One Sum to Rule Them All

Margarita Gavrilova

Based on the work with Y. Grossman, G. Papiri and S. Schacht
arXiv:2602.22320 and ongoing work

Nonleptonic Decays of Heavy Mesons, Mar 24 2026

Caltech

What we will discuss

U-spin ($d \leftrightarrow s$)

Weak Decays

Charm

$$\frac{\text{sum of CF and DCS CKM-free rates}}{\text{sum of SCS CKM-free rates}} = 1 + \mathcal{O}(\varepsilon^2)$$

Goal: understand where the second-order charm master rate sum rule comes from.

What do $\mathcal{O}(\varepsilon)$, $\mathcal{O}(\varepsilon^2)$ mean?

At the fundamental level, U-spin **breaking comes from quark-mass difference**

$$\Delta m \equiv m_s - m_d\,, \qquad \varepsilon \equiv \frac{\Delta m}{(\text{some energy scale})} \sim 30\,\%$$

What do $\mathcal{O}(\varepsilon)$, $\mathcal{O}(\varepsilon^2)$ mean?

At the fundamental level, U-spin **breaking comes from quark-mass difference**

$$\Delta m \equiv m_s - m_d\,, \qquad \varepsilon \equiv \frac{\Delta m}{(\text{some energy scale})} \sim 30\,\%$$

Then, typically, one assumes

$$\mathcal{O}(\varepsilon) \sim 30\,\%\,, \qquad \mathcal{O}(\varepsilon^2) \sim 10\,\%$$

What do $\mathcal{O}(\varepsilon)$, $\mathcal{O}(\varepsilon^2)$ mean?

At the fundamental level, U-spin **breaking comes from quark-mass difference**

$$\Delta m \equiv m_s - m_d\,, \qquad \varepsilon \equiv \frac{\Delta m}{(\text{some energy scale})} \sim 30\,\%$$

Then, typically, one assumes

~~$\mathcal{O}(\varepsilon) \sim 30\,\%\,,$~~ ~~$\mathcal{O}(\varepsilon^2) \sim 10\,\%$~~

Precise meaning **can depend** on the **process** and specific **observable** considered!

In this talk, however,

$\mathcal{O}(\varepsilon) \propto \Delta m$

$\mathcal{O}(\varepsilon^2) \propto \Delta m^2$

Formal Setup

- We assume that all external particles and the weak Hamiltonian transform in irreducible representations of U-spin.

$$I_1 \otimes I_2 \otimes \cdots \otimes I_r \otimes I_H$$

  - $I_1, \ldots, I_r$: irreps in the external states
  - I_H: irrep in the Hamiltonian

Note: mixed states and direct-sum Hamiltonians are not included in this setup.

Formal Setup

- We assume that all external particles and the weak Hamiltonian transform in irreducible representations of U-spin.

$$I_1 \otimes I_2 \otimes \cdots \otimes I_r \otimes I_H$$

  - $I_1, \ldots, I_r$: irreps in the external states
  - I_H: irrep in the Hamiltonian

Note: mixed states and direct-sum Hamiltonians are not included in this setup.

- Channels are labeled by the corresponding m-QNs

$$A(m_1, \ldots, m_r, m_H)$$

Formal Setup

- We assume that all external particles and the weak Hamiltonian transform in irreducible representations of U-spin.

$$I_1 \otimes I_2 \otimes \cdots \otimes I_r \otimes I_H$$

  - $I_1, \ldots, I_r$: irreps in the external states
  - I_H: irrep in the Hamiltonian

Note: mixed states and direct-sum Hamiltonians are not included in this setup.

- Channels are labeled by the corresponding m-QNs

$$A(m_1, \ldots, m_r, m_H)$$

- Under U-spin conjugation ($d \leftrightarrow s$), all m-QNs flip sign:

$$A(m_1, \ldots, m_r, m_H) \longrightarrow A(-m_1, \ldots, -m_r, -m_H)$$

These two channels form a U-spin pair.

Trivial Sum Rules at $\mathcal{O}(\varepsilon)$

Define schematically

$$A_\alpha \equiv A(m_1, \ldots, m_r, m_H), \qquad A_{\overline{\alpha}} \equiv A(-m_1, \ldots, -m_r, -m_H)$$

then in the symmetry limit

$$\Gamma_\alpha = \int d\Pi \, |A_\alpha|^2, \qquad \Gamma_{\overline{\alpha}} = \int d\Pi \, |A_{\overline{\alpha}}|^2$$

Note that in the symmetry limit $d\Pi = d\overline{\Pi}$

Trivial sum rules at $\mathcal{O}(\varepsilon)$

$$A_\alpha = A_{\overline{\alpha}}$$

Gronau, arXiv: hep-ph/0008292

Trivial Sum Rules at $\mathcal{O}(\varepsilon)$

Define schematically

$$A_\alpha \equiv A(m_1, \ldots, m_r, m_H), \qquad A_{\overline{\alpha}} \equiv A(-m_1, \ldots, -m_r, -m_H)$$

then in the symmetry limit

$$\Gamma_\alpha = \int d\Pi \, |A_\alpha|^2, \qquad \Gamma_{\overline{\alpha}} = \int d\Pi \, |A_{\overline{\alpha}}|^2$$

Note that in the symmetry limit
$d\Pi = d\overline{\Pi}$

Trivial sum rules at $\mathcal{O}(\varepsilon)$

$$A_\alpha = A_{\overline{\alpha}} \quad \longrightarrow \quad |A_\alpha|^2 = |A_{\overline{\alpha}}|^2$$

Gronau, arXiv: hep-ph/0008292

Trivial Sum Rules at $\mathcal{O}(\varepsilon)$

Define schematically

$$A_\alpha \equiv A(m_1, \ldots, m_r, m_H), \qquad A_{\overline{\alpha}} \equiv A(-m_1, \ldots, -m_r, -m_H)$$

then in the symmetry limit

$$\Gamma_\alpha = \int d\Pi \, |A_\alpha|^2, \qquad \Gamma_{\overline{\alpha}} = \int d\Pi \, |A_{\overline{\alpha}}|^2$$

Note that in the symmetry limit $d\Pi = d\overline{\Pi}$

Trivial sum rules at $\mathcal{O}(\varepsilon)$

$$A_\alpha = A_{\overline{\alpha}} \longrightarrow |A_\alpha|^2 = |A_{\overline{\alpha}}|^2 \longrightarrow \Gamma_\alpha = \Gamma_{\overline{\alpha}}$$

Gronau, arXiv: hep-ph/0008292

Trivial Sum Rules at $\mathcal{O}(\varepsilon)$

Define schematically

$$A_\alpha \equiv A(m_1, \ldots, m_r, m_H), \qquad A_{\overline{\alpha}} \equiv A(-m_1, \ldots, -m_r, -m_H)$$

then in the symmetry limit

$$\Gamma_\alpha = \int d\Pi \, |A_\alpha|^2, \qquad \Gamma_{\overline{\alpha}} = \int d\Pi \, |A_{\overline{\alpha}}|^2$$

Note that in the symmetry limit $d\Pi = d\overline{\Pi}$

Trivial sum rules at $\mathcal{O}(\varepsilon)$

$$A_\alpha = A_{\overline{\alpha}} \longrightarrow |A_\alpha|^2 = |A_{\overline{\alpha}}|^2 \longrightarrow \Gamma_\alpha = \Gamma_{\overline{\alpha}}$$

Gronau, arXiv: hep-ph/0008292

Are there more (non-trivial) rate sum rules in the symmetry limit?

Shmushkevich method

- developed for symmetry-limit isospin relations in strong hadronic scattering
- derives relations directly for observables $\propto |A|^2$
- no explicit Clebsch-Gordan decomposition

$\bar{N}N \to K\bar{K},2\pi$

$\bar{p}p \to K^+K^-\pi^+\pi^-$
$\bar{p}p \to K^+K^-,2\pi^0$
$\bar{p}p \to K^0\bar{K}^0\pi^+\pi^-$
$\bar{p}p \to K^0\bar{K}^0,2\pi^0$
$\bar{p}p \to K^+\bar{K}^0\pi^-\pi^0$
$\bar{p}p \to K^0K^-\pi^+\pi^0$
$\bar{p}n \to K^+K^-\pi^-\pi^0$
$\bar{p}n \to K^0K^-\pi^+\pi^-$
$\bar{p}n \to K^0K^-,2\pi^0$
$\bar{p}n \to K^0\bar{K}^0\pi^-\pi^0$
$\bar{p}n \to K^+\bar{K}^0,2\pi^-$

PHYSICAL REVIEW VOLUME 140, NUMBER 4B 22 NOVEMBER 1965

Shmushkevich's Method for a Charge-Independent Theory*

G. Pinski†, A. J. Macfarlane, and E. C. G. Sudarshan
Department of Physics, Syracuse University, Syracuse, New York
(Received 7 July 1965)

The Shmushkevich method for deducing consequences of charge independence is explained and discussed. This method, which avoids entirely the use of Clebsch-Gordan coefficients, generates linear equalities and inequalities among cross sections using only a simple counting procedure. A comprehensive list of such relations, applying to most elementary-particle reactions of interest which involve at least one pair of isospin-$\frac{1}{2}$ particles, is presented. A discussion of the various uses of these relations is given.

Charge-Independent Theory
$=$
Isospin-symmetric Theory

pure $SU(2)$ representation theory $\Rightarrow$ apply to U-spin

Shmushkevich method

$$I_1 \otimes \cdots \otimes I_r \otimes I_H$$

Define an observable proportional to the amplitude squared:

$$\Gamma(m_1, \ldots, m_r, m_H) \propto |A(m_1, \ldots, m_r, m_H)|^2$$

Shmushkevich (1955); Dushin, Shmushkevich (1956); Pinski, MacFarlane, Sudarshan (1965)

Shmushkevich method

$$I_1 \otimes \cdots \otimes I_r \otimes I_H$$

Define an observable proportional to the amplitude squared:

$$\Gamma(m_1, \ldots, m_r, m_H) \propto |A(m_1, \ldots, m_r, m_H)|^2$$

Fix one irrep I_i and its m-QN m_i, and sum over all other m-QNs:

$$\Gamma_i(m_i) \equiv \sum_{\{m_j\}_{j \neq i}} \Gamma(m_1, \ldots, m_r, m_H)$$

"Inclusive rate"

Shmushkevich (1955); Dushin, Shmushkevich (1956); Pinski, MacFarlane, Sudarshan (1965)

Shmushkevich method

$$I_1 \otimes \cdots \otimes I_r \otimes I_H$$

Define an observable proportional to the amplitude squared:

$$\Gamma(m_1, \ldots, m_r, m_H) \propto |A(m_1, \ldots, m_r, m_H)|^2$$

Fix one irrep I_i and its m-QN m_i, and sum over all other m-QNs:

$$\Gamma_i(m_i) \equiv \sum_{\{m_j\}_{j \neq i}} \Gamma(m_1, \ldots, m_r, m_H)$$

"Inclusive rate"

Shmushkevich master equation yields $2I_i$ sum rules

$$\Gamma_i(-I_i) = \Gamma_i(-I_i + 1) = \ldots = \Gamma_i(I_i)$$

Shmushkevich (1955); Dushin, Shmushkevich (1956); Pinski, MacFarlane, Sudarshan (1965)

Symmetry Argument

Consider an abstract symmetry-limit U-spin relation

$$\mathcal{S}(m_d, m_s) = 0 \quad \text{at} \quad m_d = m_s = m_{\text{av}}$$

- analytic near $m_d = m_s = m_{\text{av}}$
- symmetric under flavor exchange:

$$\mathcal{S}(m_d, m_s) = \mathcal{S}(m_s, m_d) \qquad (d \leftrightarrow s)$$

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

$$\mathcal{S}(m_d, m_s) = 0 \quad \text{at} \quad m_d = m_s = m_{\text{av}}$$
$$\mathcal{S}(m_d, m_s) = \mathcal{S}(m_s, m_d)$$

Symmetry Argument

Expand around U-spin limit

$$\Delta m \equiv m_s - m_d, \qquad m_d = m_{\text{av}} - \frac{\Delta m}{2}, \qquad m_s = m_{\text{av}} + \frac{\Delta m}{2}$$

$$\mathcal{S}\left(m_{\text{av}} - \frac{\Delta m}{2}, m_{\text{av}} + \frac{\Delta m}{2}\right) = \mathcal{S}(m_{\text{av}}, m_{\text{av}}) + \frac{\Delta m}{2}\left(\partial_{m_s} - \partial_{m_d}\right)\mathcal{S}\Big|_{\text{sym}} + O(\Delta m^2)$$

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

$$\mathcal{S}(m_d, m_s) = 0 \quad \text{at} \quad m_d = m_s = m_{\text{av}}$$
$$\mathcal{S}(m_d, m_s) = \mathcal{S}(m_s, m_d)$$

Symmetry Argument

Expand around U-spin limit

$$\Delta m \equiv m_s - m_d, \qquad m_d = m_{\text{av}} - \frac{\Delta m}{2}, \qquad m_s = m_{\text{av}} + \frac{\Delta m}{2}$$

$$\mathcal{S}\left(m_{\text{av}} - \frac{\Delta m}{2}, m_{\text{av}} + \frac{\Delta m}{2}\right) = \mathcal{S}(m_{\text{av}}, m_{\text{av}}) + \frac{\Delta m}{2}\left(\partial_{m_s} - \partial_{m_d}\right)\mathcal{S}\Big|_{\text{sym}} + O(\Delta m^2)$$

Now, exchange d and s

$$d \leftrightarrow s \qquad \Longrightarrow \qquad \Delta m \to -\Delta m, \qquad \mathcal{S} \to \mathcal{S}$$

by the definition of Δm — by our choice of $\mathcal{S}$

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

$$\mathcal{S}(m_d, m_s) = 0 \quad \text{at} \quad m_d = m_s = m_{\text{av}}$$
$$\mathcal{S}(m_d, m_s) = \mathcal{S}(m_s, m_d)$$

Symmetry Argument

Expand around U-spin limit

$$\Delta m \equiv m_s - m_d, \qquad m_d = m_{\text{av}} - \frac{\Delta m}{2}, \qquad m_s = m_{\text{av}} + \frac{\Delta m}{2}$$

$$\mathcal{S}\left(m_{\text{av}} - \frac{\Delta m}{2}, m_{\text{av}} + \frac{\Delta m}{2}\right) = \mathcal{S}(m_{\text{av}}, m_{\text{av}}) + \frac{\Delta m}{2}\left(\partial_{m_s} - \partial_{m_d}\right)\mathcal{S}\Big|_{\text{sym}} + O(\Delta m^2)$$

$\left(\partial_{m_s} - \partial_{m_d}\right)\mathcal{S}\Big|_{\text{sym}} \to 0$

Now, exchange d and s

$$d \leftrightarrow s \quad \Longrightarrow \quad \Delta m \to -\Delta m, \qquad \mathcal{S} \to \mathcal{S}$$

by the definition of Δm

by our choice of $\mathcal{S}$

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

$$\mathcal{S}(m_d, m_s) = 0 \quad \text{at} \quad m_d = m_s = m_{\text{av}}$$
$$\mathcal{S}(m_d, m_s) = \mathcal{S}(m_s, m_d)$$

Symmetry Argument

Expand around U-spin limit

$$\Delta m \equiv m_s - m_d, \qquad m_d = m_{\text{av}} - \frac{\Delta m}{2}, \qquad m_s = m_{\text{av}} + \frac{\Delta m}{2}$$

$$\mathcal{S}\left(m_{\text{av}} - \frac{\Delta m}{2}, m_{\text{av}} + \frac{\Delta m}{2}\right) = \mathcal{S}(m_{\text{av}}, m_{\text{av}}) + \frac{\Delta m}{2}\left(\partial_{m_s} - \partial_{m_d}\right)\mathcal{S}\Big|_{\text{sym}} + O(\Delta m^2)$$

0

Now, exchange d and s

$$d \leftrightarrow s \quad \Longrightarrow \quad \Delta m \to -\Delta m, \qquad \mathcal{S} \to \mathcal{S}$$

by the definition of Δm — by our choice of $\mathcal{S}$

$$\mathcal{S}(m_d, m_s) = O(\Delta m^2)$$

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

Symmetry Argument

For any $\mathcal{S}(m_d, m_s)$ such that

- $\mathcal{S}(m_d, m_s) = 0 \quad$ at $\quad m_d = m_s = m_{\text{av}}$,
- $\mathcal{S}(m_d, m_s) = \mathcal{S}(m_s, m_d)$,

we have

$$\mathcal{S}(m_d, m_s) = O(\Delta m^2)$$

Any symmetric U-spin-limit relation survives to **second order**

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

$$A_\alpha \equiv A(m_1, \ldots, m_r, m_H), \qquad A_{\overline{\alpha}} \equiv A(-m_1, \ldots, -m_r, -m_H)$$
$$\Gamma_\alpha = \int d\Pi\, |A_\alpha|^2, \qquad \Gamma_{\overline{\alpha}} = \int d\Pi\, |A_{\overline{\alpha}}|^2$$

Recipe for rate sum rules

For a decay channel α and its U-spin-conjugate channel $\overline{\alpha}$

$$\Gamma_{\overline{\alpha}}(m_d, m_s) = \Gamma_\alpha(m_s, m_d)$$

Note that for these combinations of rates to be symmetric, the rates have to be **normalized by CKM**

So we can build a symmetric combination

$$\Sigma_\alpha(m_d, m_s) \equiv \Gamma_\alpha(m_d, m_s) + \Gamma_{\overline{\alpha}}(m_d, m_s) = \Gamma_\alpha(m_d, m_s) + \Gamma_\alpha(m_s, m_d)$$

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

$$A_\alpha \equiv A(m_1, \ldots, m_r, m_H), \qquad A_{\overline{\alpha}} \equiv A(-m_1, \ldots, -m_r, -m_H)$$
$$\Gamma_\alpha = \int d\Pi\, |A_\alpha|^2, \qquad \Gamma_{\overline{\alpha}} = \int d\Pi\, |A_{\overline{\alpha}}|^2$$

Recipe for rate sum rules

For a decay channel α and its U-spin-conjugate channel $\overline{\alpha}$

$$\Gamma_{\overline{\alpha}}(m_d, m_s) = \Gamma_\alpha(m_s, m_d)$$

Note that for these combinations of rates to be symmetric, the rates have to be **normalized by CKM**

So we can build a symmetric combination

$$\Sigma_\alpha(m_d, m_s) \equiv \Gamma_\alpha(m_d, m_s) + \Gamma_{\overline{\alpha}}(m_d, m_s) = \Gamma_\alpha(m_d, m_s) + \Gamma_\alpha(m_s, m_d)$$

Therefore, **if a symmetry-limit sum rule is built from such combinations**, it is symmetric under $d \leftrightarrow s$, and **hence holds to second order**.

$$\sum_\alpha c_\alpha \Sigma_\alpha = \mathcal{O}(\varepsilon) \quad \Rightarrow \quad \sum_\alpha c_\alpha \Sigma_\alpha = \mathcal{O}(\varepsilon^2)$$

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

Shmushkevich + symmetry argument

$$I_1 \otimes \cdots \otimes I_r \otimes I_H$$

Fix one irrep I_i and its m-QN m_i, and sum over all other m-QNs: $\Gamma_i(m_i) \equiv \sum_{\{m_j\}_{j\neq i}} \Gamma(m_1, \ldots, m_r, m_H)$

Then for each I_i we can write $2I_i$ Shmushkevich sum rules:

$$\Gamma_i(-I_i) = \Gamma_i(-I_i + 1) = \ldots = \Gamma_i(I_i - 1) = \Gamma_i(I_i)$$

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

Shmushkevich + symmetry argument

$$I_1 \otimes \cdots \otimes I_r \otimes I_H$$

Fix one irrep I_i and its m-QN m_i, and sum over all other m-QNs: $\Gamma_i(m_i) \equiv \sum_{\{m_j\}_{j\neq i}} \Gamma(m_1, \ldots, m_r, m_H)$

Then for each I_i we can write $2I_i$ Shmushkevich sum rules:

$$\Gamma_i(-I_i) = \Gamma_i(-I_i+1) = \ldots = \Gamma_i(I_i-1) = \Gamma_i(I_i)$$

Take combinations symmetric under $d \leftrightarrow s$ exchange

$$\boxed{\Gamma(I_i) + \Gamma(-I_i),} \qquad \boxed{\Gamma(I_i-1) + \Gamma(-I_i+1),} \qquad \ldots$$

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

Shmushkevich + symmetry argument

$$I_1 \otimes \cdots \otimes I_r \otimes I_H$$

Fix one irrep I_i and its m-QN m_i, and sum over all other m-QNs: $\Gamma_i(m_i) \equiv \sum_{\{m_j\}_{j\neq i}} \Gamma(m_1, \ldots, m_r, m_H)$

Then for each I_i we can write $2I_i$ Shmushkevich sum rules:

$$\Gamma_i(-I_i) = \Gamma_i(-I_i+1) = \ldots = \Gamma_i(I_i-1) = \Gamma_i(I_i)$$

Take combinations symmetric under $d \leftrightarrow s$ exchange

$$\boxed{\Gamma(I_i)+\Gamma(-I_i),} \qquad \boxed{\Gamma(I_i-1)+\Gamma(-I_i+1),} \qquad \ldots$$

Therefore, for integer $I_i > 1/2$: $\Gamma(I_i)+\Gamma(-I_i) = \Gamma(I_i-1)+\Gamma(-I_i+1) = \cdots = 2\Gamma(0) + O(\varepsilon^2)$

While, for half-integer $I_i > 1/2$: $\Gamma(I_i)+\Gamma(-I_i) = \Gamma(I_i-1)+\Gamma(-I_i+1) = \cdots = \Gamma\left(\frac{1}{2}\right)+\Gamma\left(-\frac{1}{2}\right) + O(\varepsilon^2)$

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

Shmushkevich + symmetry argument

$$I_1 \otimes \cdots \otimes I_r \otimes I_H$$

Fix one irrep I_i and its m-QN m_i, and sum over all other m-QNs: $\Gamma_i(m_i) \equiv \sum_{\{m_j\}_{j\neq i}} \Gamma(m_1, \ldots, m_r, m_H)$

Then for each I_i we can write $2I_i$ Shmushkevich sum rules:

$$\Gamma_i(-I_i) = \Gamma_i(-I_i+1) = \ldots = \Gamma_i(I_i-1) = \Gamma_i(I_i)$$

Take combinations symmetric under $d \leftrightarrow s$ exchange

$$\Gamma(I_i) + \Gamma(-I_i), \qquad \Gamma(I_i-1) + \Gamma(-I_i+1), \qquad \ldots$$

Therefore, for integer $I_i > 1/2$: $\Gamma(I_i) + \Gamma(-I_i) = \Gamma(I_i-1) + \Gamma(-I_i+1) = \cdots = 2\Gamma(0) + O(\varepsilon^2)$

While, for half-integer $I_i > 1/2$: $\Gamma(I_i) + \Gamma(-I_i) = \Gamma(I_i-1) + \Gamma(-I_i+1) = \cdots = \Gamma\left(\frac{1}{2}\right) + \Gamma\left(-\frac{1}{2}\right) + O(\varepsilon^2)$

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

Hadronic charm decays

Up to relative corrections of $\mathcal{O}(\lambda^4)$, the weak charm Hamiltonian is a U-spin triplet

$$\mathscr{H}_{\text{eff}}^{U-\text{spin limit}} = f_{1,1}H_1^1 + f_{1,0}H_0^1 + f_{1,-1}H_{-1}^1$$

$$H_1^1 = (\bar{u}s)(\bar{d}c), \qquad H_0^1 = \frac{(\bar{u}s)(\bar{s}c) - (\bar{u}d)(\bar{d}c)}{\sqrt{2}}, \qquad H_{-1}^1 = -(\bar{u}d)(\bar{s}c)$$

$$f_{1,1} = V_{cd}^* V_{us} \sim \lambda^2 \qquad f_{1,0} = \frac{V_{cs}^* V_{us} - V_{cd}^* V_{ud}}{\sqrt{2}} \sim \lambda\ , \qquad f_{1,-1} = -V_{cs}^* V_{ud} \sim -1$$

(DCS) (SCS) (CF)

Hadronic charm decays

Up to relative corrections of $\mathcal{O}(\lambda^4)$, the weak charm Hamiltonian is a U-spin triplet

$$\mathcal{H}_{\text{eff}}^{U-\text{spin limit}} = f_{1,1}H_1^1 + f_{1,0}H_0^1 + f_{1,-1}H_{-1}^1$$

$$H_1^1 = (\bar{u}s)(\bar{d}c), \qquad H_0^1 = \frac{(\bar{u}s)(\bar{s}c) - (\bar{u}d)(\bar{d}c)}{\sqrt{2}}, \qquad H_{-1}^1 = -(\bar{u}d)(\bar{s}c)$$

$$f_{1,1} = V_{cd}^* V_{us} \sim \lambda^2 \qquad f_{1,0} = \frac{V_{cs}^* V_{us} - V_{cd}^* V_{ud}}{\sqrt{2}} \sim \lambda\ , \qquad f_{1,-1} = -V_{cs}^* V_{ud} \sim -1$$

(DCS) (SCS) (CF)

Therefore all weak hadronic charm decay systems have the group-theoretical structure:

$$(I_1 \otimes \cdots \otimes I_r) \otimes I_H, \qquad I_H = 1$$

→ can apply the $\mathcal{O}(\varepsilon^2)$ master sum rule to I_H

one $\mathcal{O}(\varepsilon^2)$ sum rule

all hadronic charm decays

One Sum to Rule Them All

$$(I_1 \otimes \cdots \otimes I_r) \otimes I_H$$

Master sum rule for integer $I_i > 1/2$:

$$\hat{\Gamma}(I_i) + \hat{\Gamma}(-I_i) = \hat{\Gamma}(I_i - 1) + \hat{\Gamma}(-I_i + 1) = \cdots = 2\hat{\Gamma}(0) + O(\varepsilon^2)$$

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

one $\mathcal{O}(\varepsilon^2)$ sum rule

all hadronic charm decays

One Sum to Rule Them All

$$(I_1 \otimes \cdots \otimes I_r) \otimes I_H$$

Master sum rule for integer $I_i > 1/2$:

$$\hat{\Gamma}(I_i) + \hat{\Gamma}(-I_i) = \hat{\Gamma}(I_i - 1) + \hat{\Gamma}(-I_i + 1) = \cdots = 2\hat{\Gamma}(0) + O(\varepsilon^2)$$

Apply the $\mathcal{O}(\varepsilon^2)$ master sum rule to the Hamiltonian triplet, $I_H = 1$

$$\hat{\Gamma}_H(1) + \hat{\Gamma}_H(-1) = 2 \times \frac{1}{2}\hat{\Gamma}_H(0) + \mathcal{O}(\epsilon^2)$$

(DCS) (CF) (SCS)

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

one $\mathcal{O}(\varepsilon^2)$ sum rule

all hadronic charm decays

One Sum to Rule Them All

$$(I_1 \otimes \cdots \otimes I_r) \otimes I_H$$

Master sum rule for integer $I_i > 1/2$:

$$\hat{\Gamma}(I_i) + \hat{\Gamma}(-I_i) = \hat{\Gamma}(I_i - 1) + \hat{\Gamma}(-I_i + 1) = \cdots = 2\hat{\Gamma}(0) + O(\varepsilon^2)$$

Apply the $\mathcal{O}(\varepsilon^2)$ master sum rule to the Hamiltonian triplet, $I_H = 1$

CKM-normalized rates

$$\hat{\Gamma}_H(1) + \hat{\Gamma}_H(-1) = 2 \times \frac{1}{2}\hat{\Gamma}_H(0) + \mathcal{O}(\epsilon^2)$$

(DCS) (CF) (SCS)

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

one $\mathcal{O}(\varepsilon^2)$ sum rule

all hadronic charm decays

One Sum to Rule Them All

$$\frac{\text{sum of CF and DCS CKM-free rates}}{\text{sum of SCS CKM-free rates}} = 1 + \mathcal{O}(\epsilon^2)$$

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

Testing the Sum Rules

- symmetry-limit test

$$\frac{\hat{\Gamma}(i \to f)}{\hat{\Gamma}(\bar{i} \to \bar{f})} = 1 + O(\epsilon)$$

- second-order test

$$R_H(U - \text{spin system}) \equiv \frac{\sum_{\text{CF,DCS}} \hat{\Gamma}}{\sum_{\text{SCS}} \hat{\Gamma}} = 1 + \text{O}(\epsilon^2)$$

Fully measured systems

$D^0 \to P^- P^+$

Decay	BR	CKM	Conjugate decay	BR	CKM
$D^0 \to \pi^-\pi^+$	$(1.453 \pm 0.024) \cdot 10^{-3}$	SCS	$D^0 \to K^-K^+$	$(4.08 \pm 0.06) \cdot 10^{-3}$	SCS
$D^0 \to \pi^-K^+$	$(1.362 \pm 0.025) \cdot 10^{-4}$	DCS	$D^0 \to K^-\pi^+$	$(3.945 \pm 0.030)\%$	CF

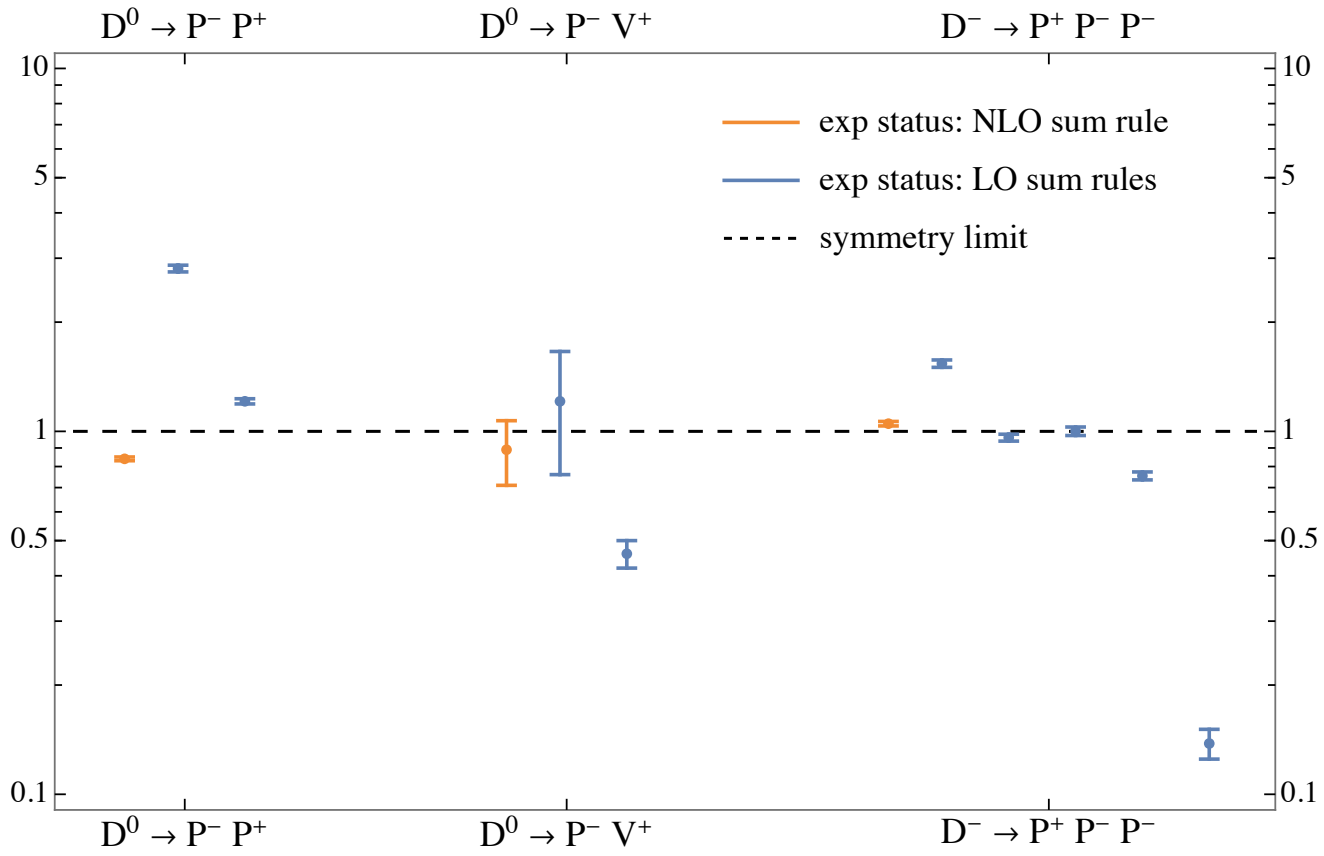
$D^0 \to P^- V^+$

Decay	BR	CKM	Conjugate decay	BR	CKM
$D^0 \to K^-\rho^+$	$(11.2 \pm 0.7)\%$	CF	$D^0 \to \pi^-K^{*+}$	$3 \times (1.28 \pm 0.47) \cdot 10^{-4}$ †	DCS
$D^0 \to K^-K^{*+}$	$3 \times (1.55 \pm 0.10) \cdot 10^{-3}$ $(S = 1.3)$ †	SCS	$D^0 \to \pi^-\rho^+$	$(1.01 \pm 0.05)\%$	SCS

$D^- \to P^+ P^- P^-$

Decay	BR	CKM	Conjugate decay	BR	CKM
$D^- \to \pi^+K^-\pi^-$	$(4.91 \pm 0.09) \cdot 10^{-4}$	DCS	$D_s^- \to K^+\pi^-K^-$	$(5.45 \pm 0.08) \cdot 10^{-2}$	CF
$D^- \to K^+K^-K^-$	$(6.14 \pm 0.11) \cdot 10^{-5}$	DCS	$D_s^- \to \pi^+\pi^-\pi^-$	$(1.090 \pm 0.014) \cdot 10^{-2}$	CF
$D^- \to K^+\pi^-\pi^-$	$(9.38 \pm 0.16) \cdot 10^{-2}$	CF	$D_s^- \to \pi^+K^-K^-$	$(1.293 \pm 0.027) \cdot 10^{-4}$	DCS
$D^- \to K^+K^-\pi^-$	$(9.68 \pm 0.18) \cdot 10^{-3}$	SCS	$D_s^- \to \pi^+\pi^-K^-$	$(6.23 \pm 0.10) \cdot 10^{-3}$	SCS
$D^- \to \pi^+\pi^-\pi^-$	$(3.27 \pm 0.09) \cdot 10^{-3}$	SCS	$D_s^- \to K^+K^-K^-$	$(2.18 \pm 0.20) \cdot 10^{-4}$	SCS

Fully measured systems



Grossman & Robinson (2013)

MG, Y. Grossman, S. Schacht, arXiv:2602.22320

Systems with missing data

$D^0 \to V^- P^+$

$D^0 \to K^{*-}\pi^+$	$3 \times (1.82 \pm 0.30)\%$ $(S = 2.2)$ †	CF	$D^0 \to \rho^- K^+$	$< (3.06 \pm 0.16) \cdot 10^{-4}$ ‡	DCS
$D^0 \to K^{*-}K^+$	$3 \times (5.54 \pm 0.37) \cdot 10^{-4}$ †	SCS	$D^0 \to \rho^- \pi^+$	$(5.15 \pm 0.26) \cdot 10^{-3}$	SCS

Second-order master sum rule allows to derive prediction for the missing BR: $\mathscr{B}(D^0 \to \rho^- K^+) = (2.08 \pm 0.30) \times 10^{-4}$

$C_b^+ \to L_b^+ P^+ P^-$

Decay	BR	CKM	Conjugate decay	BR	CKM
$\Lambda_c^+ \to \Sigma^+ K^- K^+$	$(3.6 \pm 0.4) \cdot 10^{-3}$	CF	$\Xi_c^+ \to p\pi^-\pi^+$	?	DCS
$\Lambda_c^+ \to \Sigma^+ \pi^- \pi^+$	$(4.54 \pm 0.20)\%$	CF	$\Xi_c^+ \to pK^-K^+$	?	DCS
$\Lambda_c^+ \to \Sigma^+ \pi^- K^+$	$(2.04 \pm 0.26) \cdot 10^{-3}$	SCS	$\Xi_c^+ \to pK^-\pi^+$	$(6.2 \pm 3.0) \cdot 10^{-3}$	SCS
$\Lambda_c^+ \to pK^-\pi^+$	$(6.35 \pm 0.25)\%$	CF	$\Xi_c^+ \to \Sigma^+\pi^-K^+$	?	DCS
$\Lambda_c^+ \to pK^-K^+$	$(1.08 \pm 0.05) \cdot 10^{-3}$	SCS	$\Xi_c^+ \to \Sigma^+\pi^-\pi^+$	$(1.4 \pm 0.8)\%$	SCS
$\Lambda_c^+ \to p\pi^-\pi^+$	$(4.67 \pm 0.24) \cdot 10^{-3}$	SCS	$\Xi_c^+ \to \Sigma^+K^-K^+$	$(4.3 \pm 2.5) \cdot 10^{-3}$	SCS
$\Lambda_c^+ \to p\pi^-K^+$	$(1.13 \pm 0.17) \cdot 10^{-4}$	DCS	$\Xi_c^+ \to \Sigma^+K^-\pi^+$	$(2.7 \pm 1.2)\%$	CF

Systems with missing data

Decay	BR	CKM	Conjugate decay	BR	CKM
$\Xi_c^0 \to \Xi^- \pi^+$	$(1.43 \pm 0.27)\%$	CF	$\Xi_c^0 \to \Sigma^- K^+$	?	DCS
$\Xi_c^0 \to \Sigma^- \pi^+$	?	SCS	$\Xi_c^0 \to \Xi^- K^+$	$(3.9 \pm 1.1) \cdot 10^{-4}$	SCS
$\Xi_c^0 \to p K^-$	?	SCS	$\Xi_c^0 \to \Sigma^+ \pi^-$	?	SCS
$\Xi_c^0 \to \Sigma^+ K^-$	$(1.8 \pm 0.4) \cdot 10^{-3}$	CF	$\Xi_c^0 \to p \pi^-$	?	DCS
$\Xi_c^0 \to \Delta^- \pi^+$	?	DCS	$\Xi_c^0 \to \Omega^- K^+$	?	CF
$\Xi_c^0 \to \Sigma^{*-} \pi^+$	?	SCS	$\Xi_c^0 \to \Xi^{*-} K^+$	?	SCS
$\Xi_c^0 \to \Xi^{*-} \pi^+$	?	CF	$\Xi_c^0 \to \Sigma^{*-} K^+$	?	DCS

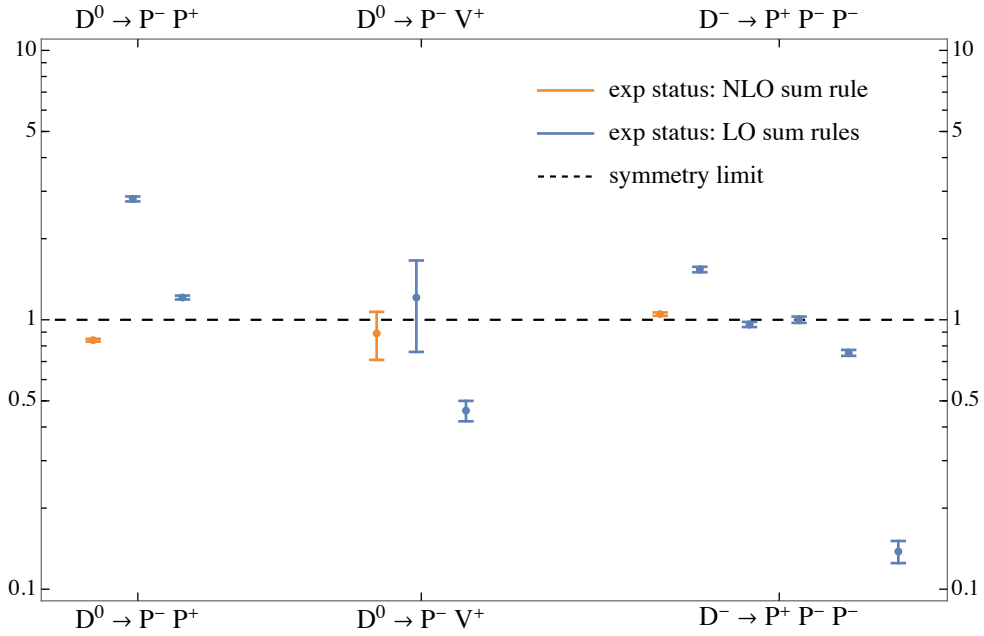
$\frac{1}{2} \otimes \frac{3}{2} \otimes 1$

Two irreps $> 1/2 \longrightarrow$ **two $\mathcal{O}(\varepsilon^2)$ sum rules!**

Takeaway

- Second-order master sum rule in charm follows from a few **simple, general observations**
- These ideas can be applied **beyond charm, beyond U-spin, beyond weak decays**
- In data **second-order sum rules can work strikingly well**, even when symmetry-limit relations appear badly broken

$$\frac{\text{sum of CF and DCS CKM-free rates}}{\text{sum of SCS CKM-free rates}} = 1 + \mathcal{O}(\varepsilon^2)$$



Outlook

- Accumulate more second-order sum rules
  → study patterns in symmetry breaking
- Learn from data
  → possible probes of underlying dynamics, perhaps even new physics
- Crucial open question: what do $\mathcal{O}(\varepsilon)$, $\mathcal{O}(\varepsilon^2)$ actually mean?