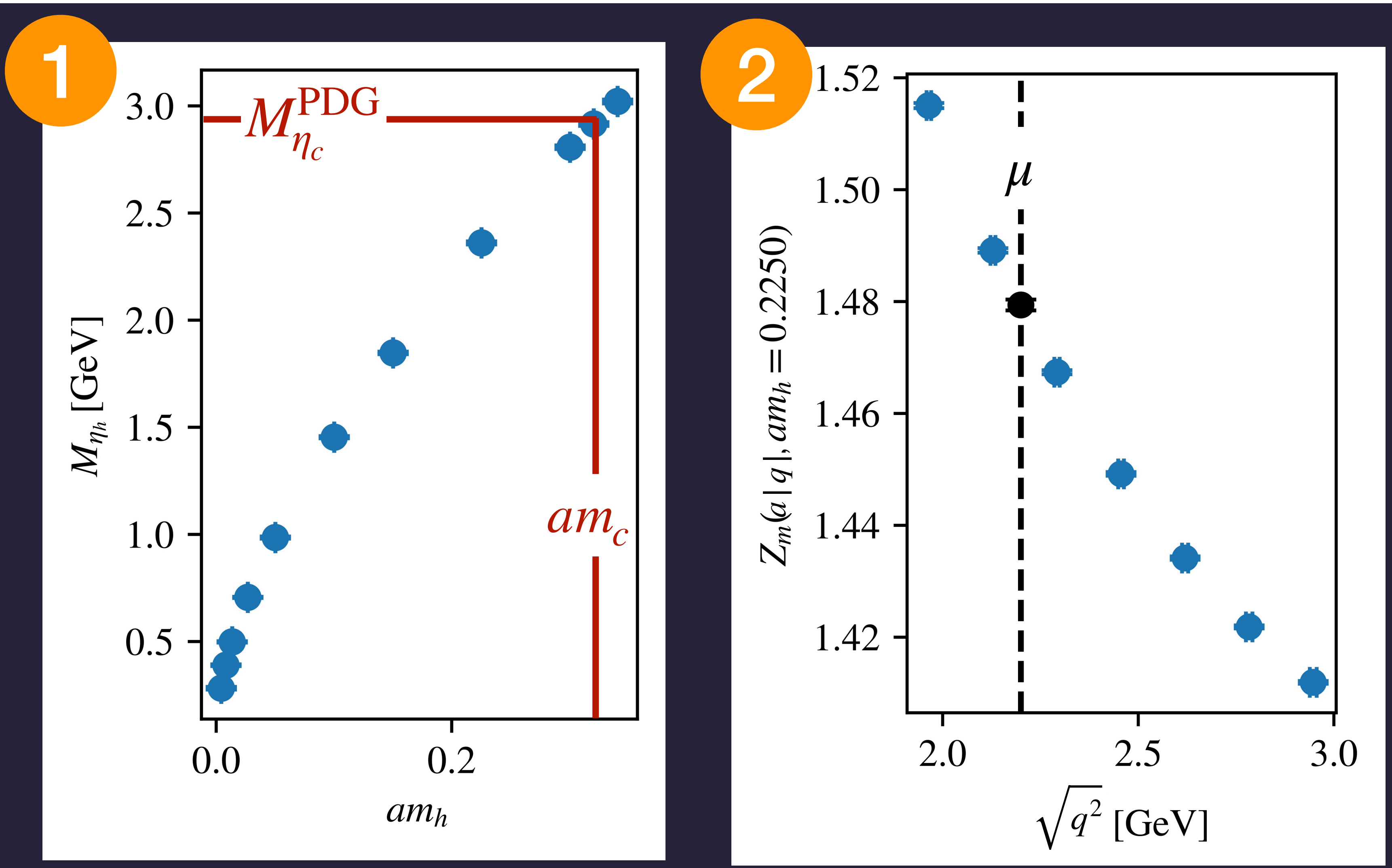

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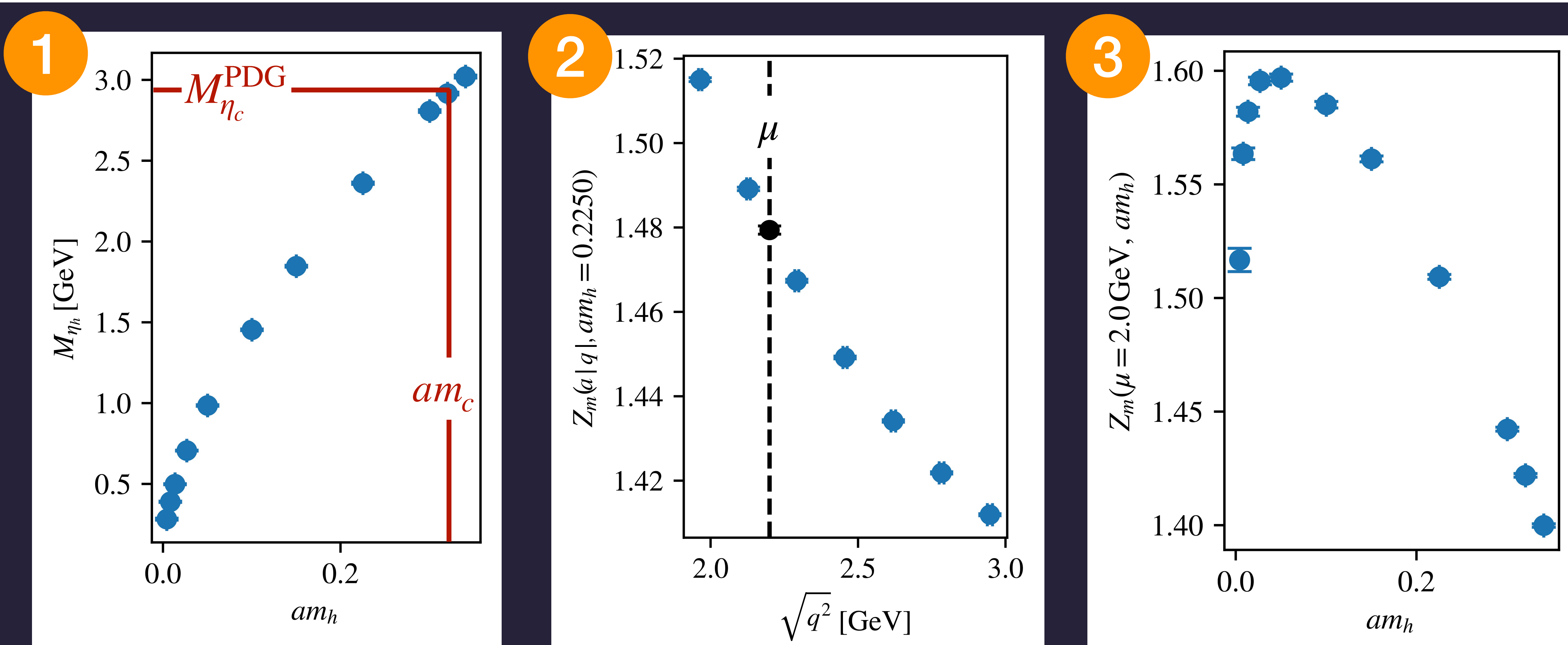

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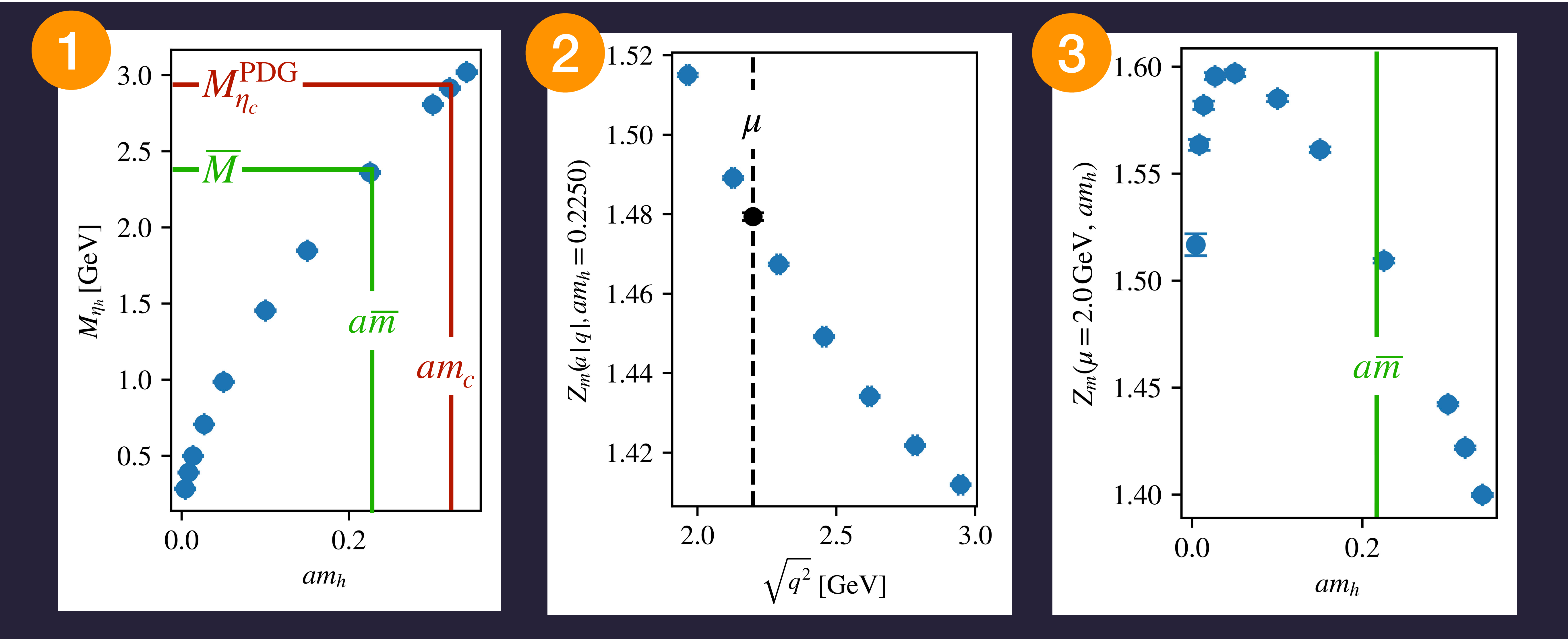
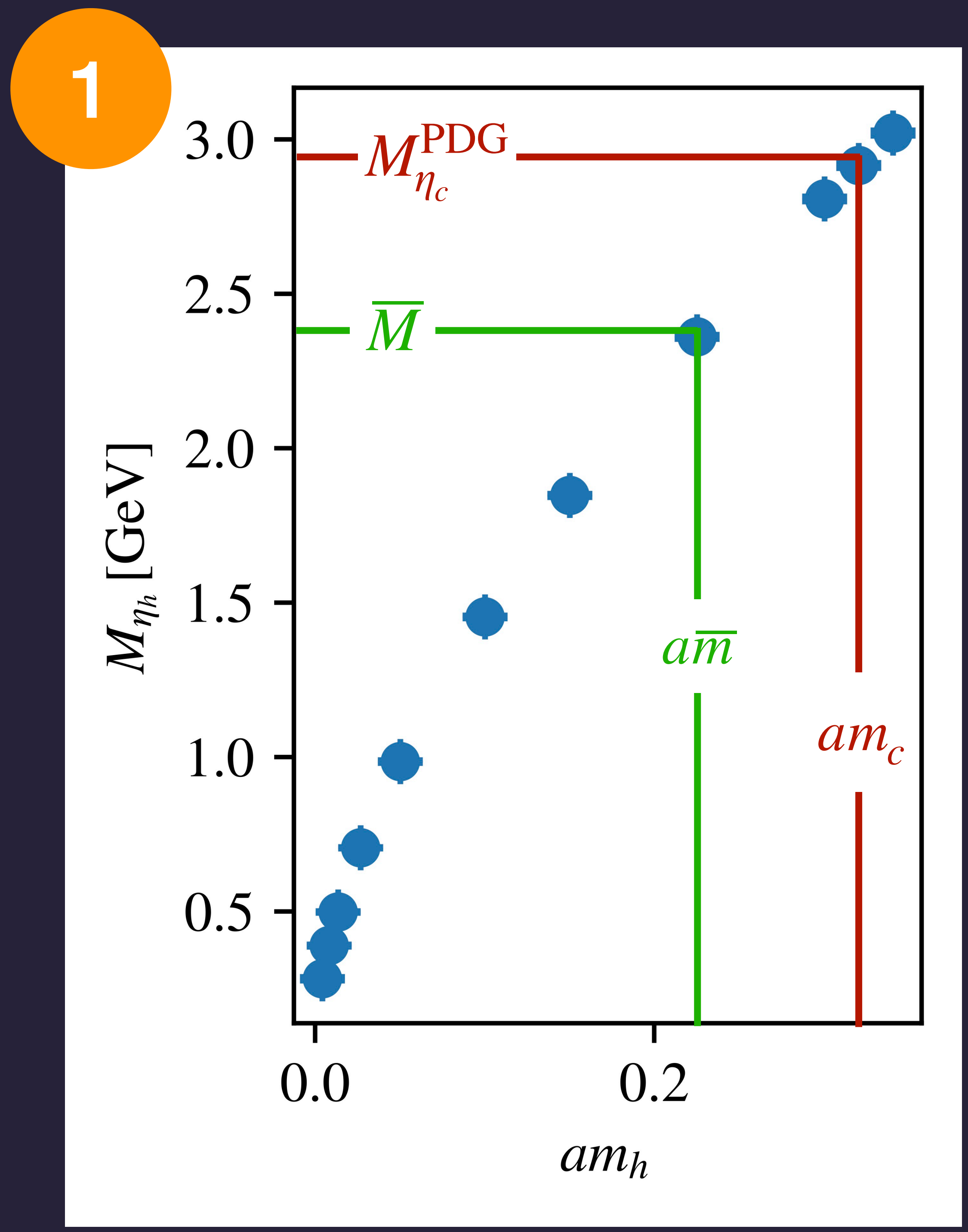


Reference scales

$$m_{c,R}(\mu, \overline{m}_R) = \lim_{a\to 0} Z_m(a\mu, a\overline{m})\, m_c$$

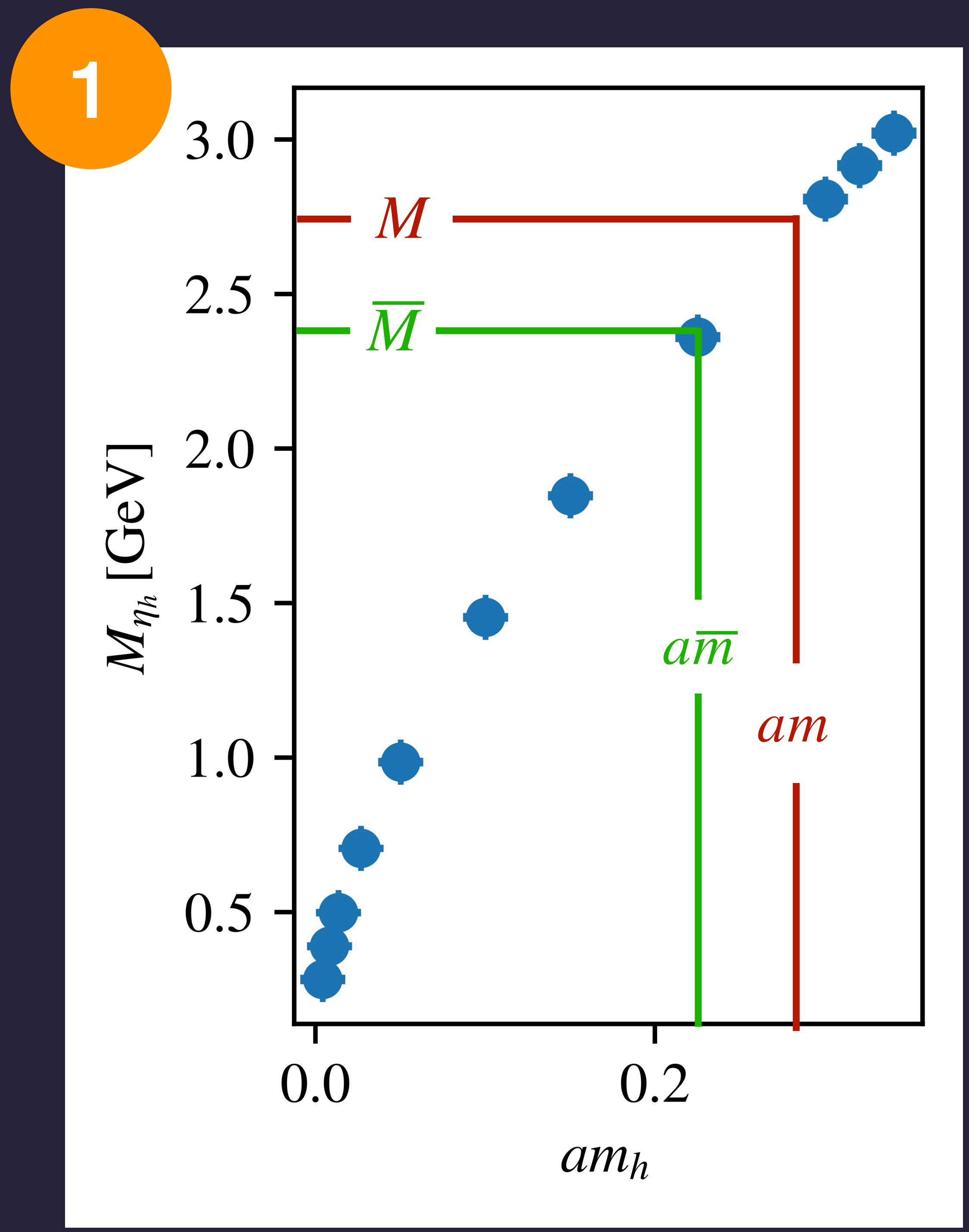


Reference scales

$$m_R(\mu, \overline{m}_R) = \lim_{a\to 0} Z_m(a\mu, a\overline{m})\, m$$

choose $\overline{M}$

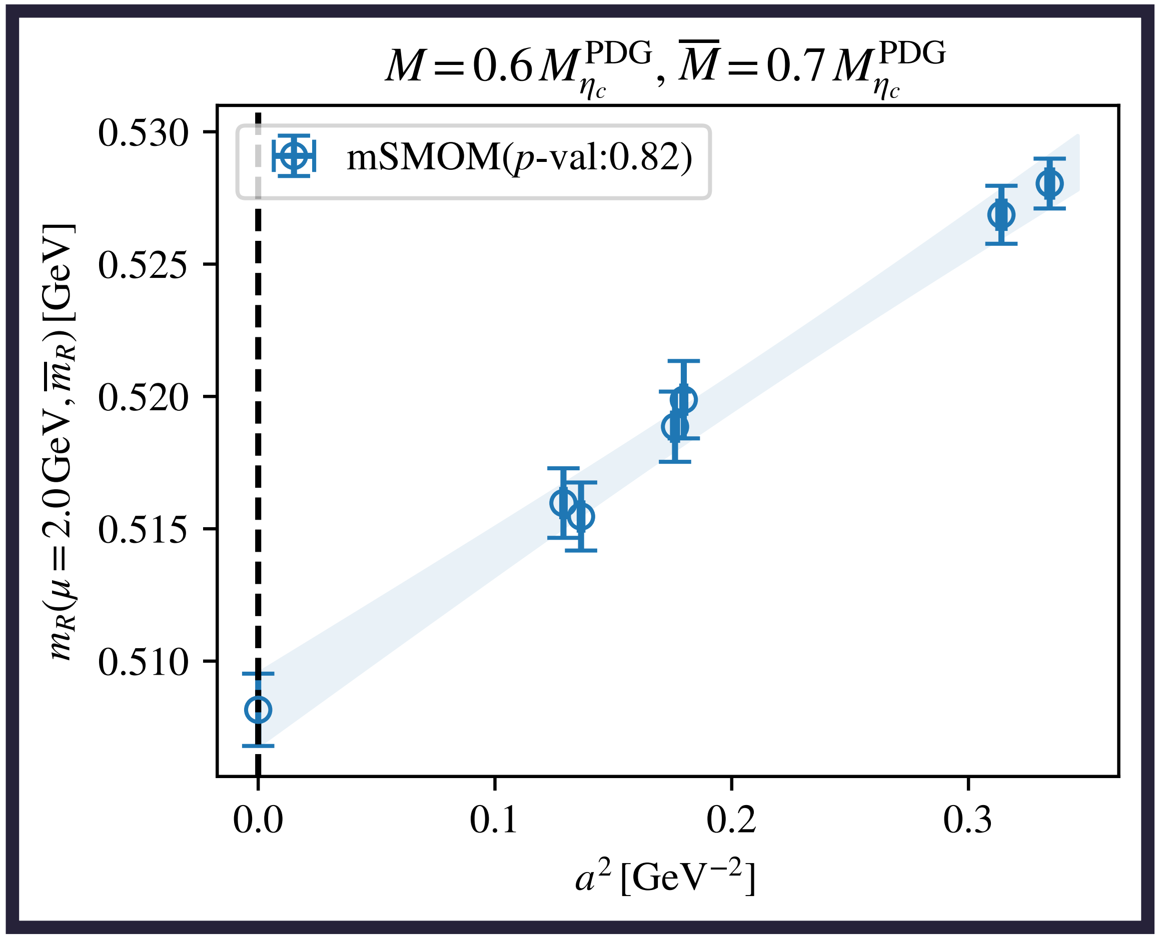
choose M



Continuum extrapolation
renormalised charm quark mass

Step 1. Choose $(M, \mu, \overline{M})$

$$m_R(\mu, \overline{m}_R) = \lim_{a \to 0} Z_m(a\mu, a\overline{m})\, m$$



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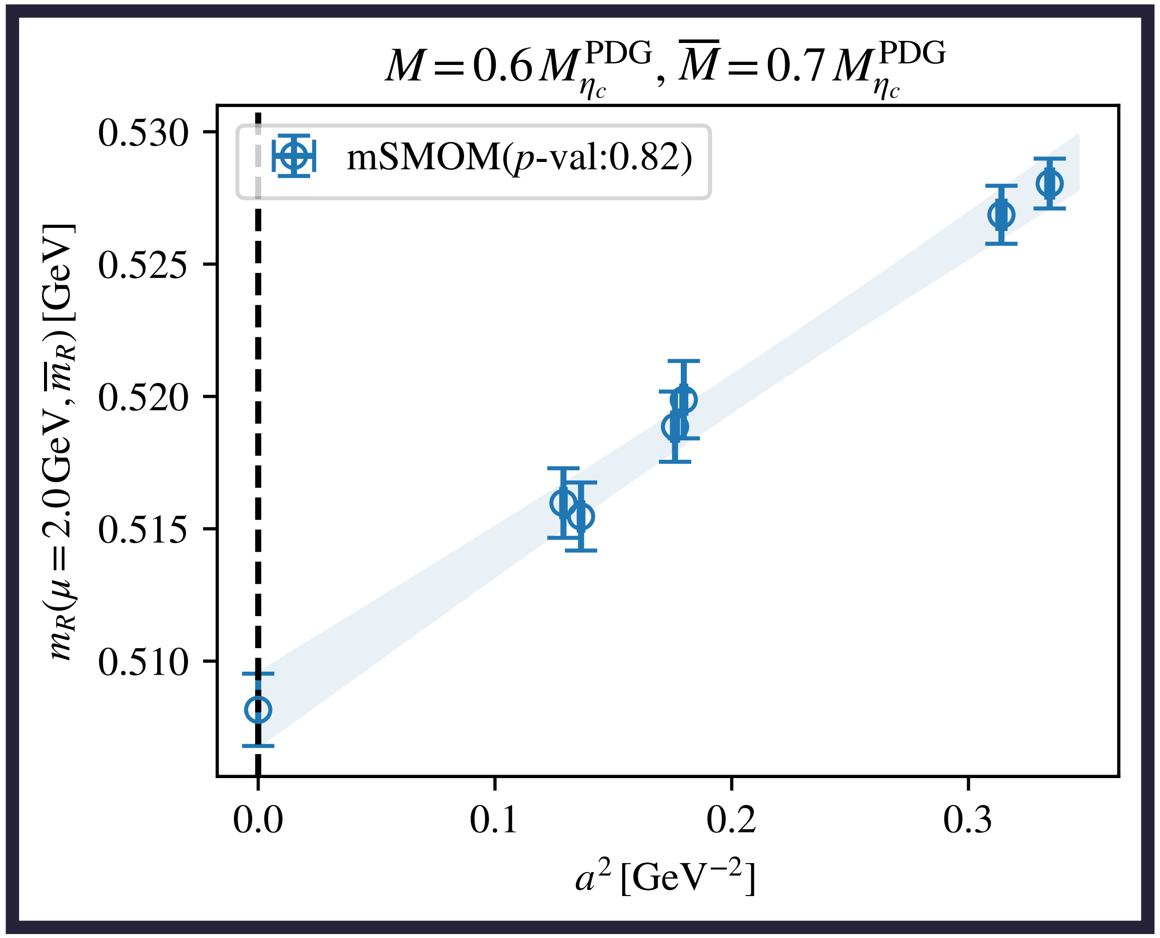
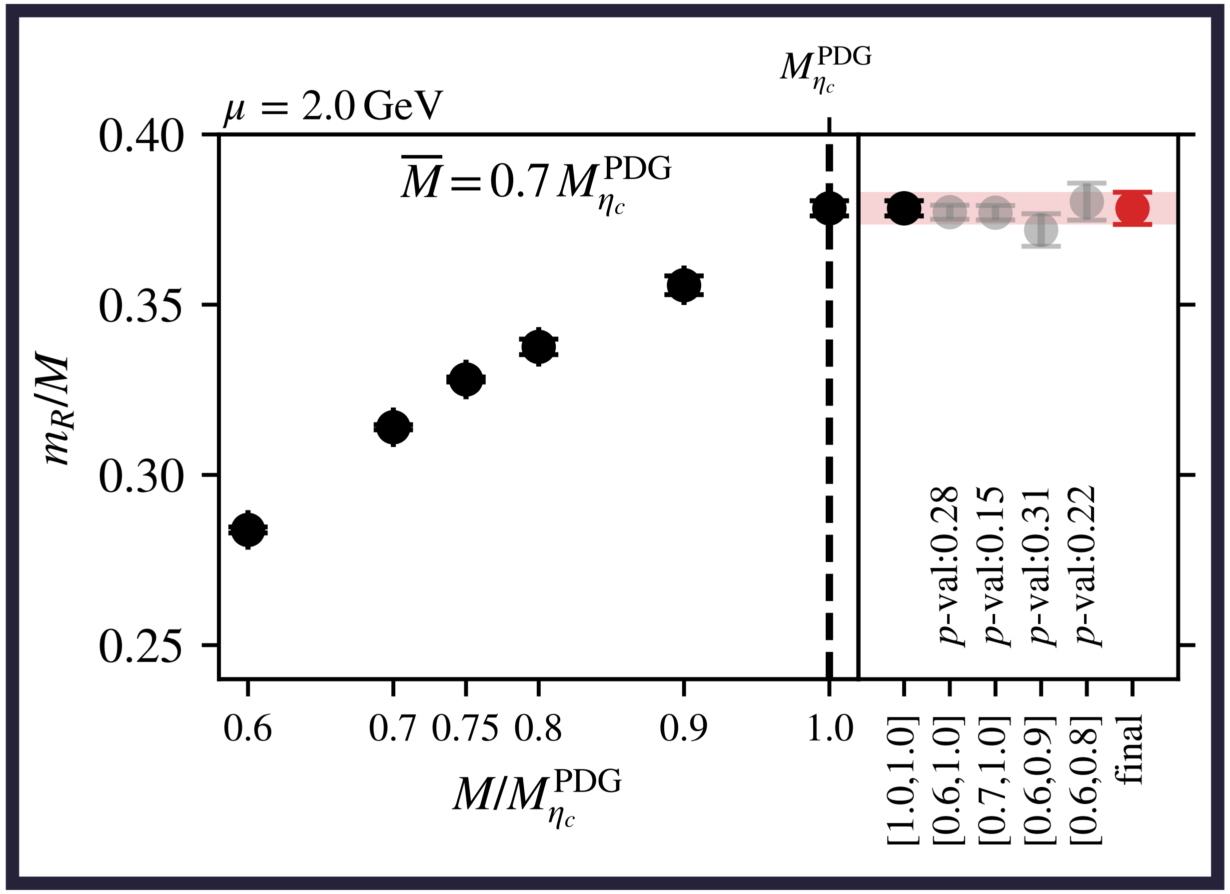
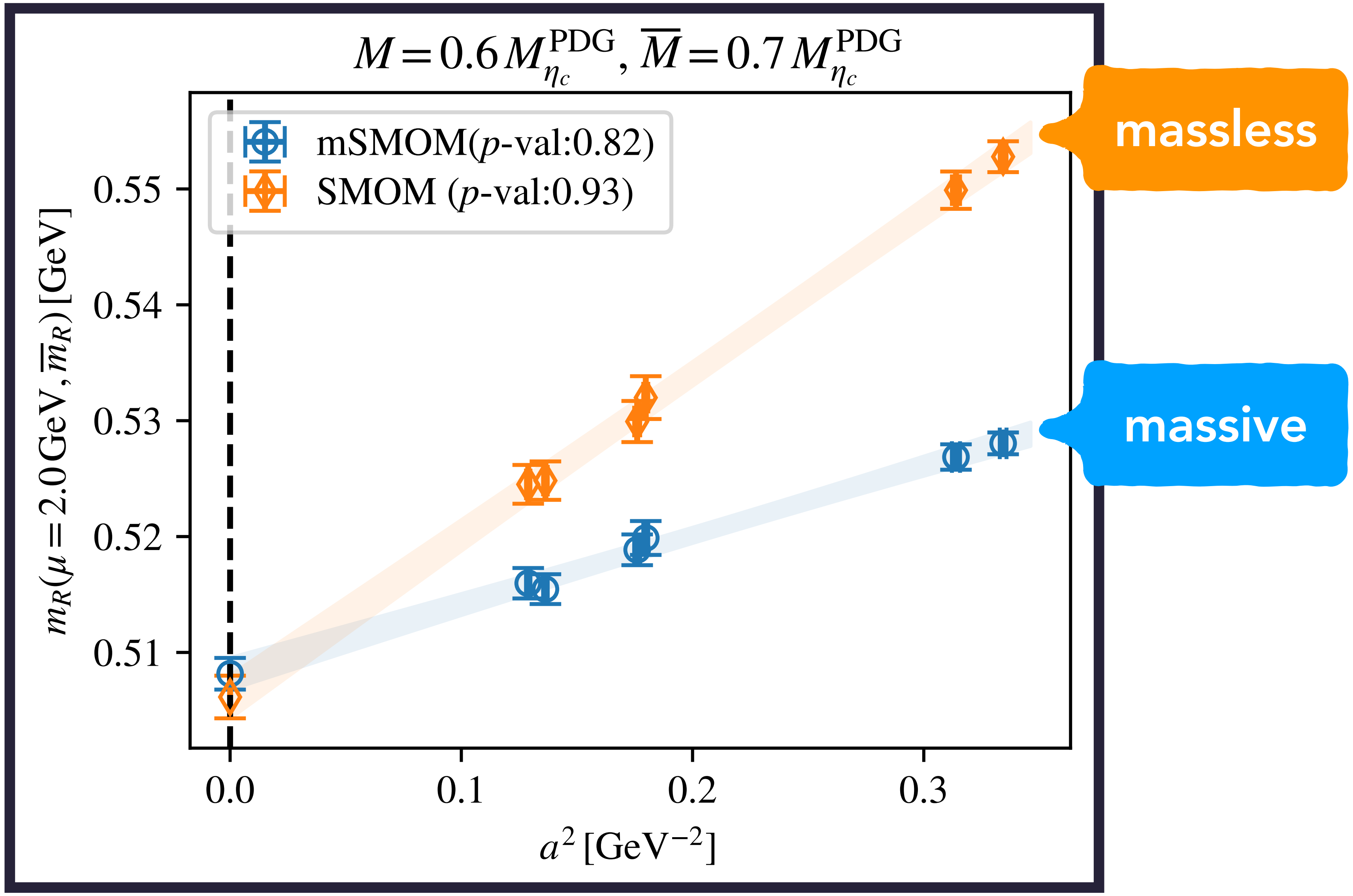


Step 2. Extrapolate to physical charm scale

$$m_{c,R}(\mu, \overline{m}_R) = \lim_{M\to M_{\eta_c}^{\mathrm{PDG}}} m_R(\mu, \overline{m})$$



Absorption of lattice artefacts
SMOM vs mSMOM

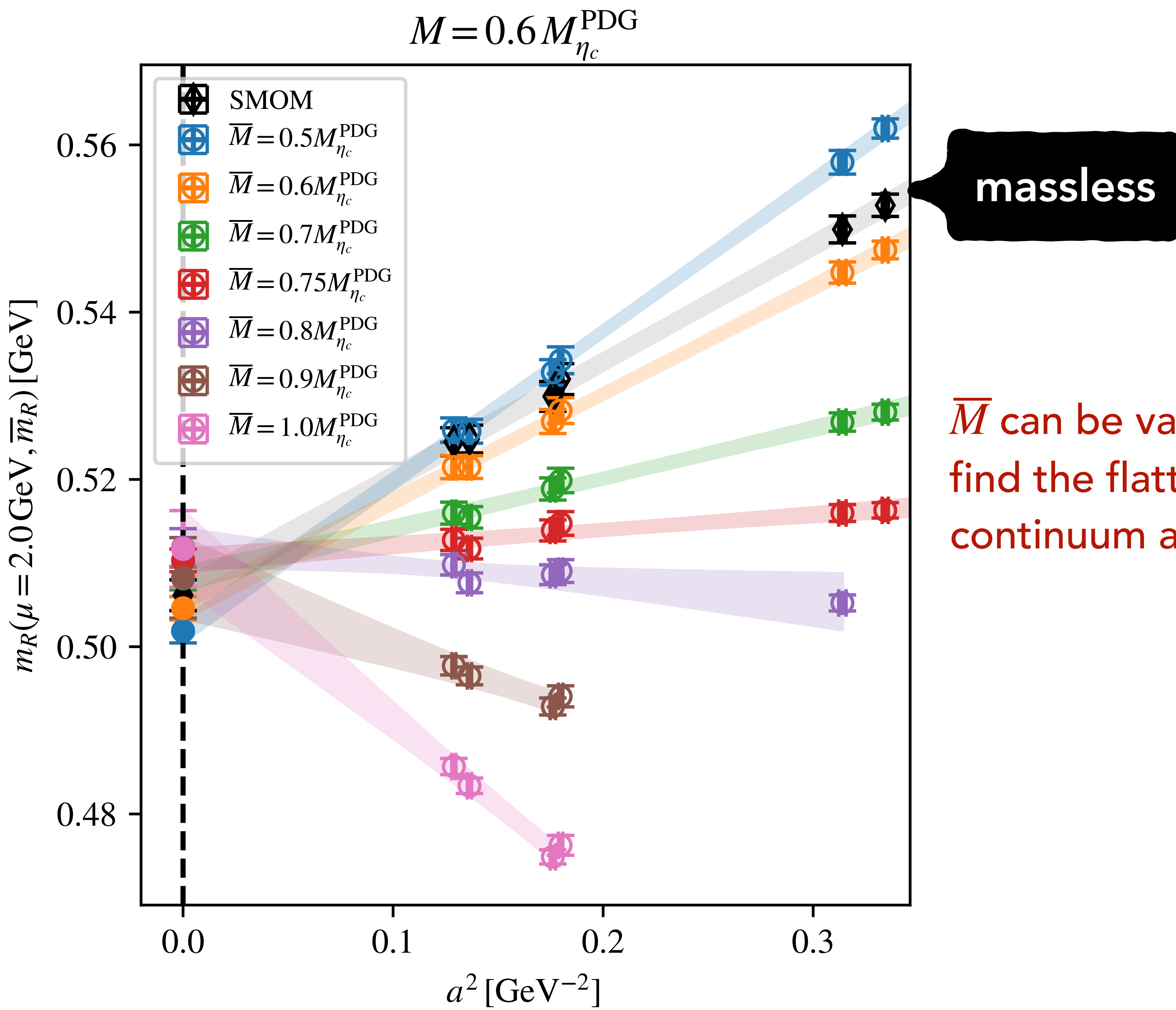


A flatter approach to the continuum using the massive scheme

$\Longrightarrow$ reduction in discretisation effects

Absorption of lattice artefacts

tuning mSMOM reference mass using $\overline{M}$



$\overline{M}$ can be varied to find the flattest continuum approach

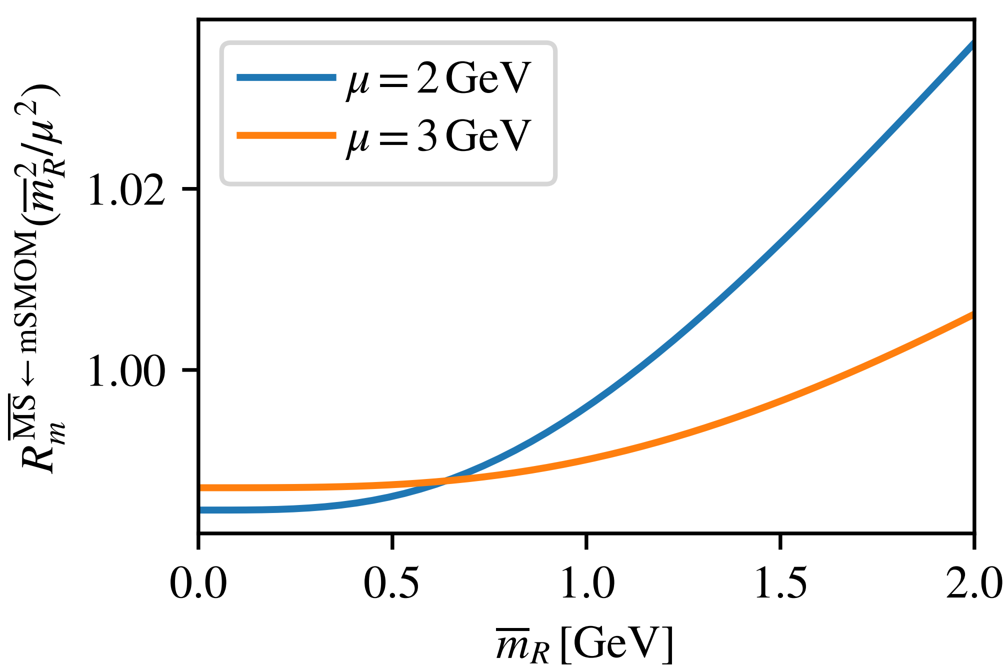
Matching to $\overline{\text{MS}}$
renormalised charm quark mass

Step 3: Perturbative matching to $\overline{\text{MS}}$ using $(\mu, \overline{m}_R)$

$$m_{c,R}^{\overline{\text{MS}}}(\mu) = R_m^{\overline{\text{MS}}\leftarrow\text{mSMOM}}\left(\frac{\overline{m}_R^2}{\mu^2}\right) m_{c,R}^{\text{mSMOM}}(\mu, \overline{m}_R)$$

- Conversion factors computed to 1-loop in Landau gauge: $(u = \overline{m}_R^2/\mu^2)$

$$R_m^{\overline{\text{MS}}\leftarrow\text{mSMOM}}(u) = 1 + \frac{\alpha}{4\pi}C_F\left[-4 + \frac{3}{2}C_0(u) + 3\ln(1+u) - 3u\ln\left(\frac{u}{1+u}\right)\right]$$



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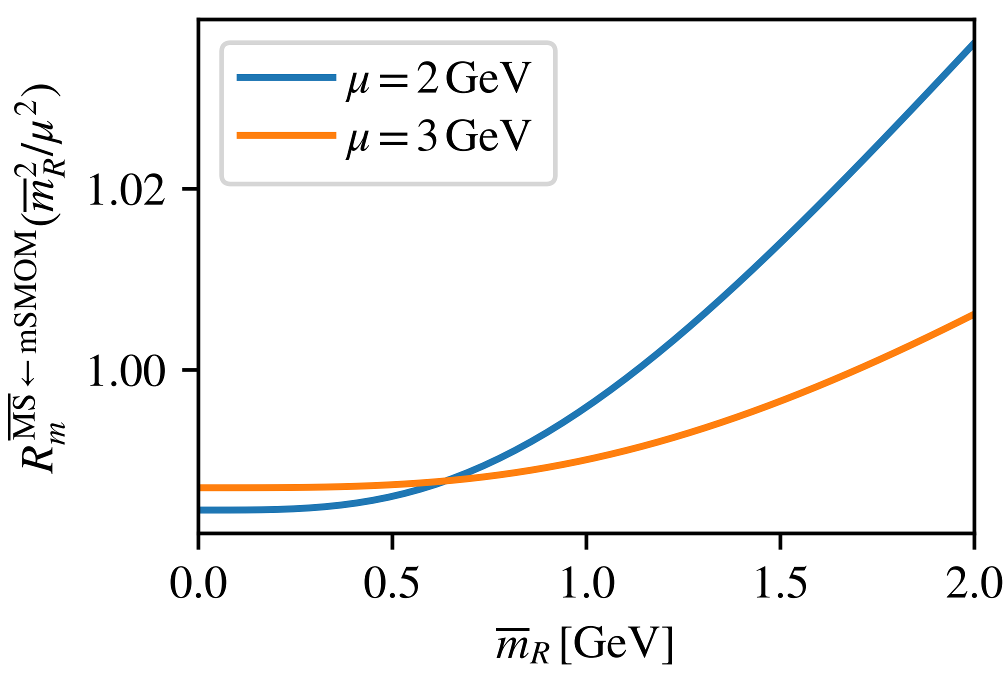
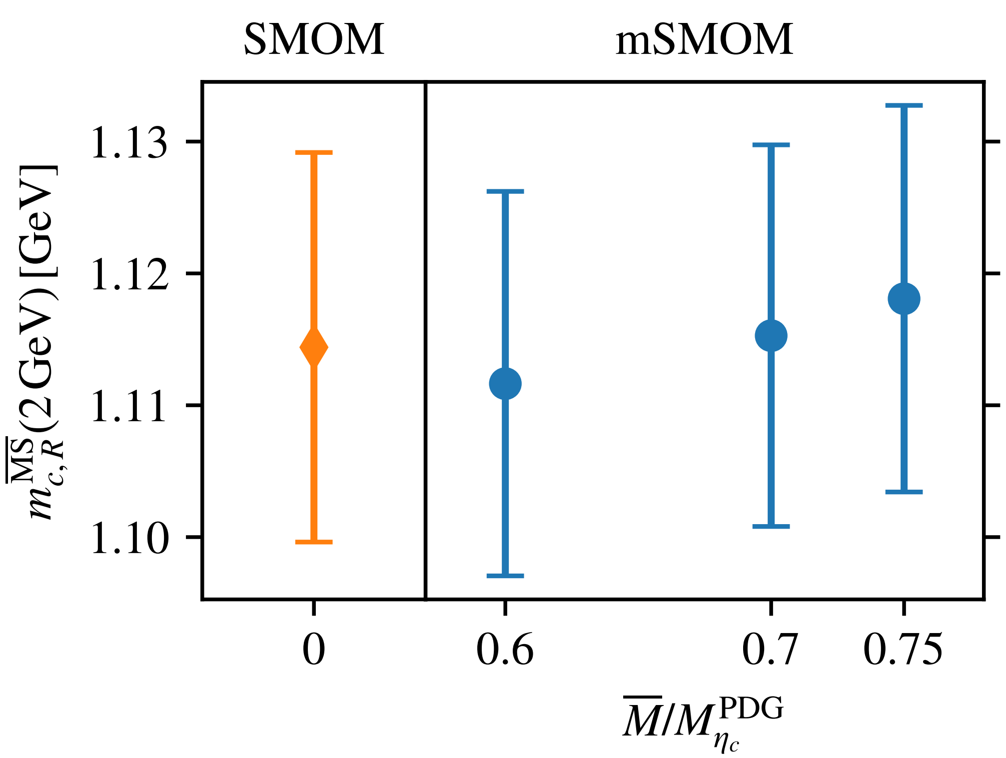
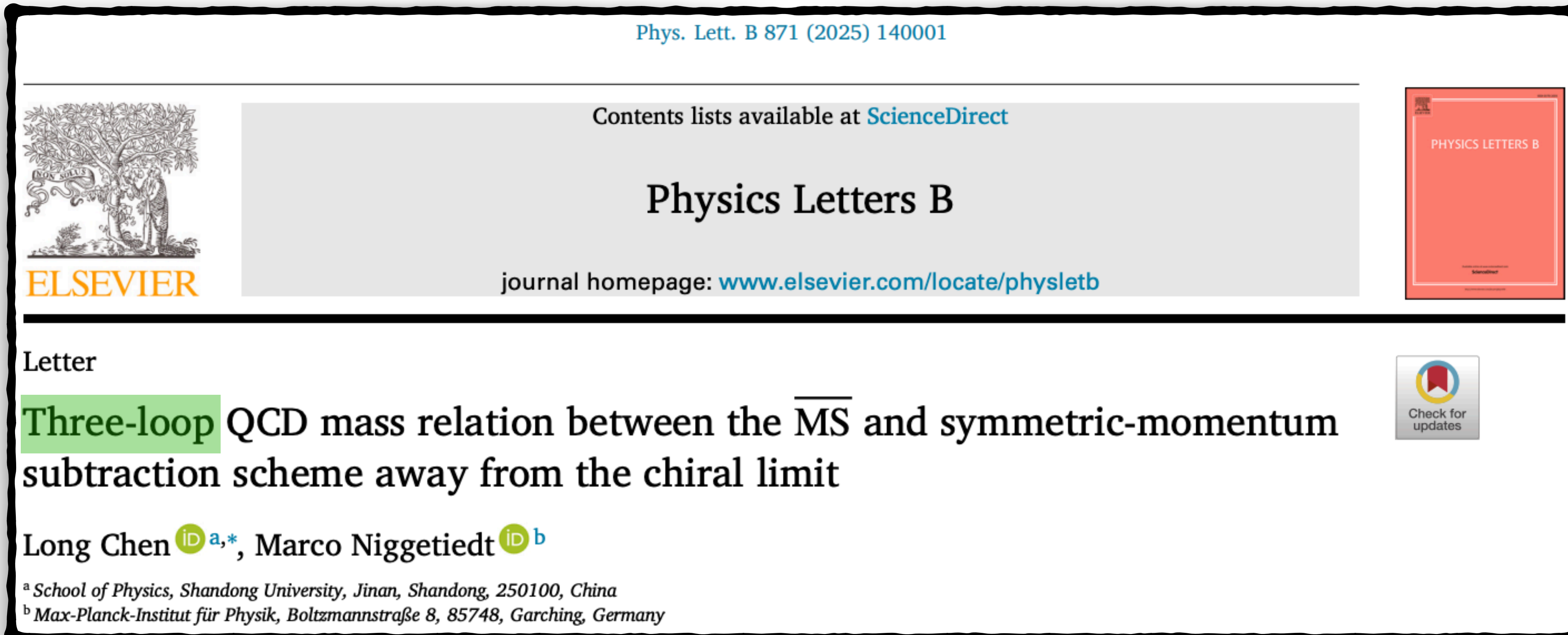




Matching to $\overline{\mathrm{MS}}$ renormalised charm quark mass

NEW UPDATE



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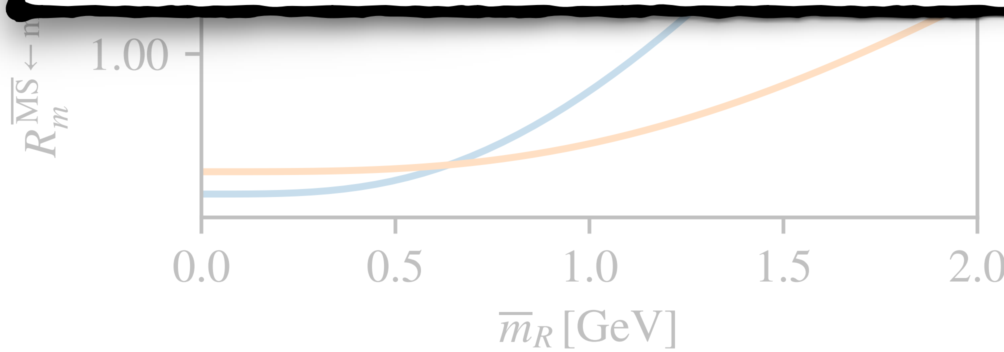
Letter

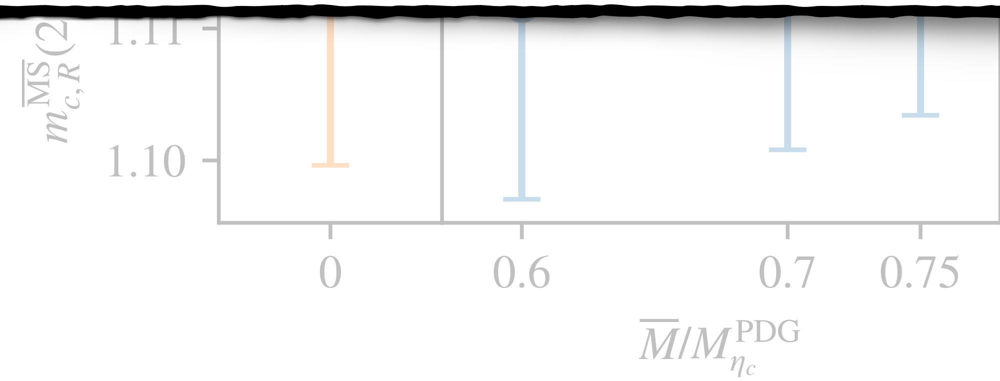
Three-loop QCD mass relation between the $\overline{\mathrm{MS}}$ and symmetric-momentum subtraction scheme away from the chiral limit

Long Chen[a,*], Marco Niggetiedt[b]

[a] School of Physics, Shandong University, Jinan, Shandong, 250100, China

[b] Max-Planck-Institut für Physik, Boltzmannstraße 8, 85748, Garching, Germany





Final results

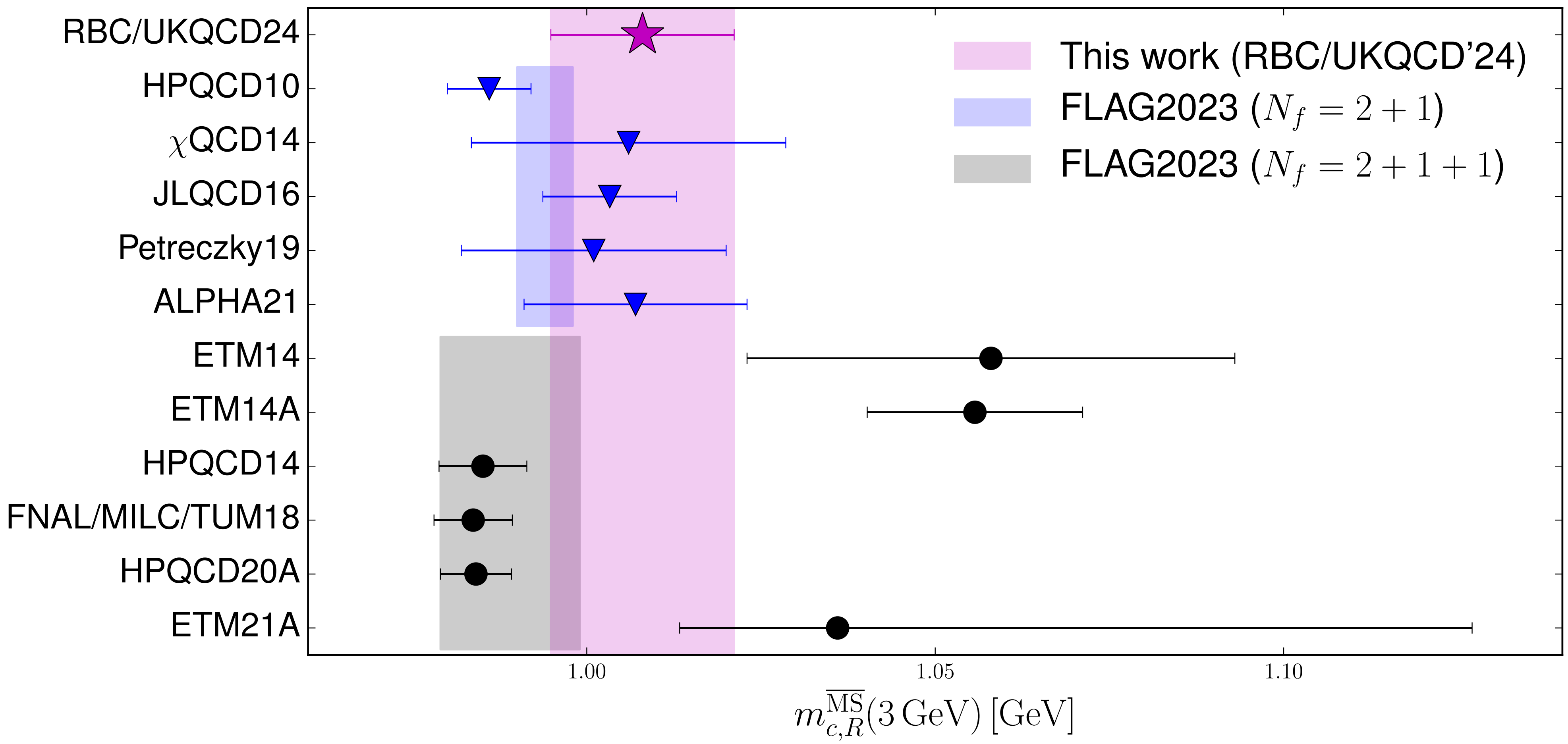
renormalised charm quark mass

✓ Full error budget with statistical + systematic + PT matching error

$$m_{c,R}^{\overline{\mathrm{MS}}}(2\,\mathrm{GeV}) = 1.115(7)(12)(4)\,\mathrm{GeV}$$

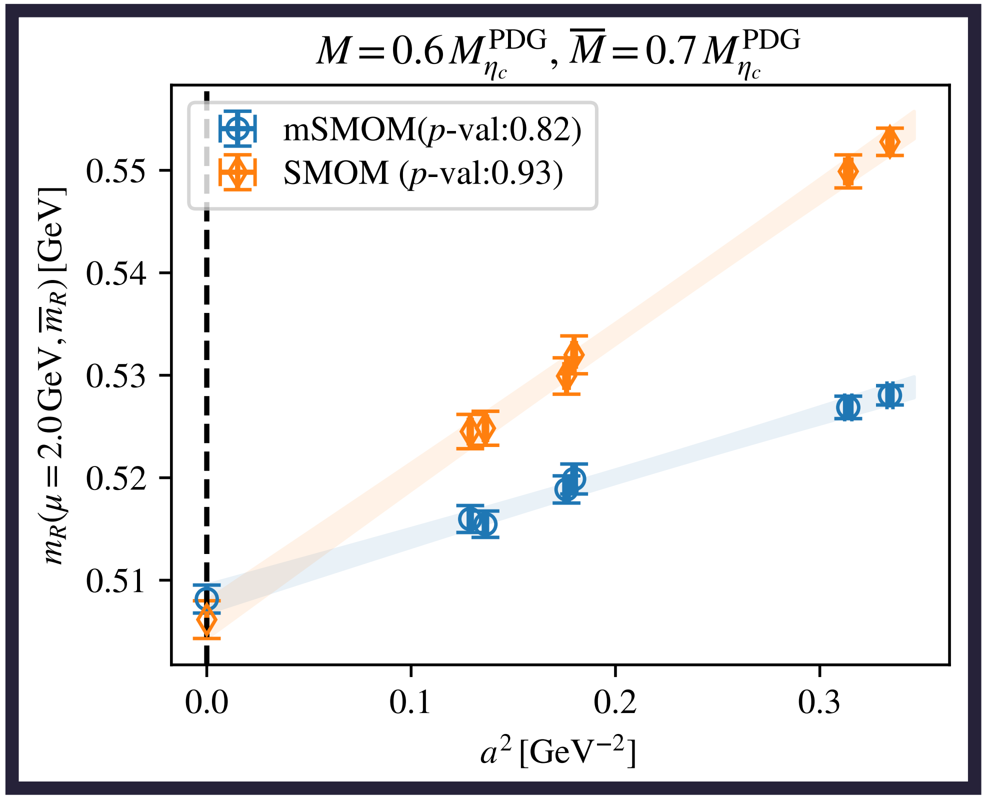
$$m_{c,R}^{\overline{\mathrm{MS}}}(3\,\mathrm{GeV}) = 1.008(6)(11)(4)\,\mathrm{GeV}$$

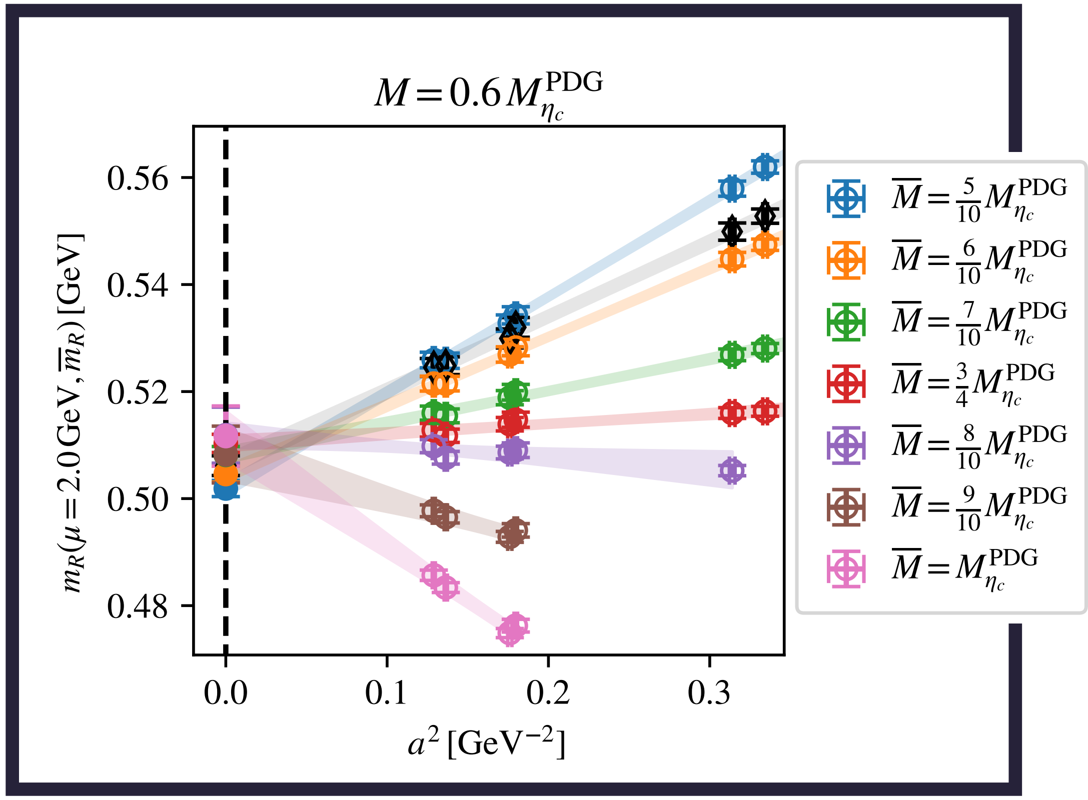
$$m_{c,R}^{\overline{\mathrm{MS}}}(m_{c,R}^{\overline{\mathrm{MS}}}) = 1.292(5)(10)(4)\,\mathrm{GeV}$$



Main takeaways

- Massive renormalisation accounts for am_c-sized discretisation effects
- RI/mSMOM scheme: renormalised charm quark mass





Reduced discretisation effects using the massive scheme

Can tune scheme mass for optimal reduction in discretisation error

(observable-dependent)

Renormalised quark masses using gradient flow

Matthew Black

In collaboration with:
R. Harlander, A. Hasenfratz, A. Rago, O. Witzel

[arXiv:2506.16327]
[in review, PRD]

Higgs-Maxwell Workshop, 11th February 2026

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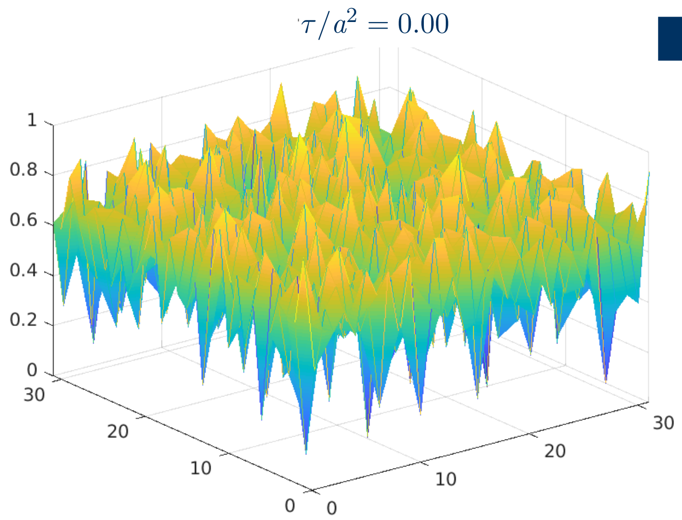
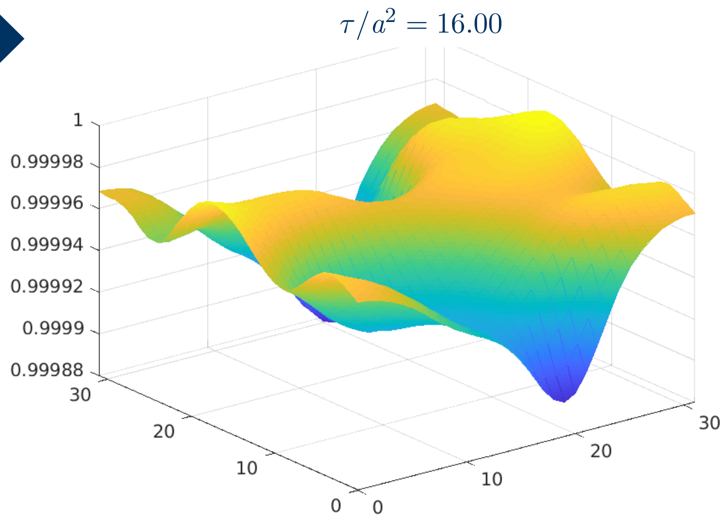
  - Can we find a renormalisation procedure with an easier matching window?

Gradient Flow

- Introduced by [Narayanan, Neuberger '06] [Lüscher '10] [Lüscher '13]
  - ➥ Scale setting ($\sqrt{8t_0}$), RG β-function, Λ parameter
- Introduce auxiliary dimension, **flow time** τ as a way to regularise the UV
- Well-defined damping of UV fluctuations





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- For use in renormalisation:
  - Flowed gauge fields automatically renormalised
  - Flowed fermionic fields require additional multiplicative wavefunction renormalisation factor Z_χ
    - Automatically a massive renormalisation procedure
  - Flowed matrix elements of composite operators automatically renormalised (up to Z_χ)
    - appropriately chosen ratios will cancel Z_χ

- Well-studied for e.g. energy-momentum tensor [Makino, Suzuki '14] [Harlander, Kluth, Lange '18]
  hadronic vacuum polarisation [Harlander, Lange, Neumann '20]
  parton distribution functions [Francis et al. '25]
  four-quark operators [Black et al. '24] [Black et al. 26XX.XXXX]
  and more...

- Re-express effective Hamiltonian in terms of 'flowed' operators:

$$\mathcal{H}_{\text{eff}} = \sum_n C_n \mathcal{O}_n = \sum_n \widetilde{C}_n(\tau)\widetilde{\mathcal{O}}_n(\tau).$$

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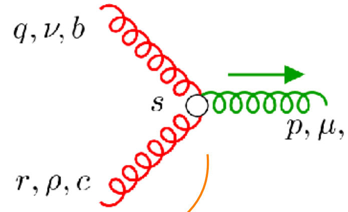

- Relate to regular operators in 'short-flow-time expansion':

$$\widetilde{\mathcal{O}}_n(\tau) = \sum_m \zeta_{nm}(\tau)\mathcal{O}_m + O(\tau)$$

'flowed' MEs calculated on lattice
renormalised along flow time

matching matrix
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new Feynman
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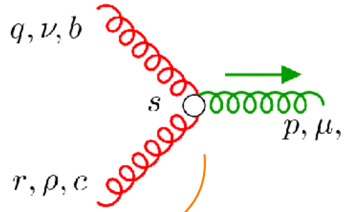

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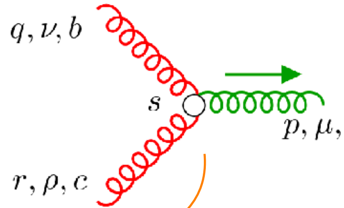

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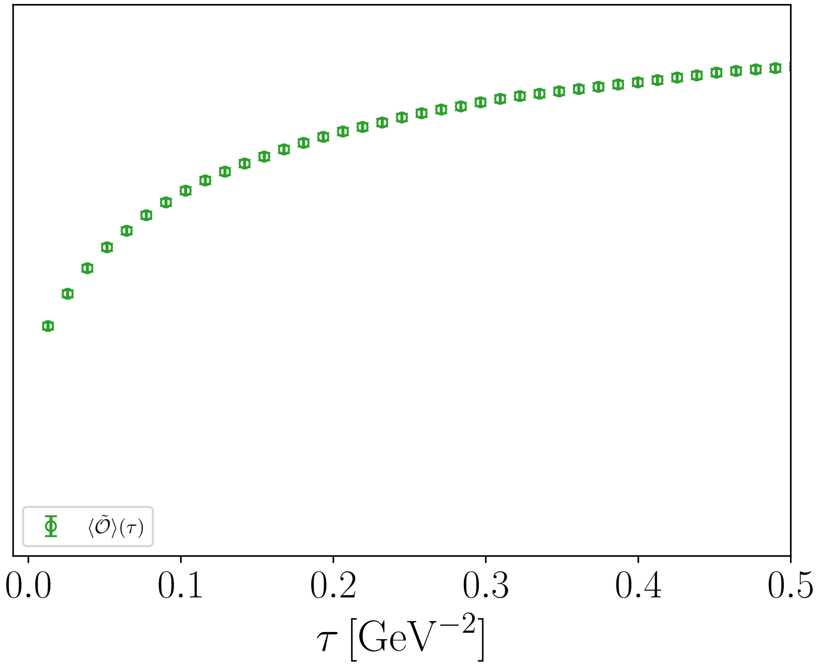
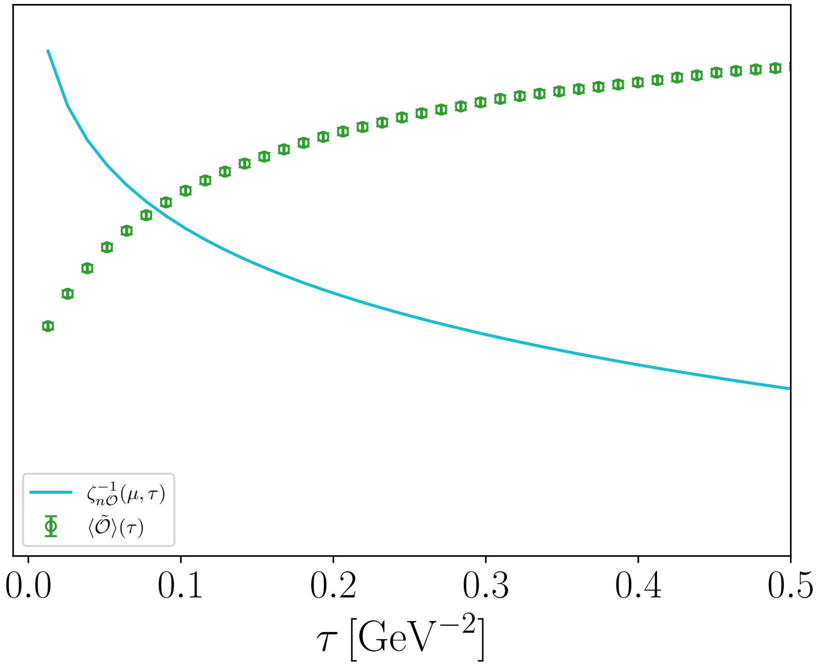
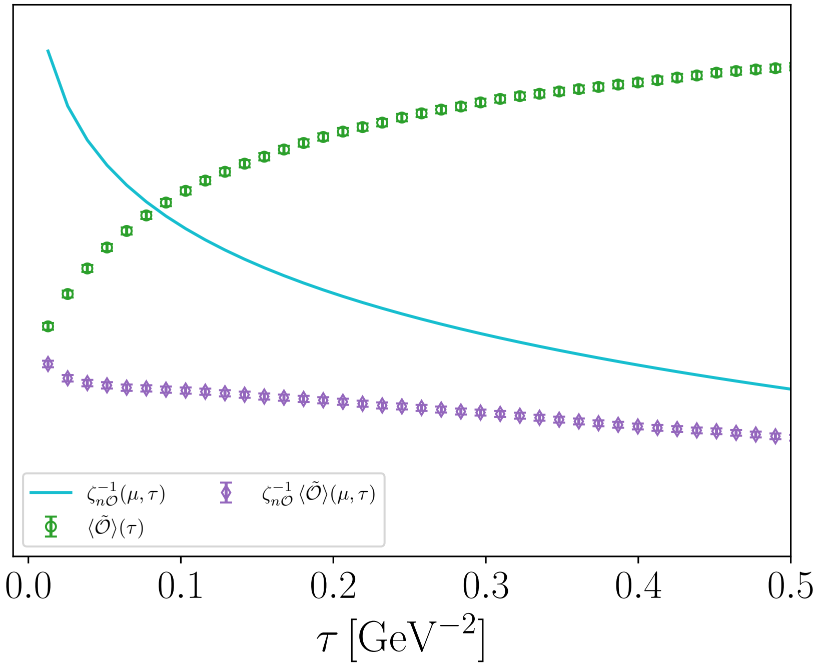
- Corrections $O(\tau^n, (m_q^2\tau)^n)$ appear ➡ keep τ small
- Flow-time logarithms $\log^n(\mu^2\tau)$ cancelled to perturbative order n ➡ τ cannot be **too** small



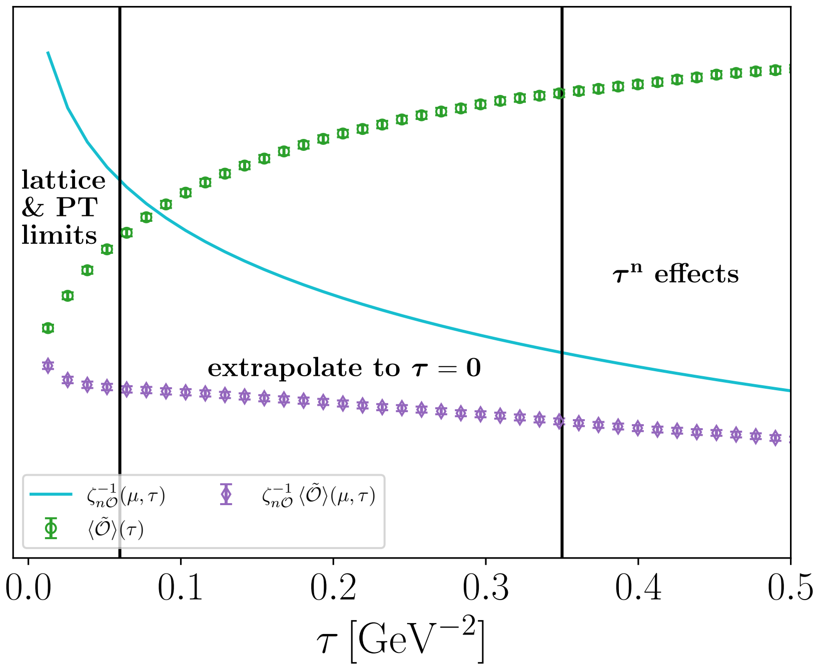
➤ Measure flowed matrix element $\langle\mathcal{O}\rangle(\tau)$ on the lattice



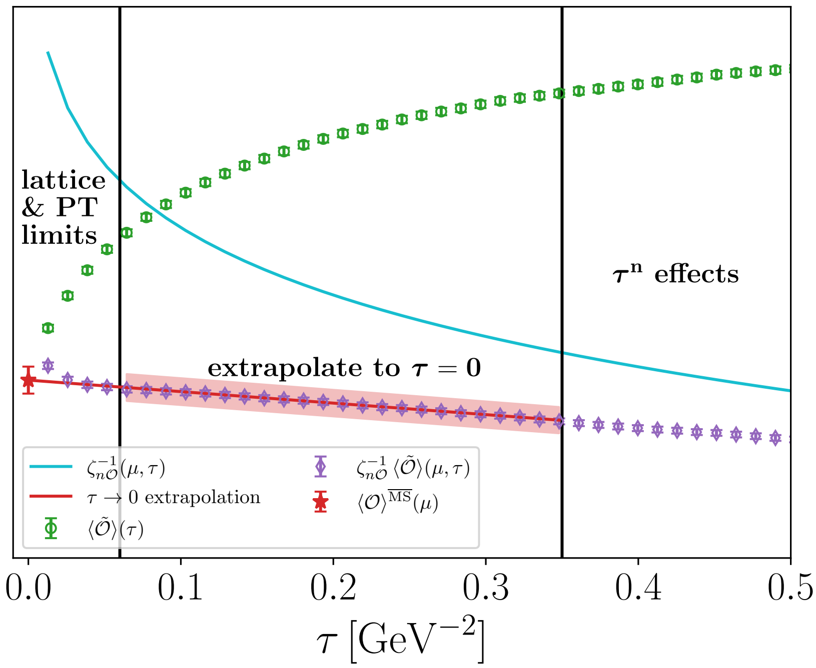
- Calculate perturbative matching coefficients $\zeta_{n\mathcal{O}}^{-1}(\mu,\tau)$



- Combine for ‘matched’ operator dependent on flow time τ and renormalisation scale μ



- Larger systematic effects at extremities



- Take $\tau \to 0$ result ➡ $\langle\mathcal{O}\rangle^{\overline{\mathrm{MS}}}(\mu)$

Analysis and Results

	L	T	a^{-1}/GeV	am_s^{phys}	am_c^{phys}	M_π/MeV	srcs $\times$ N_{conf}
C1	24	64	1.7848	0.03224	0.64	340	32×101
C2	24	64	1.7848	0.03224	0.64	433	32×101
M1	32	64	2.3833	0.02477	0.45	302	32×79
M2	32	64	2.3833	0.02477	0.45	362	32×89
M3	32	64	2.3833	0.02477	0.45	411	32×68
F1S	48	96	2.785	0.02167	0.37	267	24×98

[Allton et al. '08]
[Aoki et al. '10]
[Blum et al. '14]
[Boyle et al. '17]

- Functional form of GF lattice two-point correlation function:

$$\langle \mathcal{O}_J(t;\tau)\, \mathcal{O}_I^\dagger(t=0;\tau=0)\rangle \overset{t\to\infty}{\approx} \frac{\langle \mathcal{O}_J\rangle(\tau)\langle \mathcal{O}_I^\dagger\rangle(\tau=0)}{2M_{\mathrm{PS}}^{(rs)}}$$

for pseudoscalar meson PS containing quarks r and s

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- Recall matrix elements of pseudoscalar ($P = q_1\gamma_5\bar{q}_2$) and axial-pseudoscalar ($A^0 = q_1\gamma_0\gamma_5\bar{q}_2$) currents:

$$Z_P\langle 0|P|\mathrm{PS}\rangle = \frac{Z_A}{Z_m}\frac{f_{\mathrm{PS}}^{(rs)}M_{\mathrm{PS}}^{(rs)\,2}}{m^{(r)}+m^{(s)}} \qquad\qquad Z_A\langle 0|A^0|\mathrm{PS}\rangle = f_{\mathrm{PS}}^{(rs)}M_{\mathrm{PS}}^{(rs)}$$

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- Form ratios to extract quark mass:

$$\left(m_{GF}^{(r)}(\tau)+m_{GF}^{(s)}(\tau)\right) = M_{\mathrm{PS}}^{(rs)}R_{rs}^{\mathcal{O}}(\tau) \quad \text{with} \quad R_{rs}^{\mathcal{O}}(\tau) = \lim_{t\to\infty} -\frac{\langle A^0(t;\tau)\,\mathcal{O}(t=0;\tau=0)\rangle_{rs}}{\langle P(t;\tau)\,\mathcal{O}(t=0;\tau=0)\rangle_{rs}}$$

  ➡ ratio can be chosen for either $\mathcal{O} = A^0, P$

  ➡ require GF smearing radius $\sqrt{8\tau} \ll t$ to not destroy signal

- In practice, finite lattices cause time-dependence in ratio, so form double ratio to cancel it:

$$R_{rs}^{AP}(\tau) = \lim_{t\to\infty} \sqrt{\frac{\langle A^0(t;\tau)\, A^0(t=0;\tau=0)\rangle_{rs}}{\langle P(t;\tau)\, A^0(t=0;\tau=0)\rangle_{rs}} \frac{\langle A^0(t;\tau)\, P(t=0;\tau=0)\rangle_{rs}}{\langle P(t;\tau)\, P(t=0;\tau=0)\rangle_{rs}}}$$

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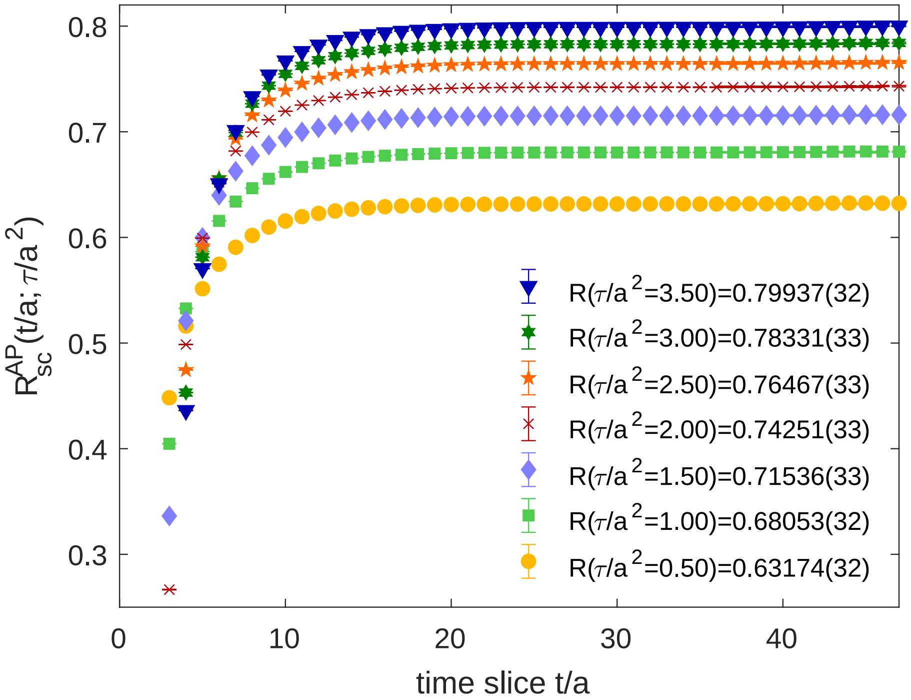
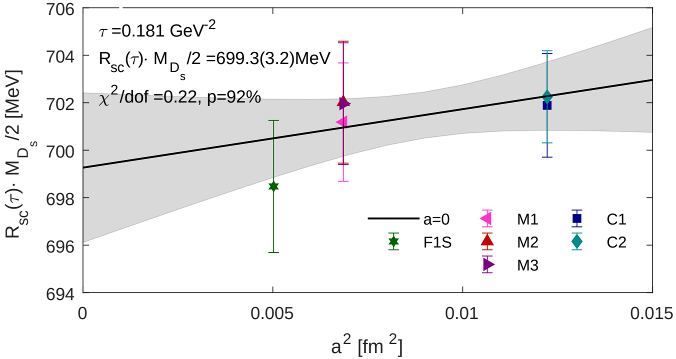
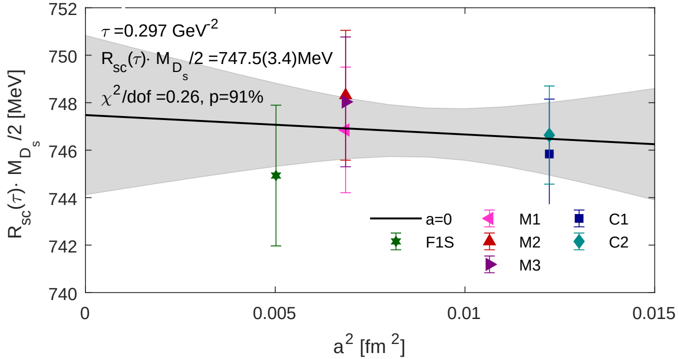
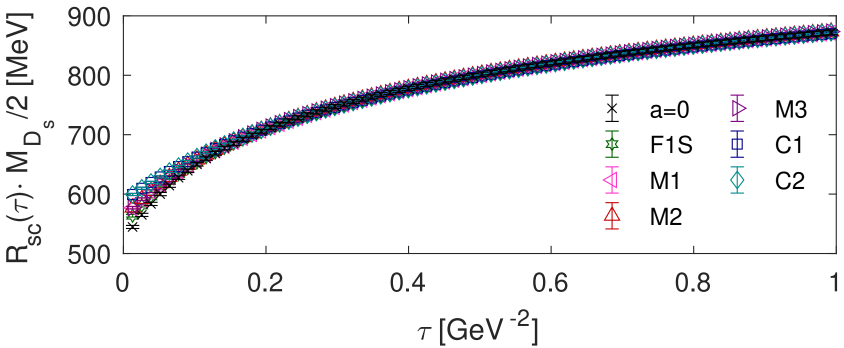
- Ens: F1S, $a^{-1} = 2.785$ GeV
- Stat errors smaller than symbols
- Correlated fits for $t \in (36, 46)$
- Repeat for each ensemble

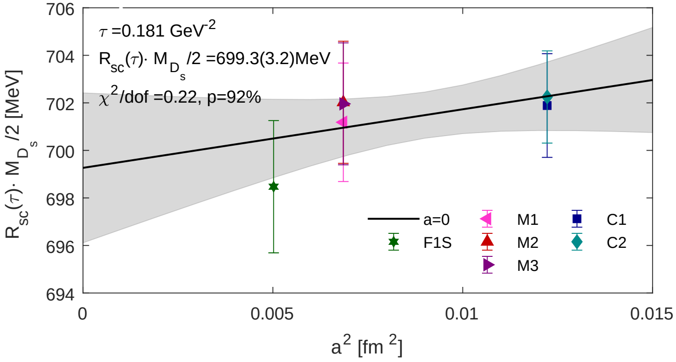


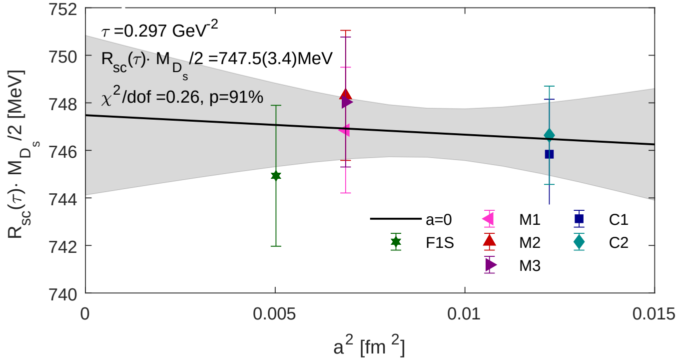




- ➤ Most discretisation effects quickly absorbed at $t > 0$
- ➤ GF defines renormalised trajectory
  - ➥ take $a \to 0$ for $\tau > 0$ with no additional renormalisation







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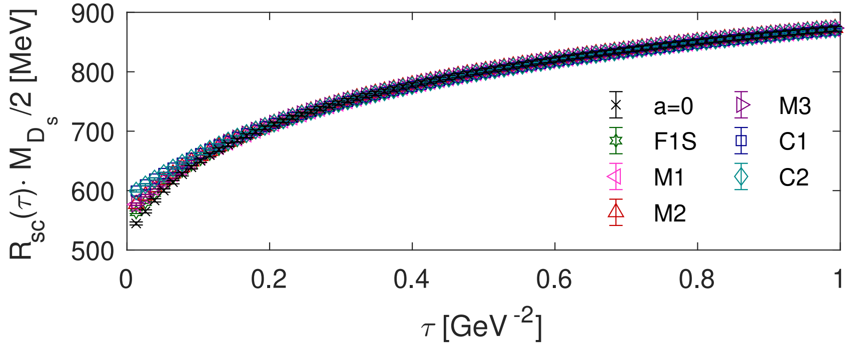
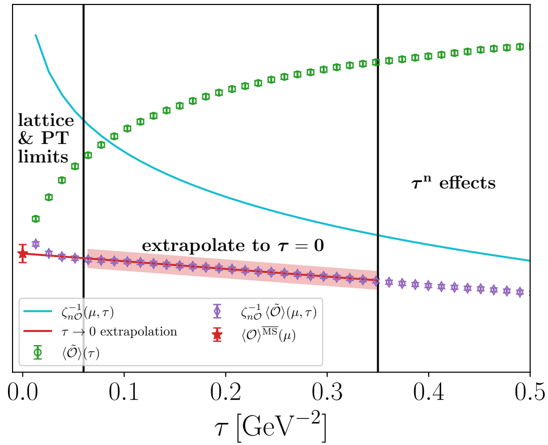
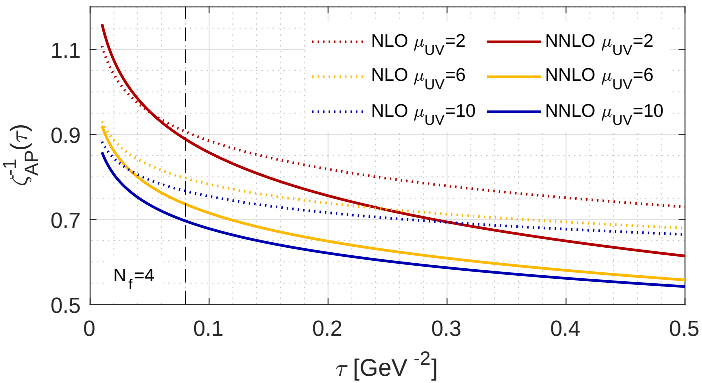


- ➤ This is a fully renormalised quark mass, using the scale τ instead of μ



- GF renormalised quark masses can be related to $\overline{\mathrm{MS}}$ scheme using SFTX

$$m_{\overline{\mathrm{MS}}}(\mu_{\mathrm{UV}}) = \lim_{\tau\to 0} \zeta^{-1}_{AP}(\mu_{\mathrm{UV}},\tau)\, m_{GF}(\tau)$$



- ζ^{-1}_{AP} calculated in perturbation theory to NNLO [Borgulat et al. '23]
- Combine and take $\tau \to 0$ limit!

➤ Consider NLO and NNLO matching

➤ Choose two fit ansätze:

$$f_1(\tau) = \left(c_2 \log(\mu^2\tau) + c_1\right)\tau + c_0$$
$$f_2(\tau) = c_2\tau^2 + c_1\tau + c_0$$

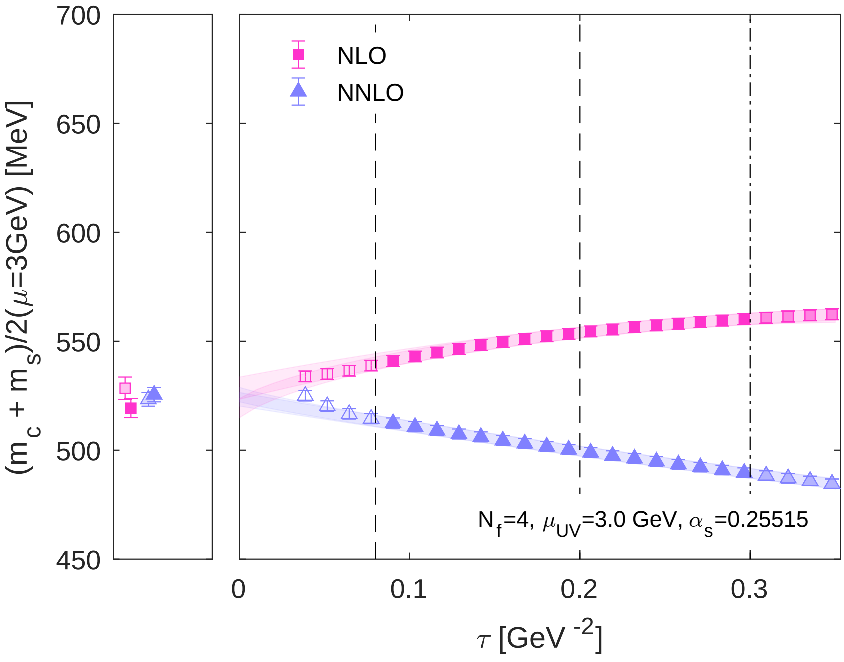
➤ Pick SFTX 'valid' region

➡ some freedom + plenty data

$\tau_{\text{min}} \in (0.08, 0.2)\,\text{GeV}^{-2}$; $\tau_{\text{max}} = 0.3\,\text{GeV}^{-2}$

$\tau_{\text{max}} \in (0.25, 0.35)\,\text{GeV}^{-2}$; $\tau_{\text{max}} = 0.14\,\text{GeV}^{-2}$

➤ Quark mass scales $O(m_c^2\tau)$ *may* contaminate for some flow times



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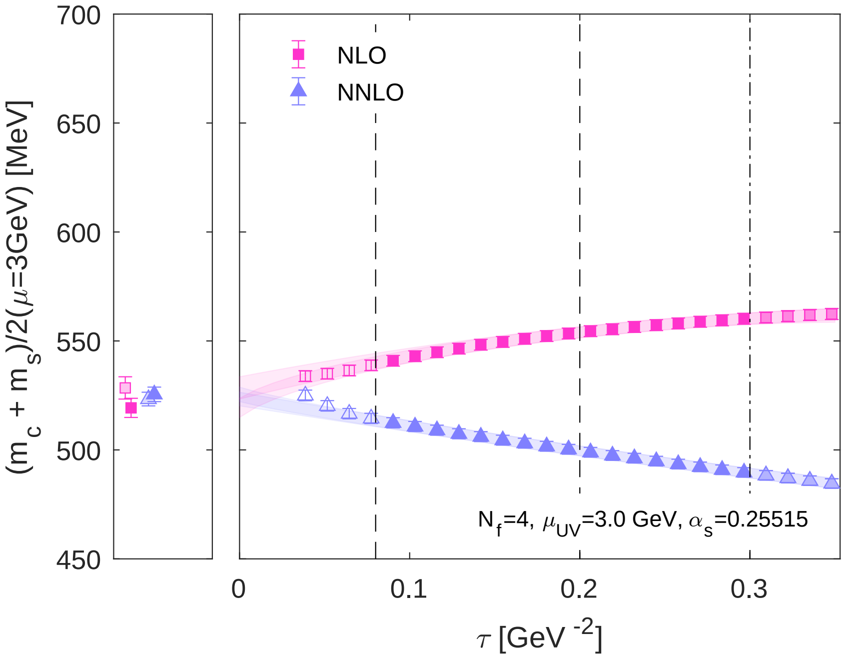
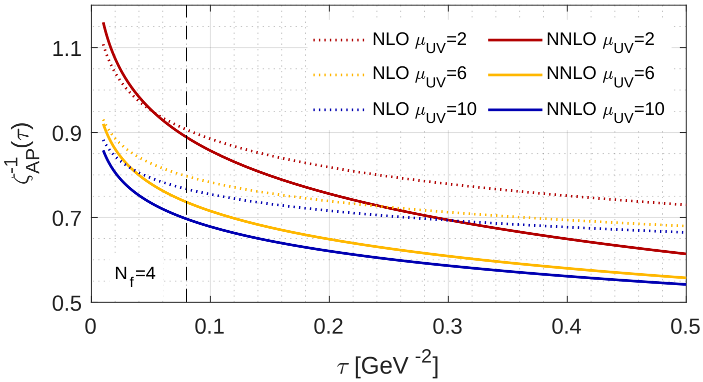
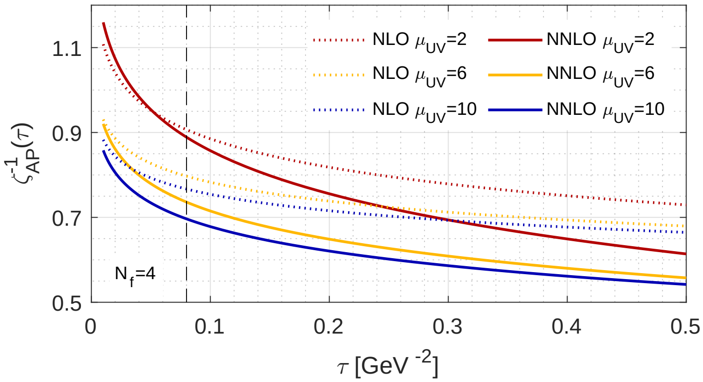


Can we improve this extrapolation?


NLO μ_{UV}=2
NNLO μ_{UV}=2
NLO μ_{UV}=6
NNLO μ_{UV}=6
NLO μ_{UV}=10
NNLO μ_{UV}=10
$\zeta^{-1}_{AP}(\tau)$
N_f=4
τ [GeV $^{-2}$]
1.1
0.9
0.7
0.5
0
0.1
0.2
0.3
0.4
0.5

- Define a flow-time RGE:

$$\tau \partial_\tau \tilde{\mathcal{O}}(\tau) = \tilde{\gamma}(a_s(\mu), L_{\mu\tau}) \tilde{\mathcal{O}}(\tau)$$



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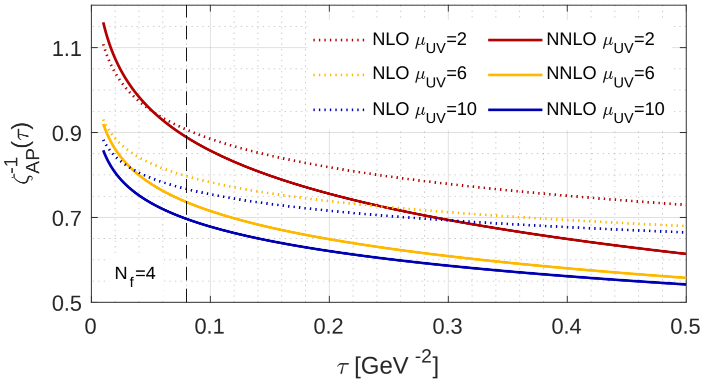
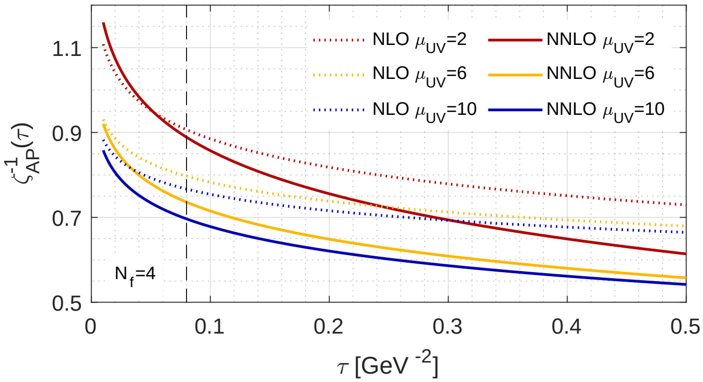
➤ We can resum GF logs in ζ_{AP}:

$$m_{\overline{\mathrm{MS}}}(\mu_{\mathrm{UV}}) = \lim_{\tau\to 0}\left(\zeta_{AP}^{\mathrm{imp}}(\mu_{\mathrm{UV}}, \tau_\mu)\right)^{-1} m_{GF}(\tau)$$





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- perturbative GF anomalous dimension γ_m^{GF} [Borgulat et al. '23]

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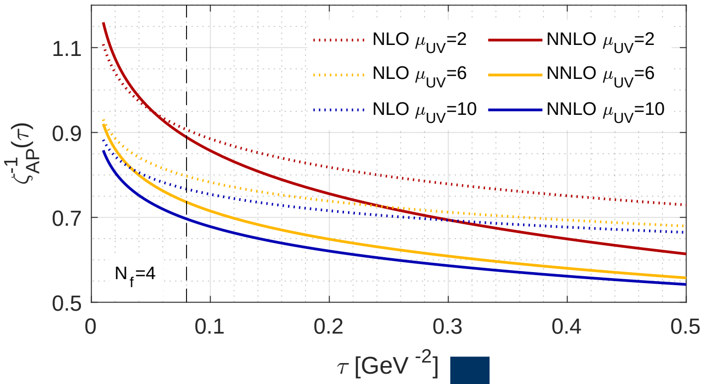
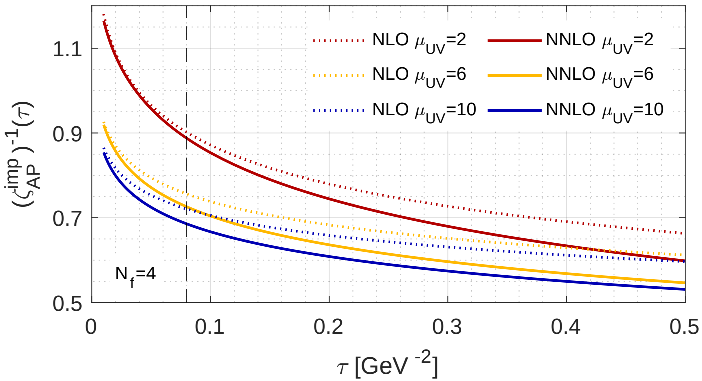
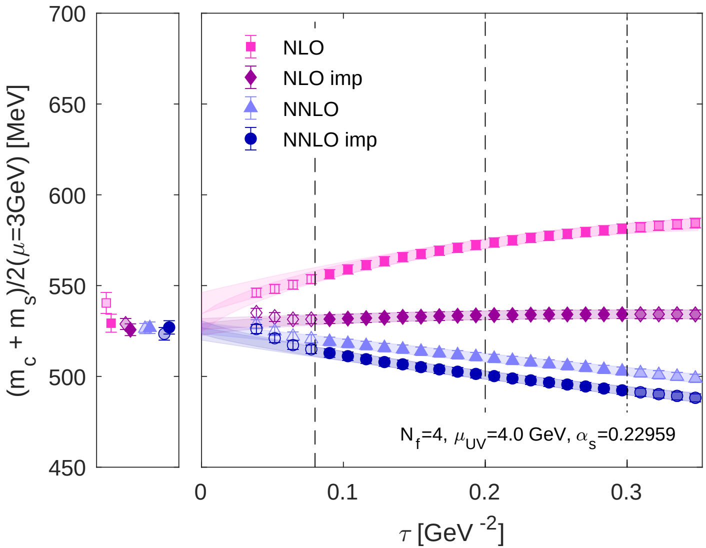




- ➤ Less flow-time curvature
  - ➡ contaminating logs are now 0
- ➤ $O(m_c^2\tau)$ effects more suppressed at τ_μ
  - ➡ compatible final results show $O(m_c^2\tau)$ under control ✔



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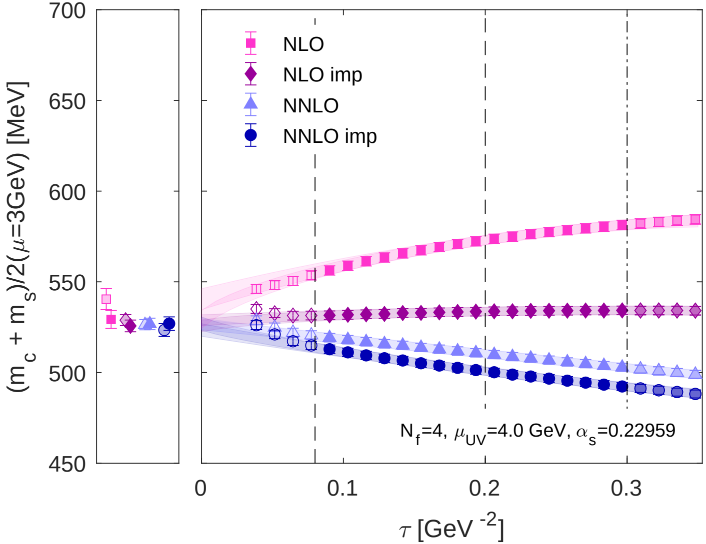
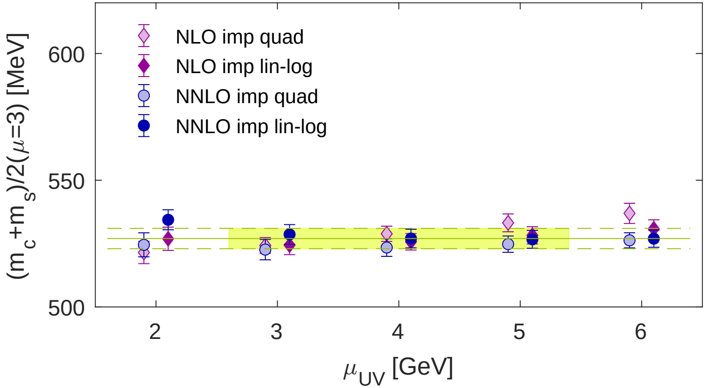
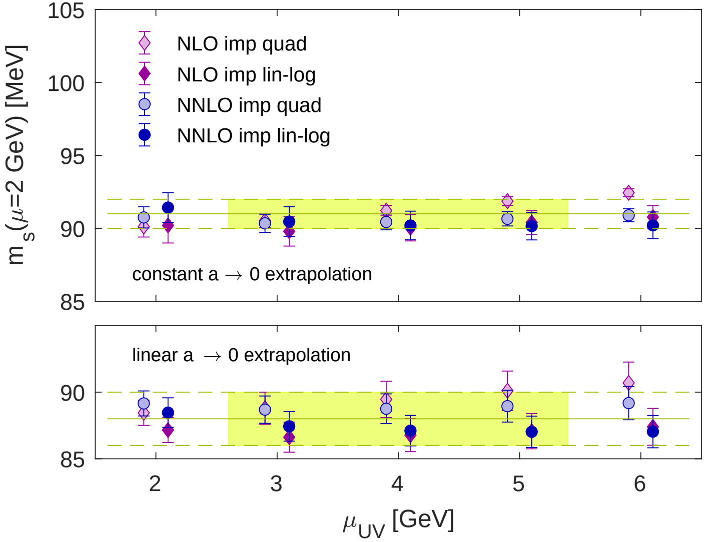
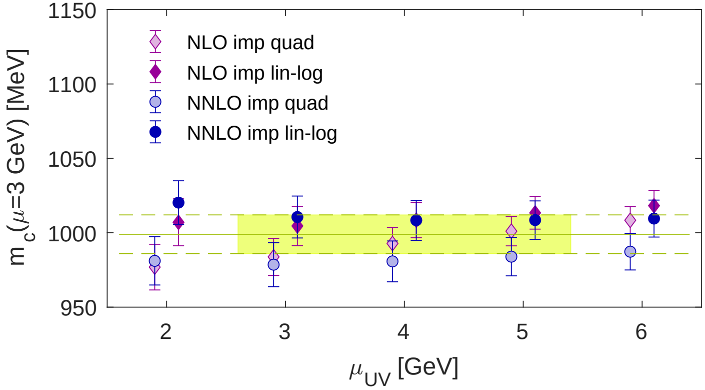


- ➤ Vary $\mu_{\mathrm{UV}} = 2, 3, 4, 5, 6$ GeV
  - ➡ run back to $\mu = 3$ GeV in $\overline{\mathrm{MS}}$ @ 4-loop
- ➤ Take correlated average from results at $\mu_{\mathrm{UV}} = 3, 4, 5$ GeV







- $m_c^{\eta_c}(\mu = 3\,\mathrm{GeV}) = 999(14)\,\mathrm{MeV}$

- $m_s^{\eta_s}(\mu = 2\,\mathrm{GeV}) = 89(3)\,\mathrm{MeV}$
- $(m_c + m_s)/2(\mu = 3\,\mathrm{GeV}) = 527(4)\,\mathrm{MeV}$

- $m_c(\mu = 3\,\mathrm{GeV}) = 974(8)\,\mathrm{MeV}$
- $\dfrac{m_c}{m_s} = 12.1(4)$

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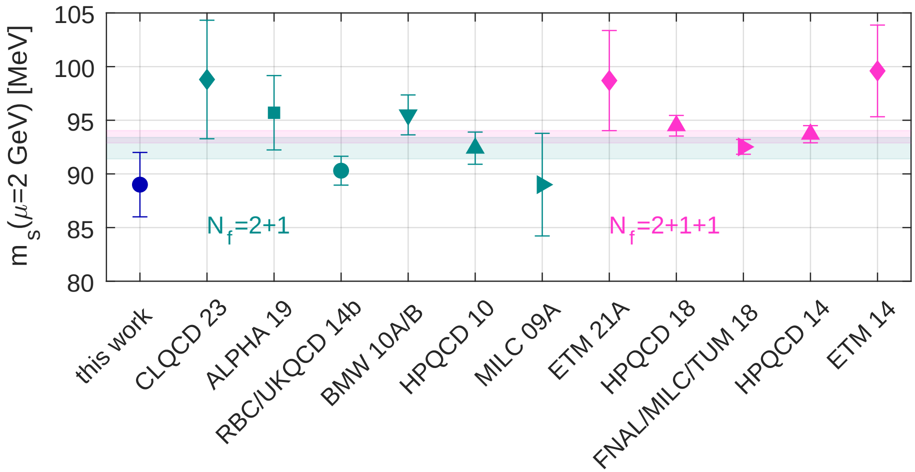

- $m_c^{\eta_c}(\mu = 3\,\mathrm{GeV}) = 999(14)\,\mathrm{MeV}$
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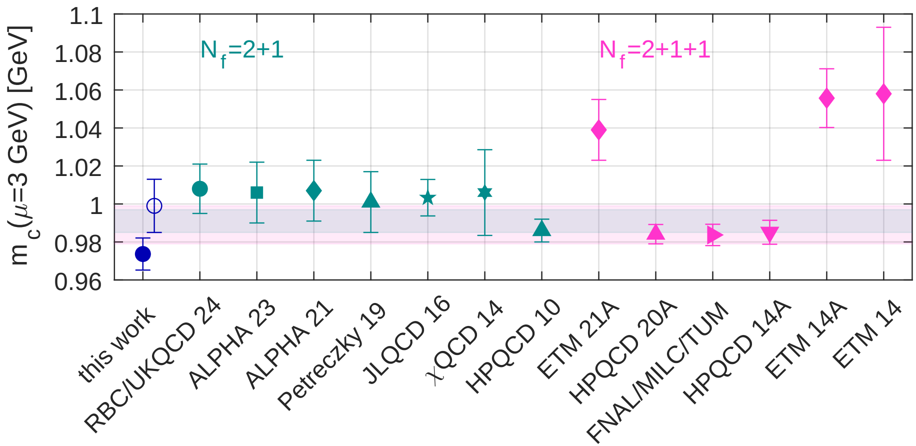

Summary

- GF+SFTX is a powerful concept and we are just starting to leverage its potential
- Simple prescription to obtain renormalised quark masses
  - Automatically a massive renormalisation scheme
  - Improves discretisation effects; no gauge fixing; easily attainable matching window

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Gradient Flow Workshop

- 12^{th} – 15^{th} May, 2026 @ Higgs Centre
- Bringing together experts from both lattice and perturbation theory
- Higgs Colloquium on α_s determinations from Martin Lüscher
- Talks by Robert Harlander, Anna Hasenfratz, Alberto Ramos, more...
- This is a growing topic – come along, learn more, join in!

`https://indico.ph.ed.ac.uk/e/gradientflow26`