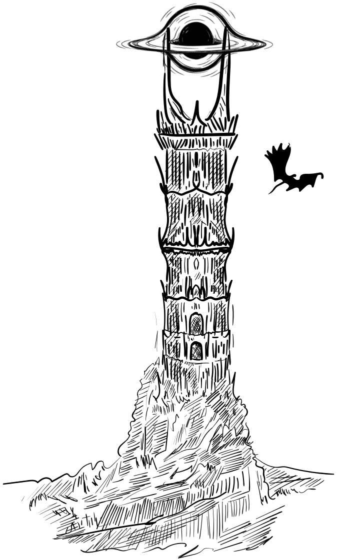
Extracting zero-damped modes using exact WKB

...in collaboration with Prisco Lo Chiatto,
Sebastian Schenk,
and Felix Yu

MAX-PLANCK-INSTITUT
FÜR PHYSIK

Effective Theories for Nonperturbative Physics, IPPP
Nils Wagner (TU Munich) 1^{st} September, 2026

Outline of the talk



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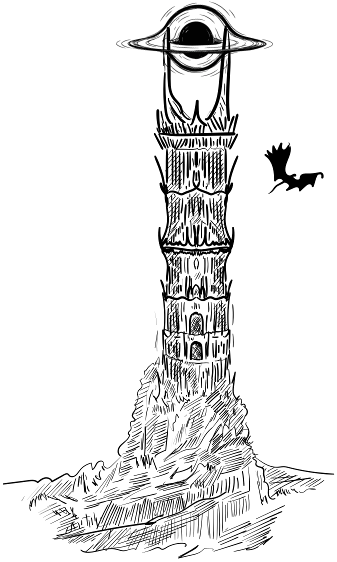
1 What are "zero-damped modes"?

- *Motivation: Black hole mergers*
- *QNMs & ZDMs*
- *Scalar perturbations in a RN background*



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   - *Connection formulae*
   - *The harmonic oscillator reloaded*

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- *Exact ZDM quantization condition*
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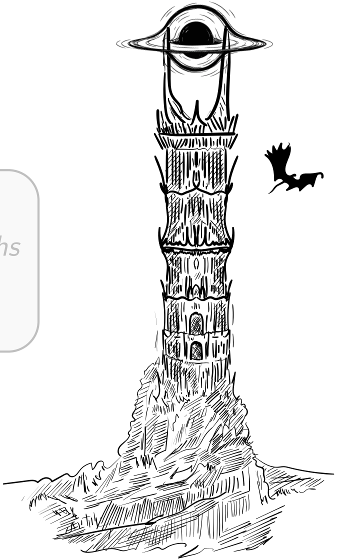

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LIGO & Virgo Collaboration (2016), *Phys. Rev. Lett.* *116*(6), FIG. 2

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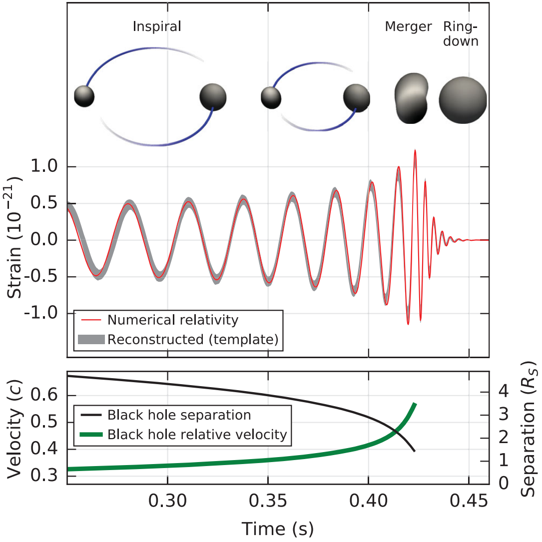

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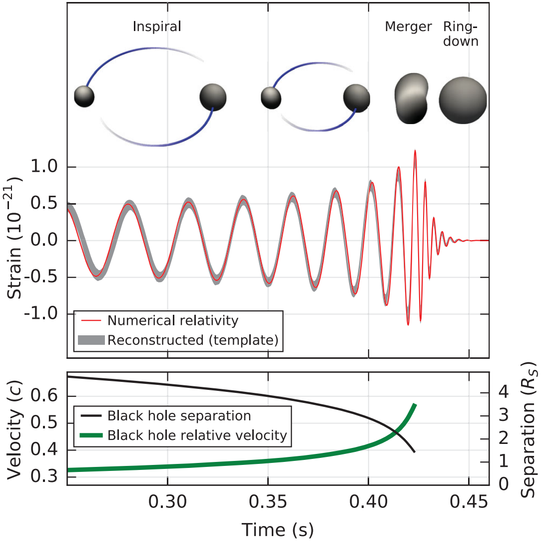

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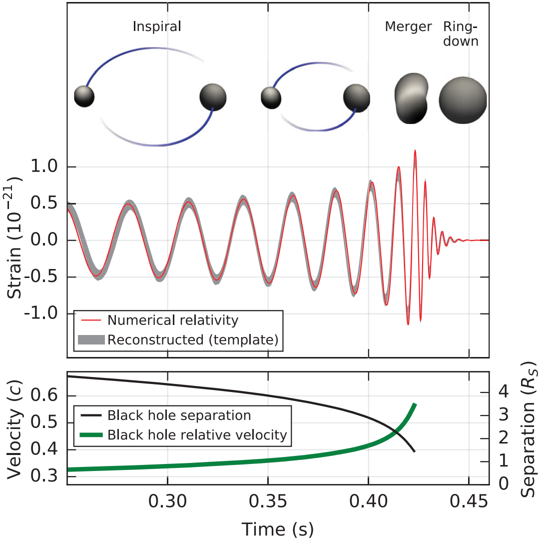

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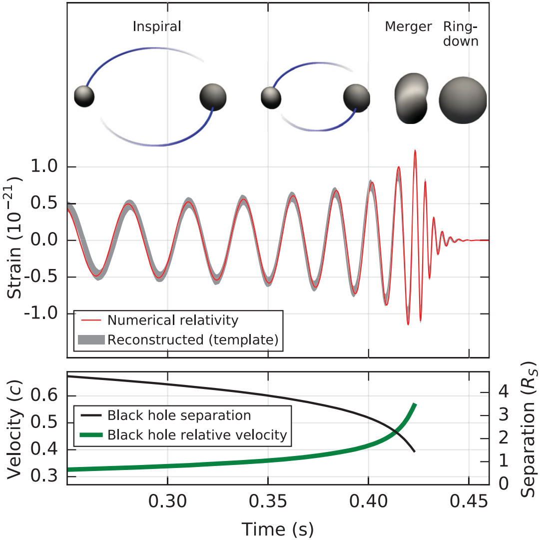

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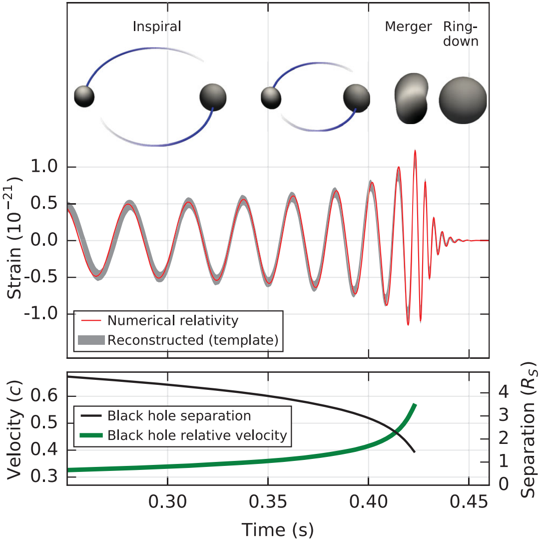
Upshot: Final ringdown phase directly probes the characteristics of the remnant BH



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- Instate physical boundary conditions to obtain the **discrete eigenmodes ω**

Sketch of BH perturbation theory

Regge & Wheeler (1957), *Phys. Rev. 108*
Teukolsky (1973), *Astrophys. J. 185*

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Upshot

The characteristic frequencies ω encode the properties of the background BH

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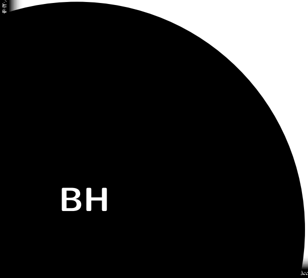

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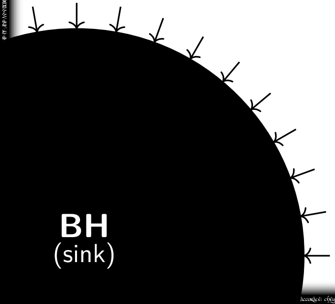

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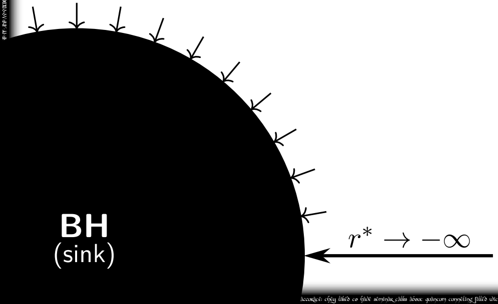

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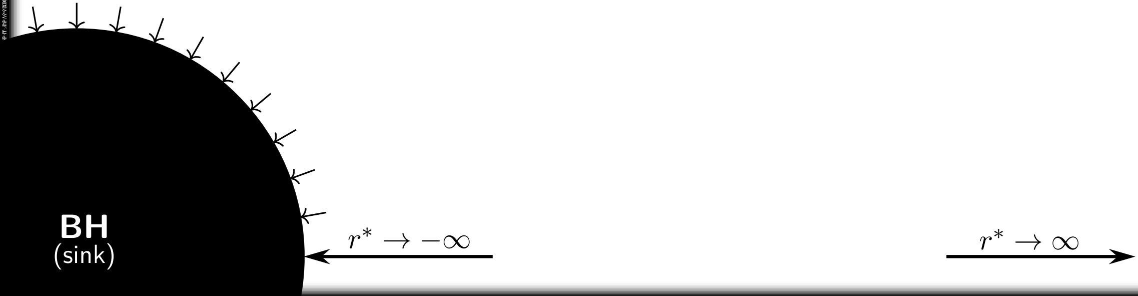

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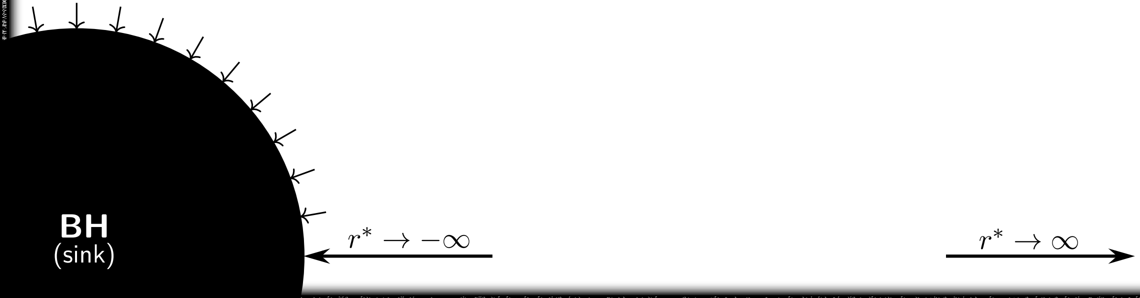

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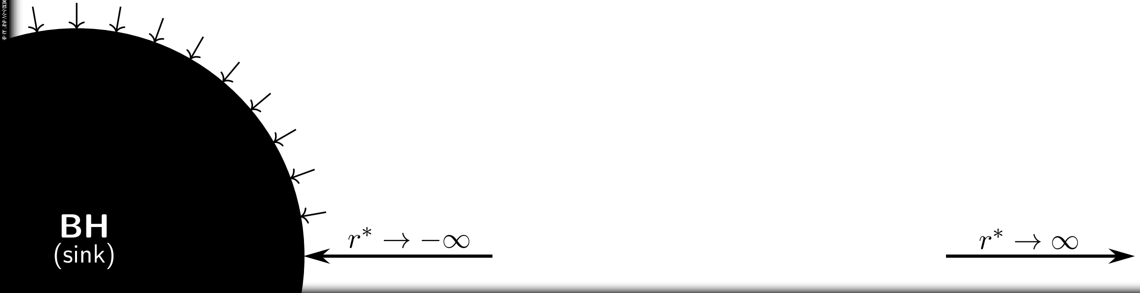

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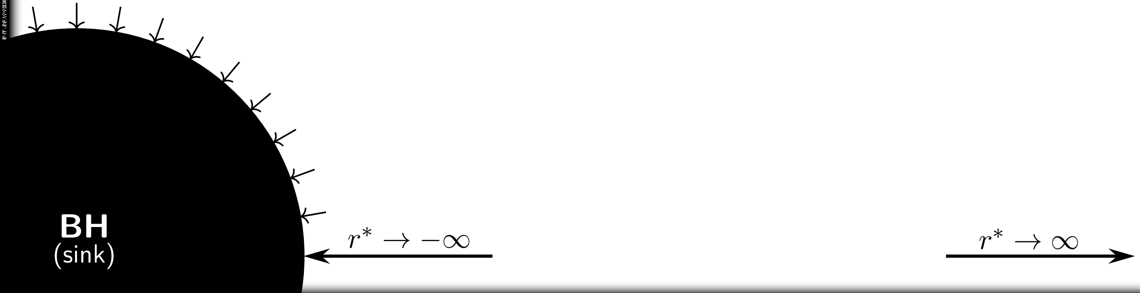

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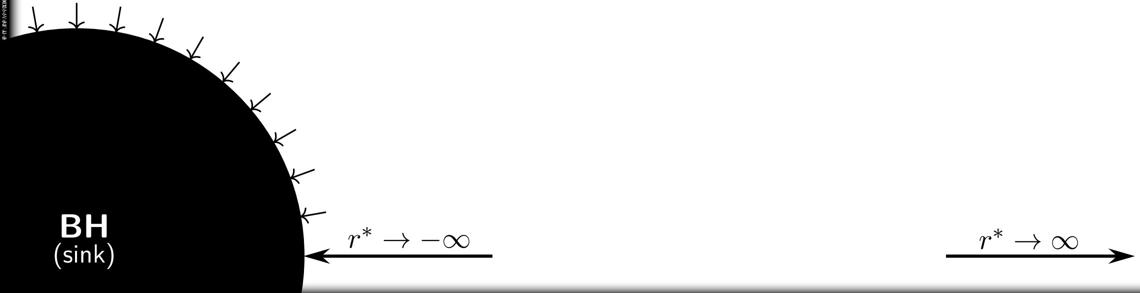

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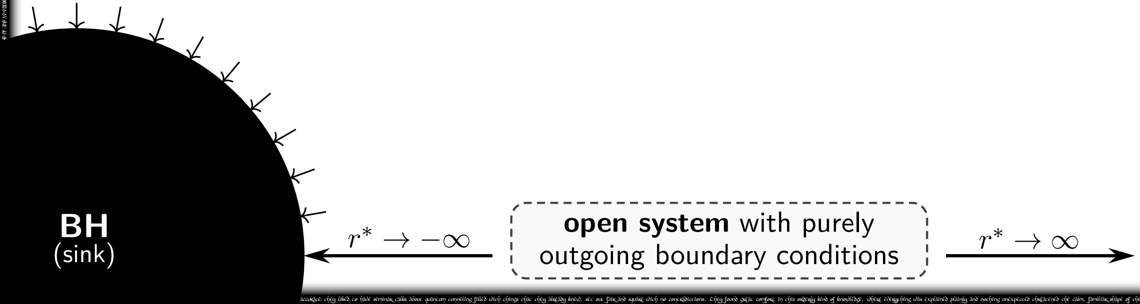

Boundary conditions & QNMs

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The **quasinormal modes** $\omega = \omega_{\mathrm{R}} - i\omega_{\mathrm{I}}$ are **complex**, as the eigenvalue problem is **non-Hermitian**.

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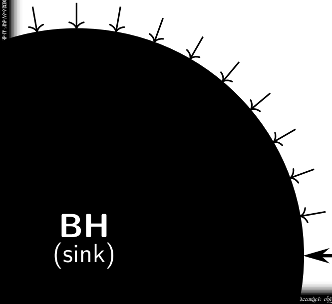

$\longleftarrow r^* \to -\infty$ **open system** with purely outgoing boundary conditions $r^* \to \infty \longrightarrow$

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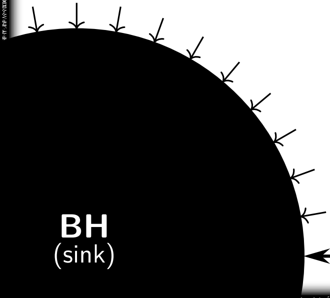

Physical interpretation of $\omega_{\mathrm{R/I}}$:

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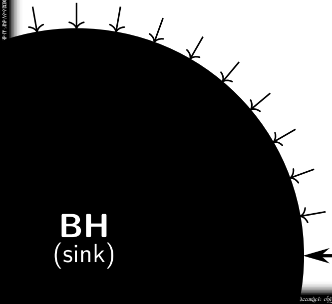

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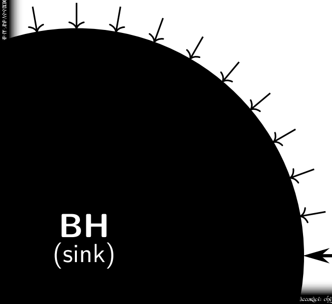

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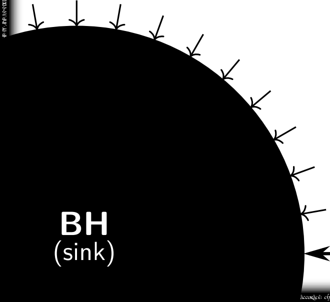

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Near extremality, the QNM spectrum develops interesting features...

More on QNM spectra

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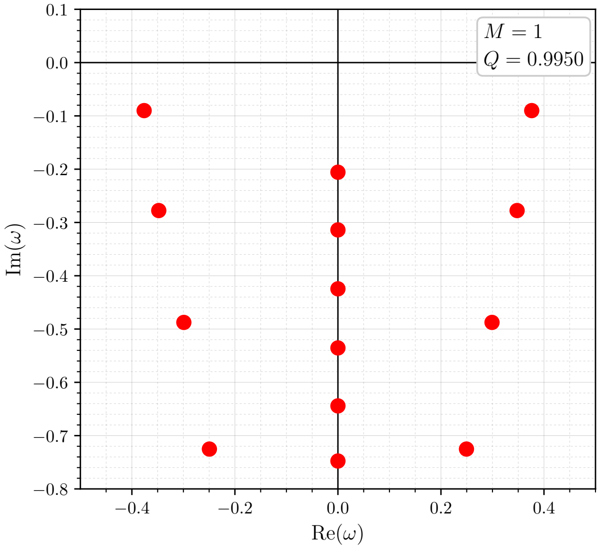

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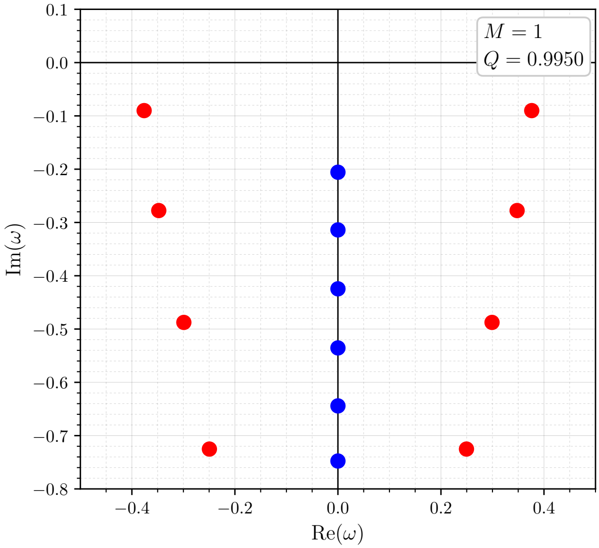

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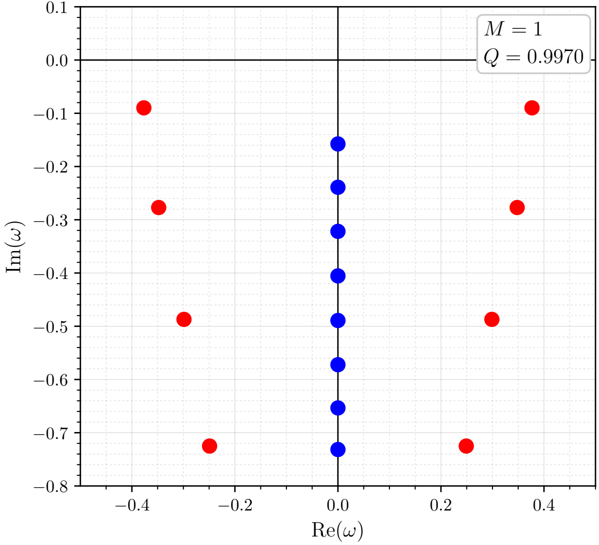

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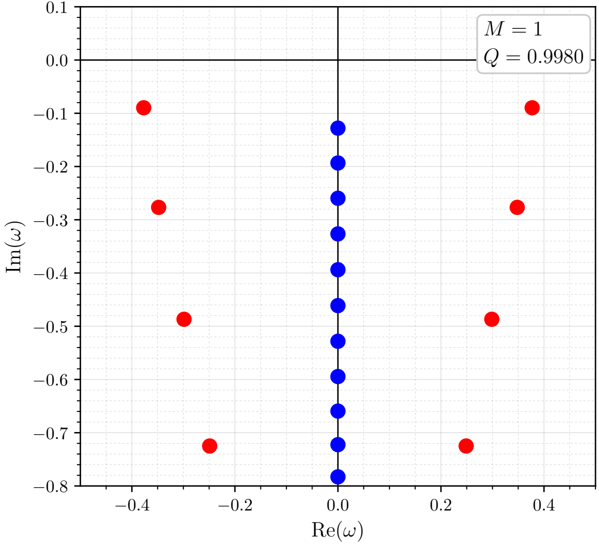

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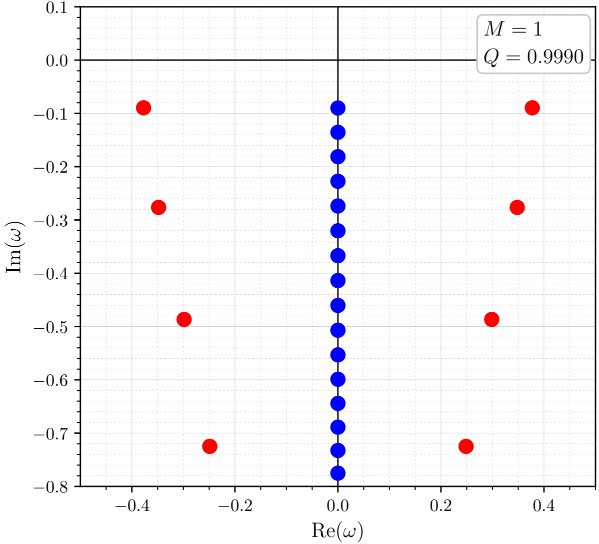

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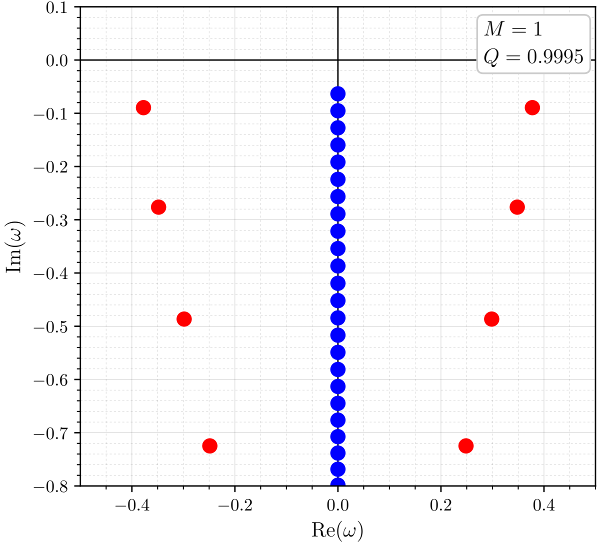

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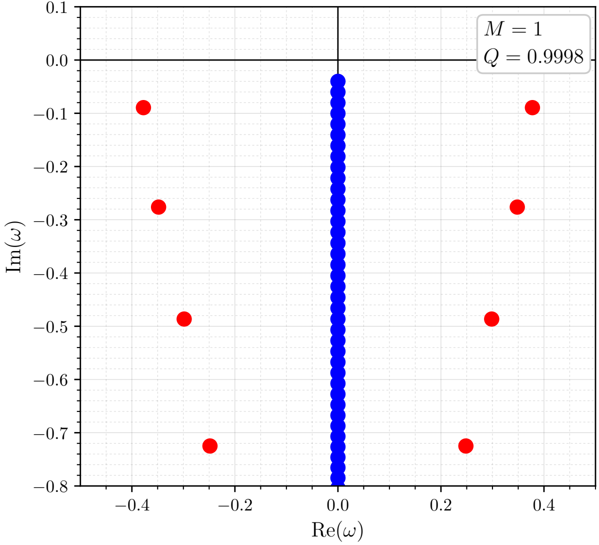

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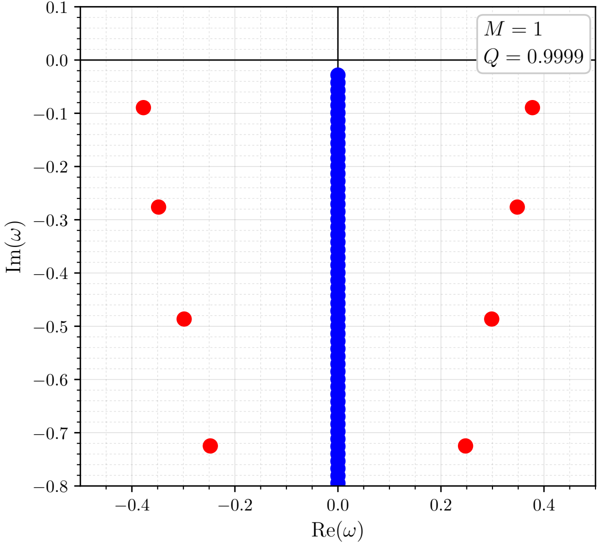

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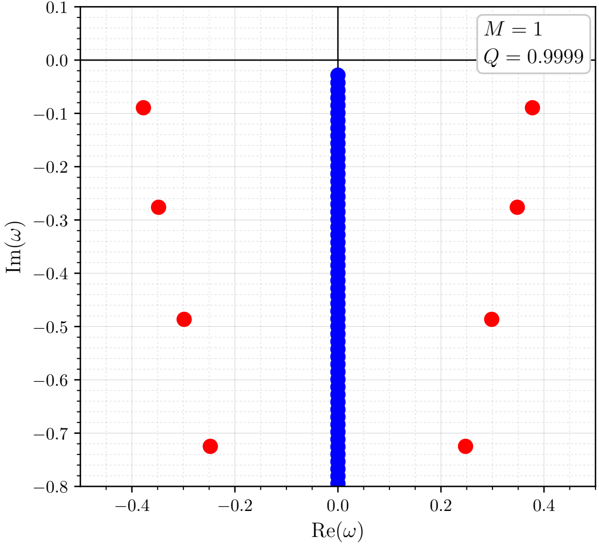

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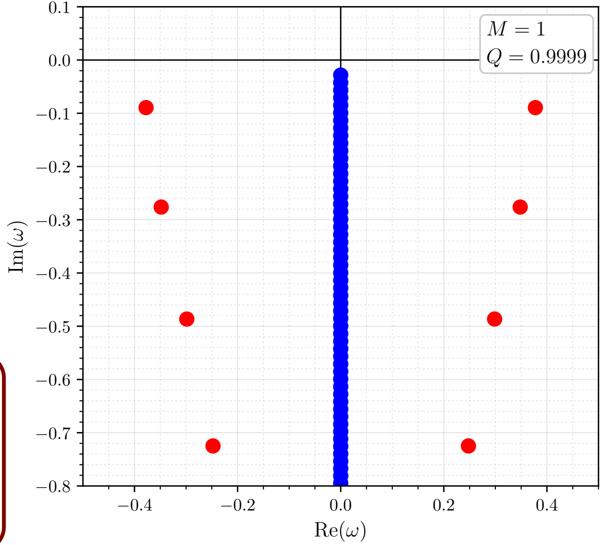

ZDMs

Zero-damped modes, characterized by $\mathrm{Im}(\omega) \to 0$, are very **long-lived** perturbations that influence the late-time ringdown.

Scalar perturbations in a near-extremal RN background

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Outline of the talk

1. **What are "zero-damped modes"?**
   - *Motivation: Black hole mergers*
   - *QNMs & ZDMs*
   - *Scalar perturbations in a RN background*
2. **Exact WKB in a Nutshell**
   - *Formal WKB solutions & Stokes graphs*
   - *Connection formulae*
   - *The harmonic oscillator reloaded*
3. **Applying EWKB to the RN case**
   - *Exact ZDM quantization condition*
   - *Determining the Voros symbols*
   - *ZDM spectrum at NNLO*
4. **Summary**

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“Formal” WKB solutions:

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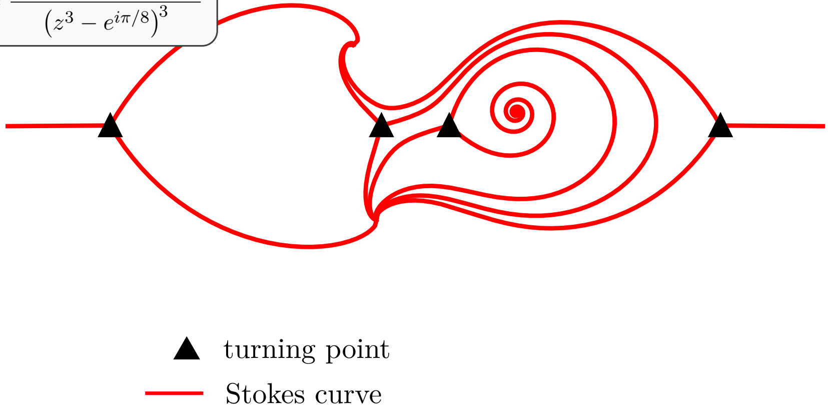

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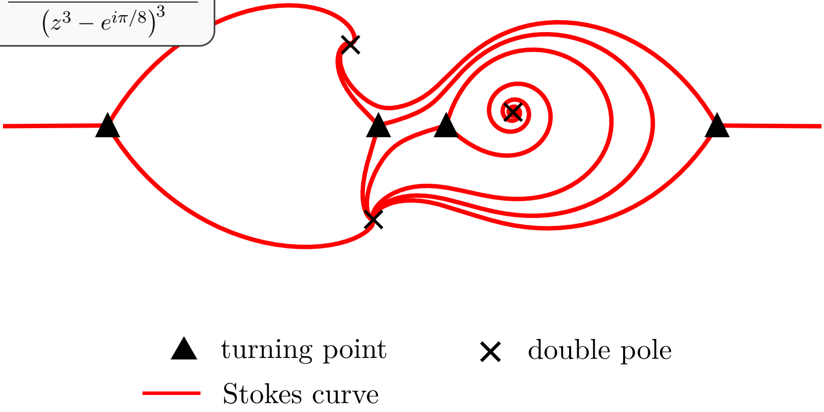

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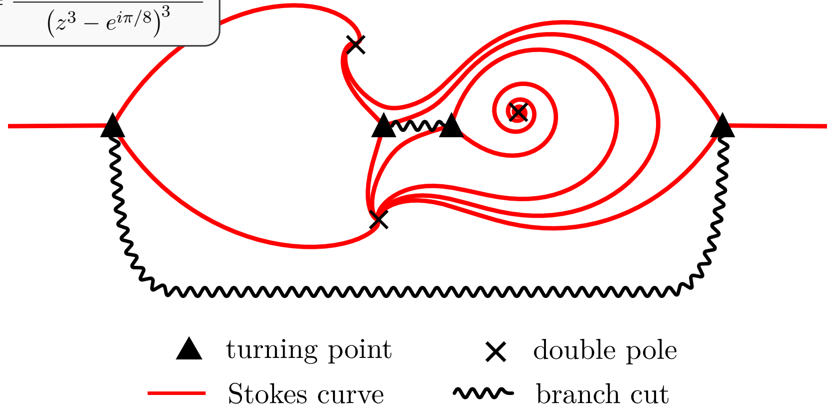

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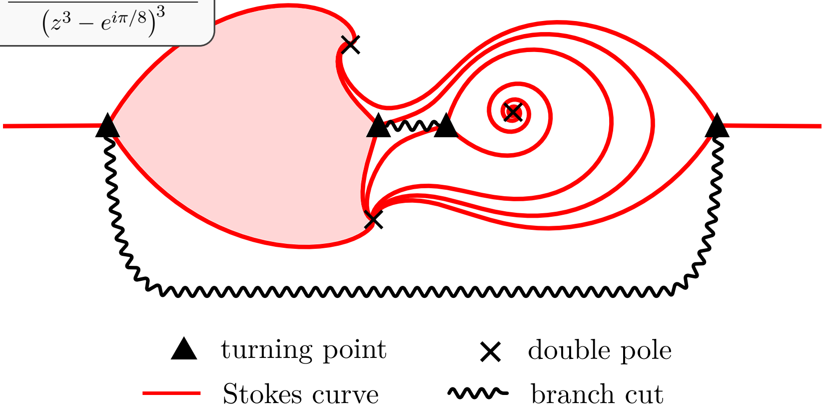

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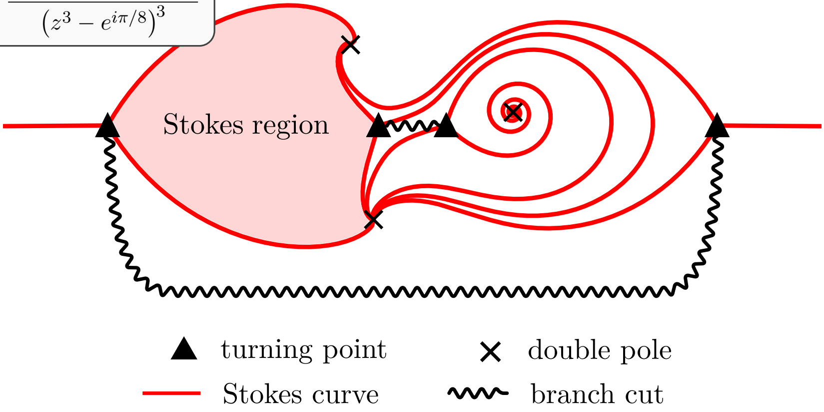

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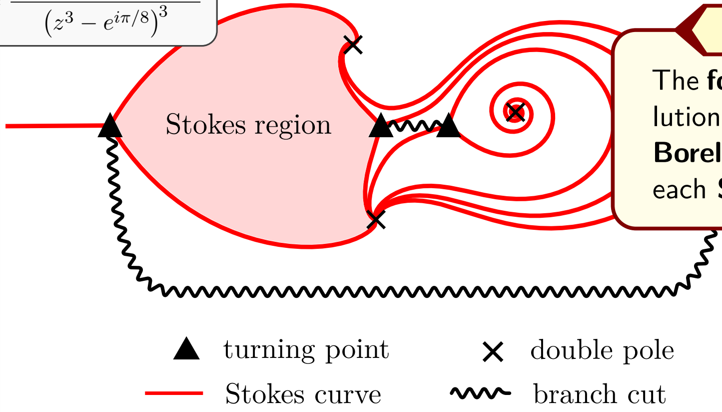

Upshot

The **formal** WKB solutions $\Psi_\pm(z)$ can be **Borel-resummed** in each **Stokes region**.

Crossing formulae

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Local structure around a simple turning point:

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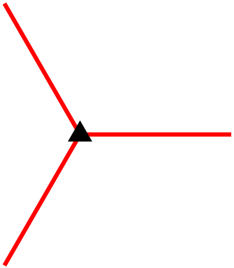

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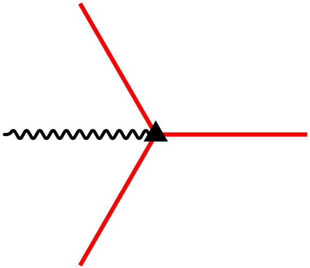

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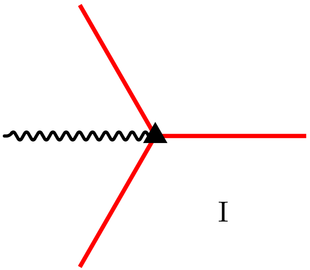

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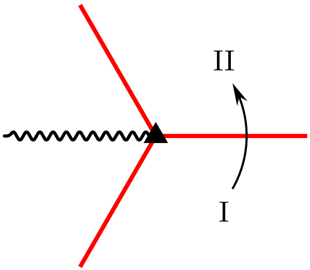

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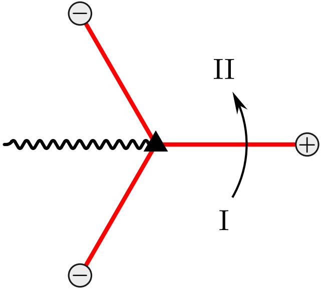

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Voros (1983), *Ann. Inst. Henri Poincaré 39*(3)
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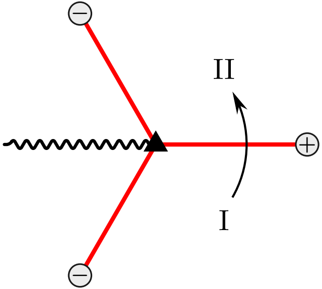

$$\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}_{\mathrm{II}} = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}_{\mathrm{I}}$$

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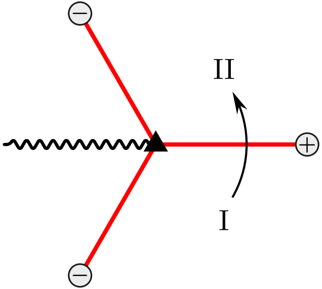

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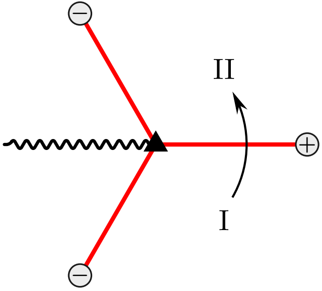

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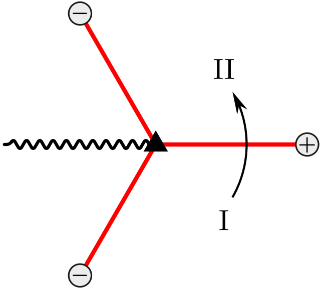

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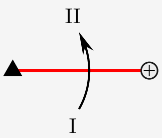

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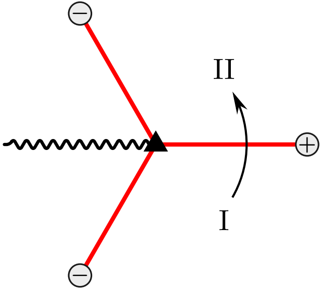

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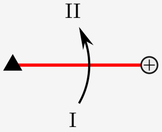

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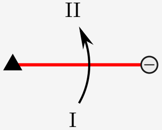

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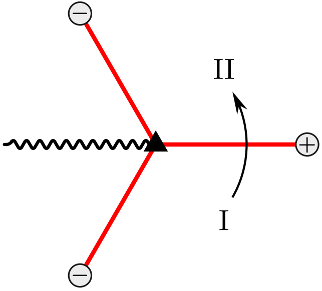

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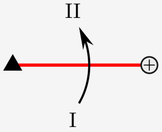

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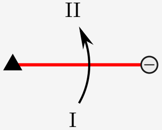

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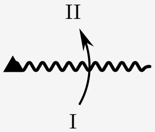

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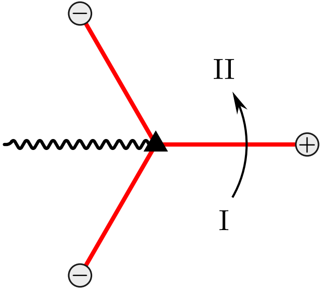

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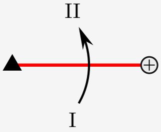

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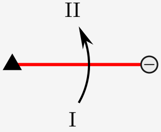

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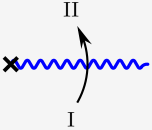

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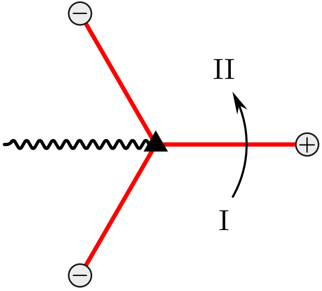

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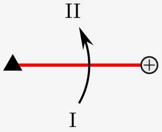

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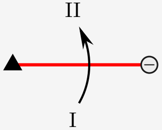

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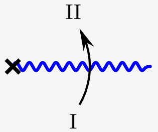

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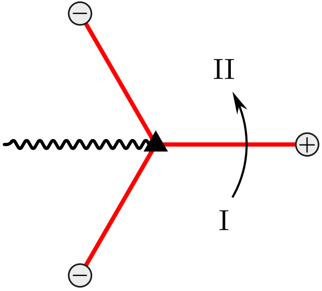

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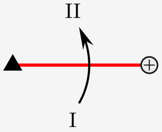

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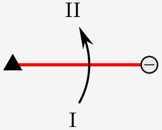

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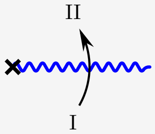

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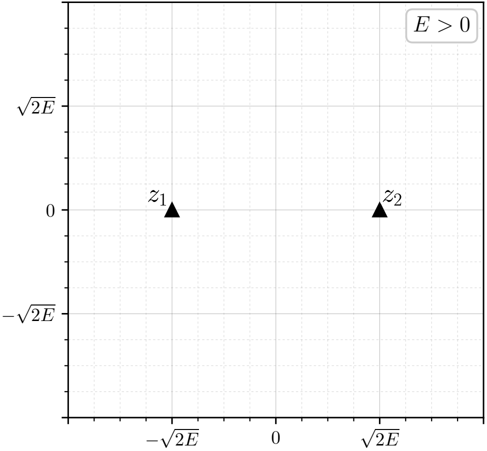

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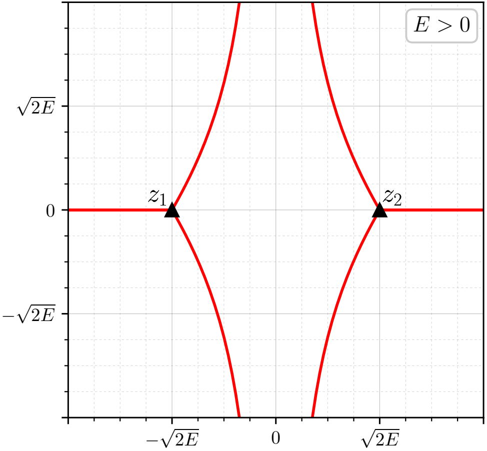

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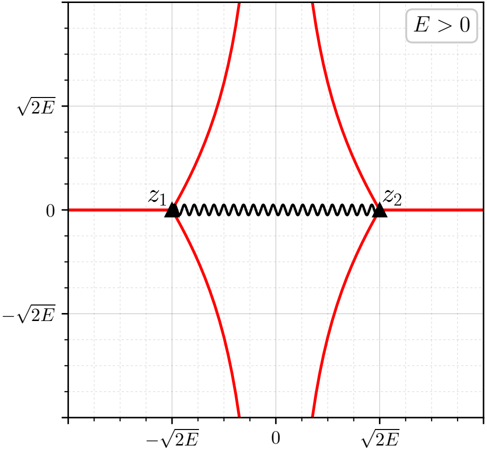

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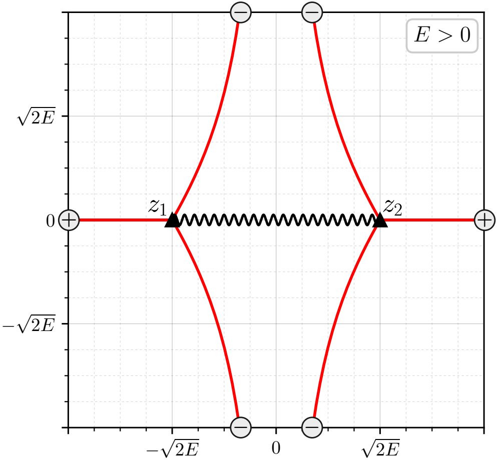

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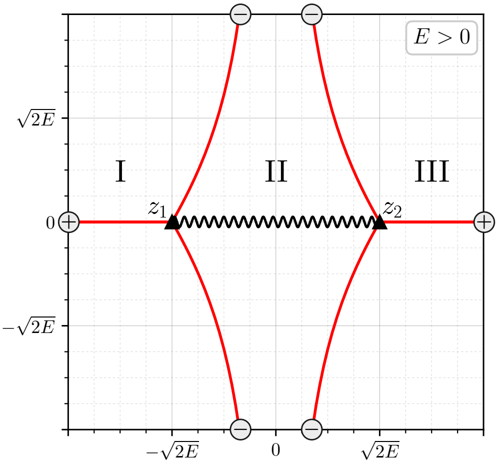

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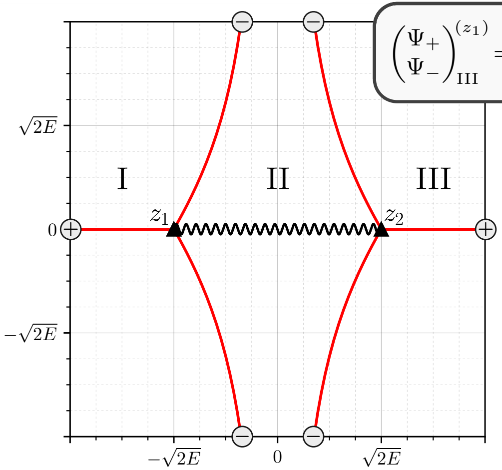

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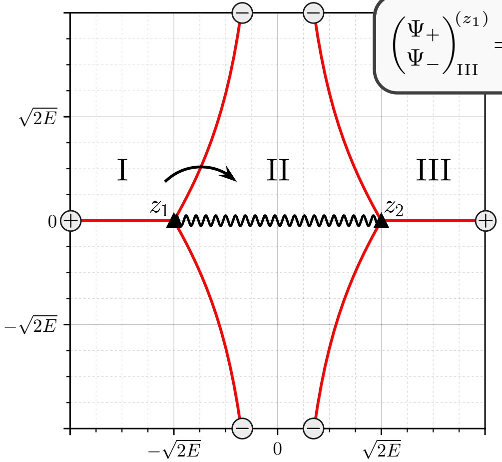

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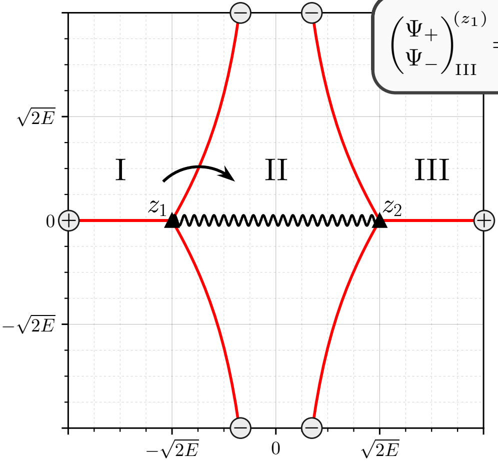

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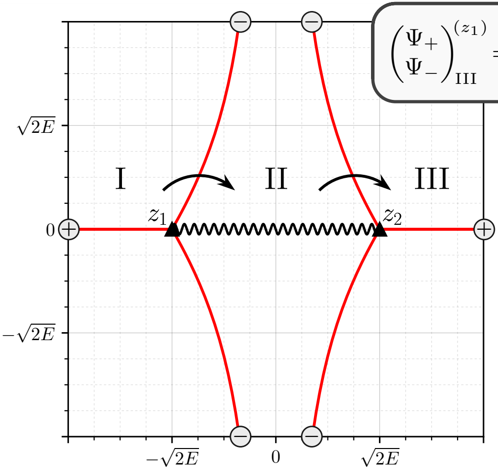

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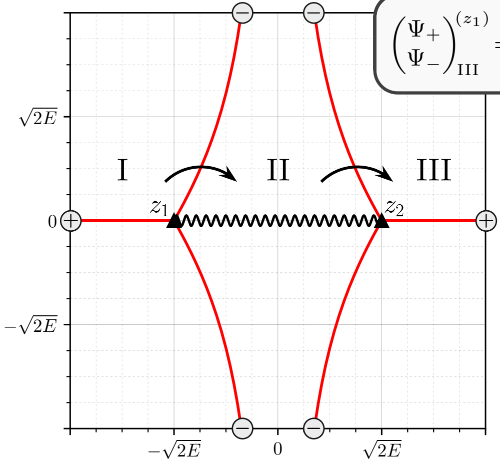

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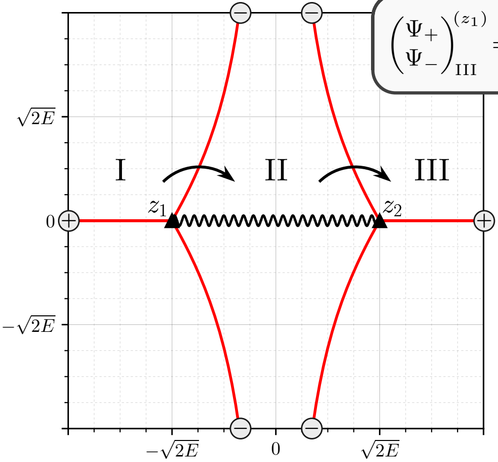

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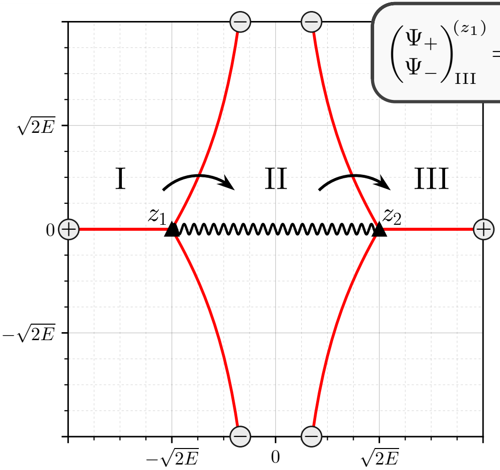

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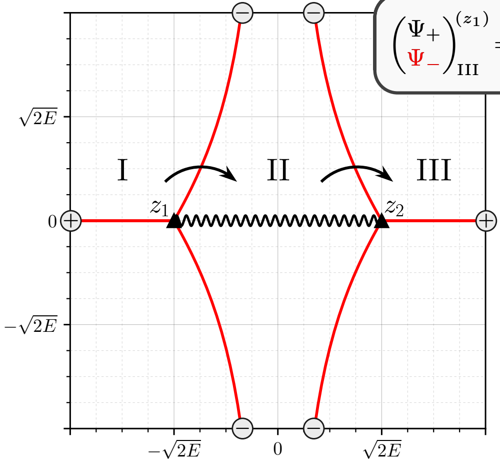

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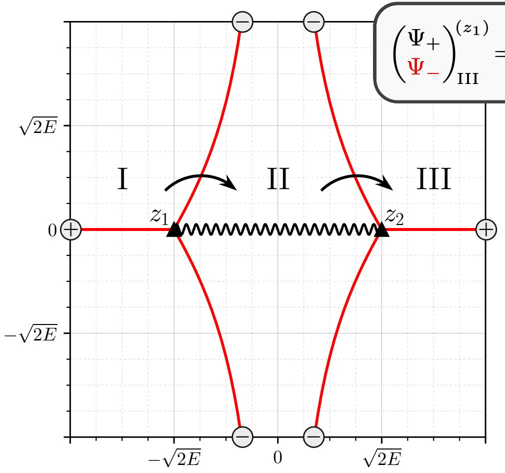

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$$=\begin{pmatrix}1 & 0\\ i(1+\mathcal{V}_{12}^{-1}) & 1\end{pmatrix}\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}^{(z_1)}_{\mathrm{I}}$$

Exact quantization condition:

$$\mathcal{V}_{12}\overset{!}{=}-1$$

The harmonic oscillator returns

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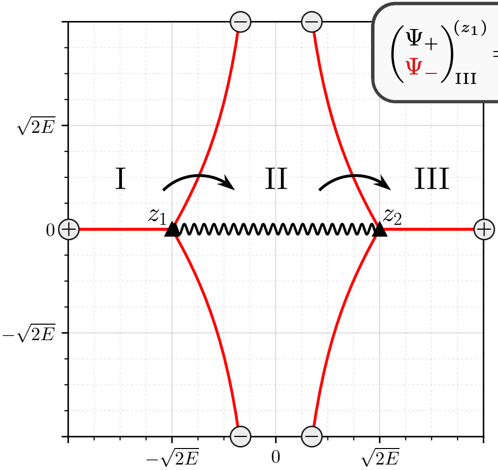

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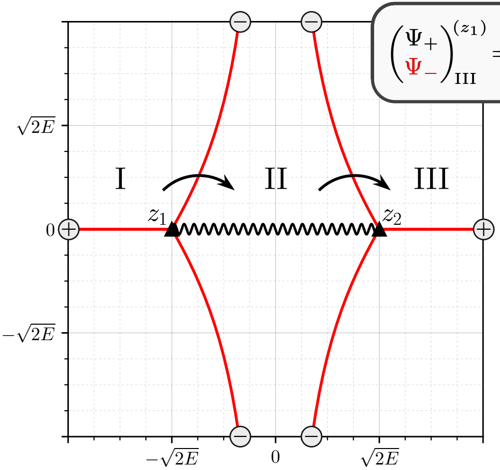

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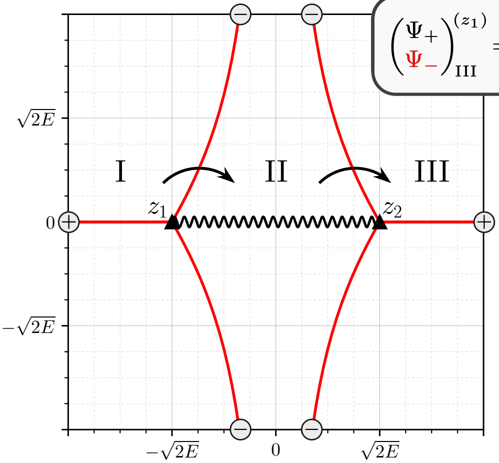

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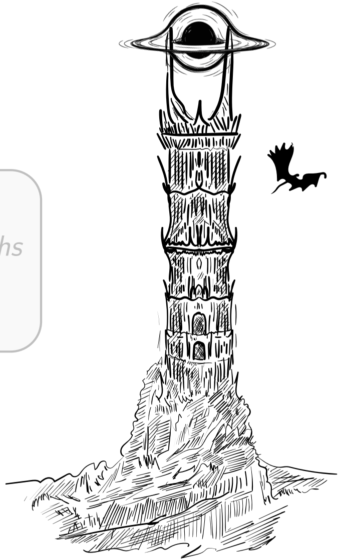
Outline of the talk

1. **What are "zero-damped modes"?**
   - *Motivation: Black hole mergers*
   - *QNMs & ZDMs*
   - *Scalar perturbations in a RN background*
2. **Exact WKB in a Nutshell**
   - *Formal WKB solutions & Stokes graphs*
   - *Connection formulae*
   - *The harmonic oscillator reloaded*
3. **Applying EWKB to the RN case**
   - *Exact ZDM quantization condition*
   - *Determining the Voros symbols*
   - *ZDM spectrum at NNLO*
4. **Summary**



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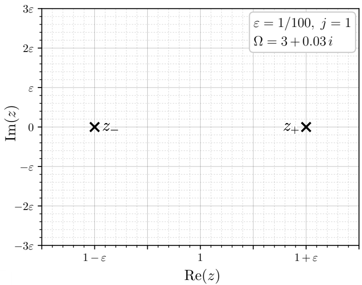

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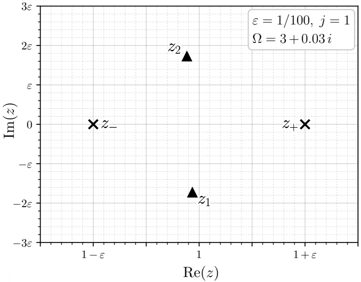

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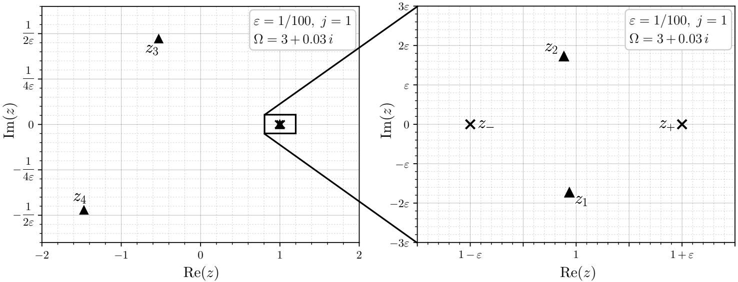

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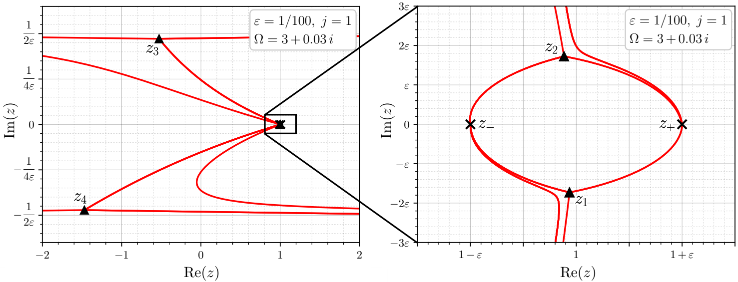

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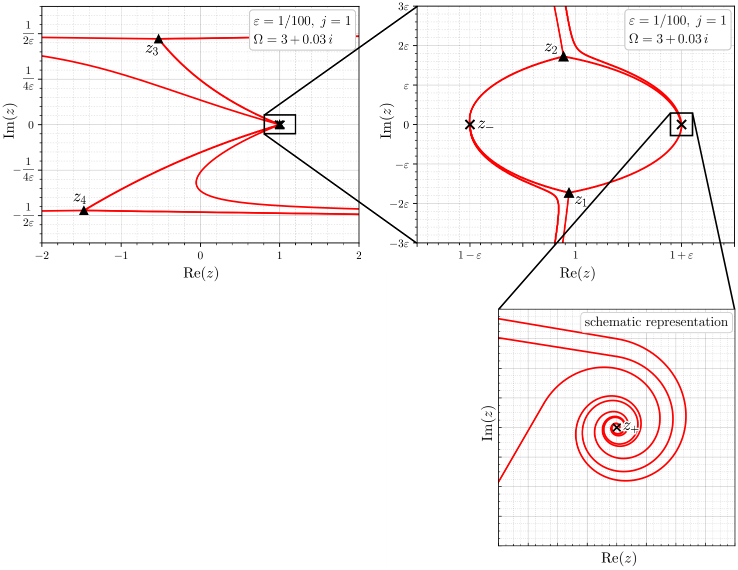

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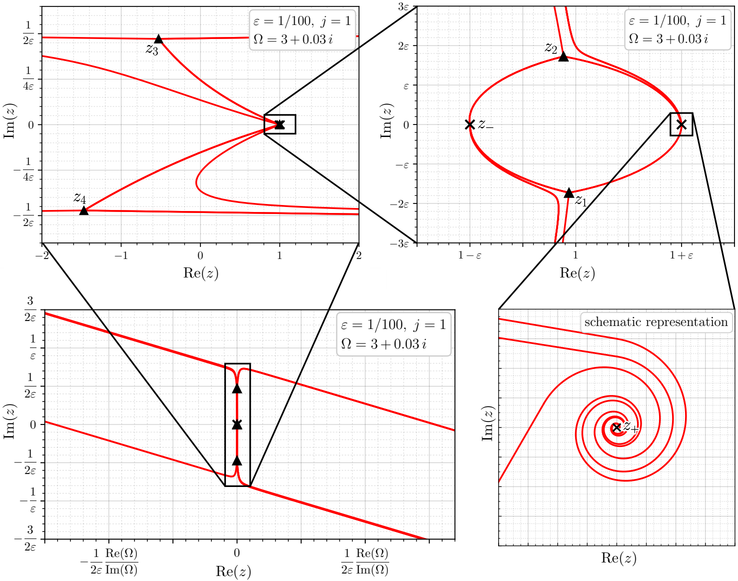

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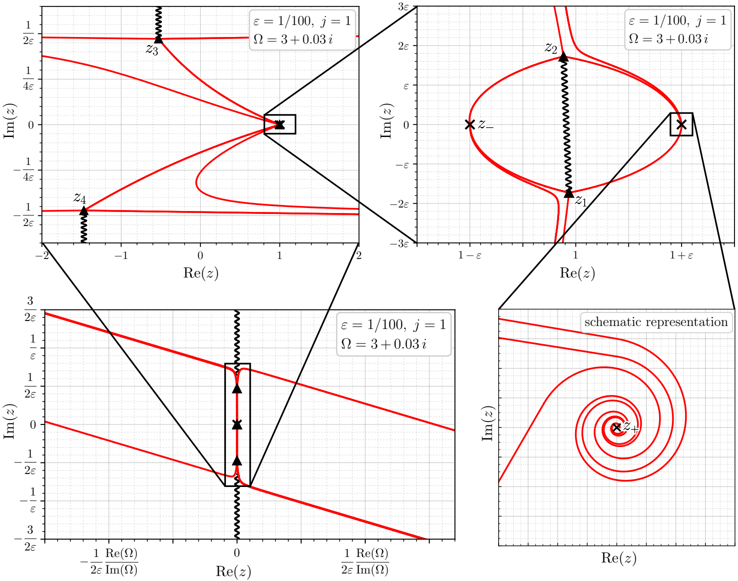

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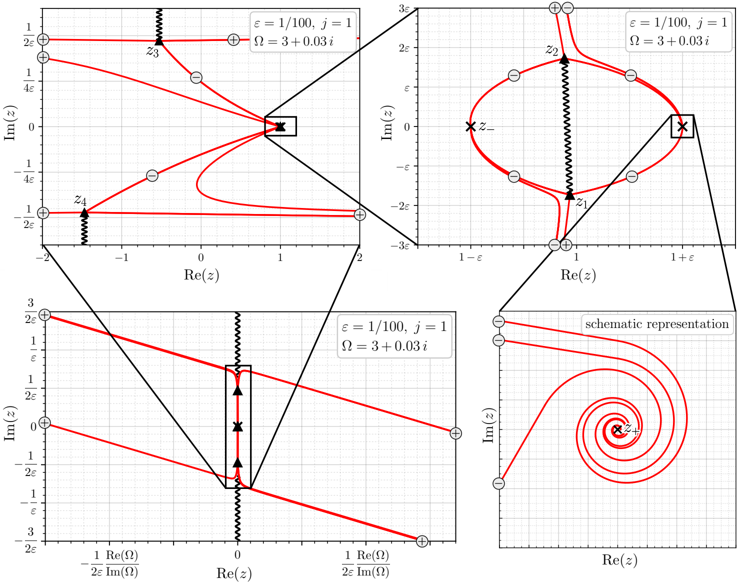

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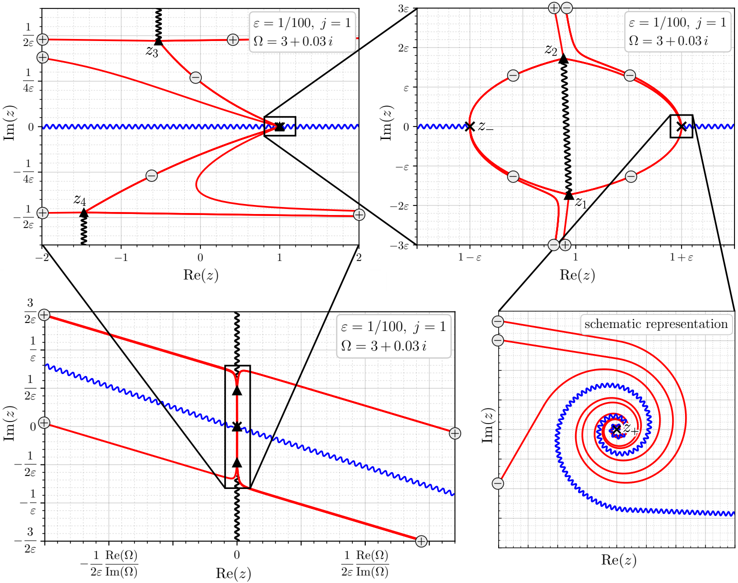
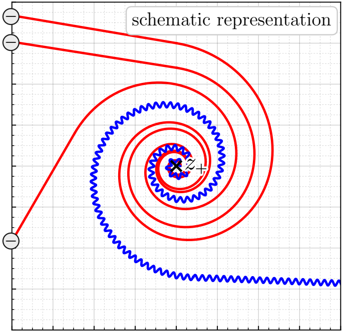


Exact quantization condition



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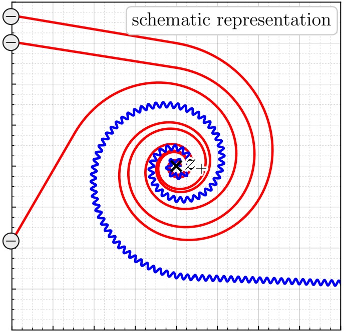

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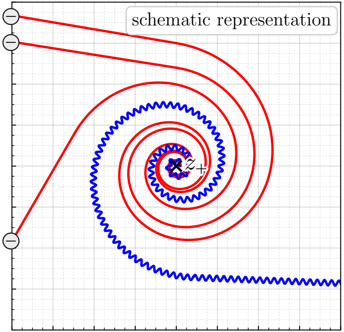

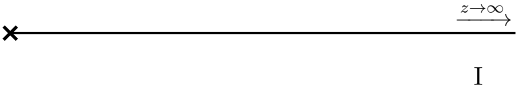

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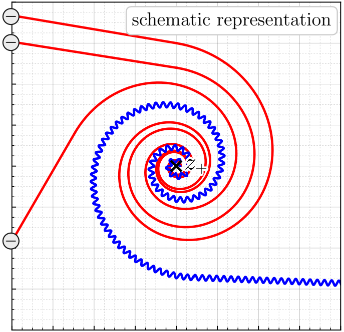

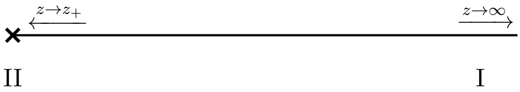

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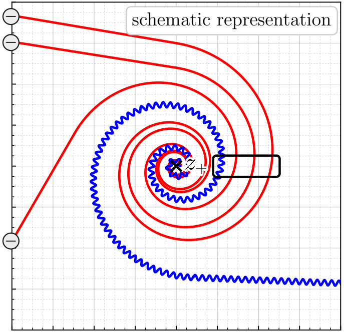

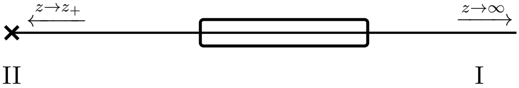

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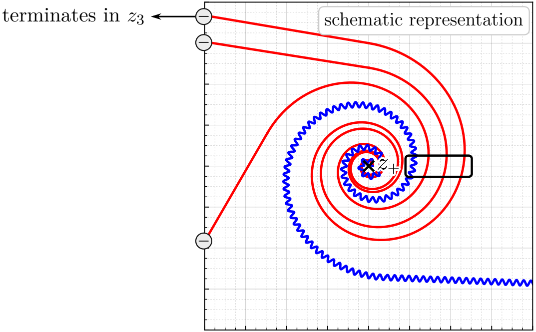

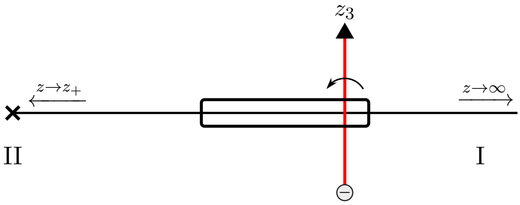

$$\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}_{\mathrm{II}}^{(z_3)} =$$

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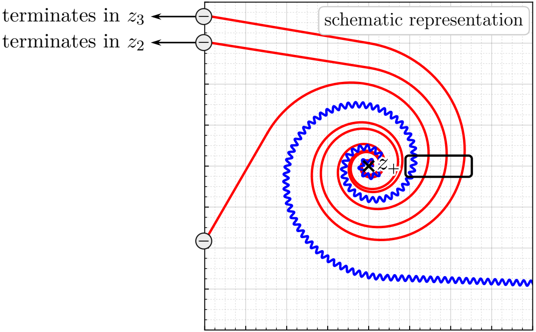

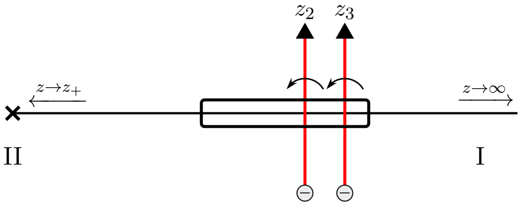

$$\begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}_{\mathrm{II}}^{(z_3)} = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \begin{pmatrix} \mathcal{V}_{23}^{1/2} & 0 \\ 0 & \mathcal{V}_{23}^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix}_{\mathrm{I}}^{(z_3)}$$

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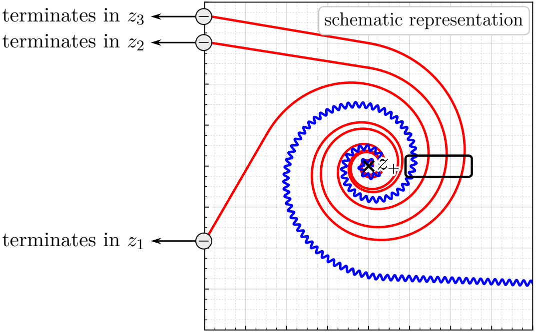

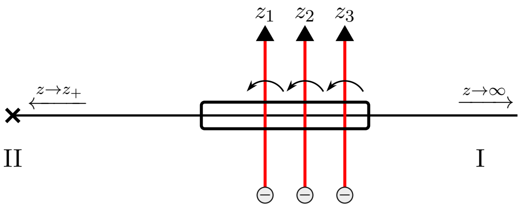

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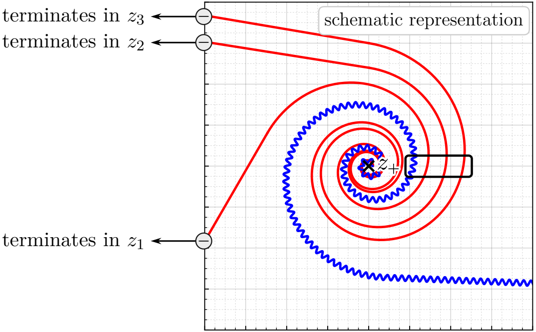

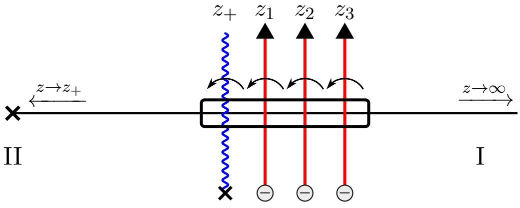

$$\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}^{(z_3)}_{\mathrm{II}} = \begin{pmatrix}\nu_+ & 0\\ 0 & \nu_-\end{pmatrix}\begin{pmatrix}1 & 0\\ i & 1\end{pmatrix}$$

$$\times \begin{pmatrix}\mathcal{V}_{12}^{1/2} & 0\\ 0 & \mathcal{V}_{12}^{-1/2}\end{pmatrix}\begin{pmatrix}1 & 0\\ i & 1\end{pmatrix}\begin{pmatrix}\mathcal{V}_{23}^{1/2} & 0\\ 0 & \mathcal{V}_{23}^{-1/2}\end{pmatrix}\begin{pmatrix}1 & 0\\ i & 1\end{pmatrix} \qquad \begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}^{(z_3)}_{\mathrm{I}}$$

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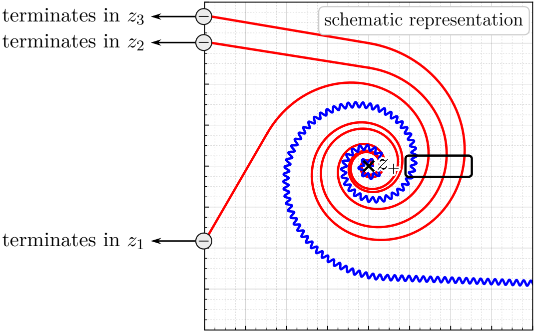

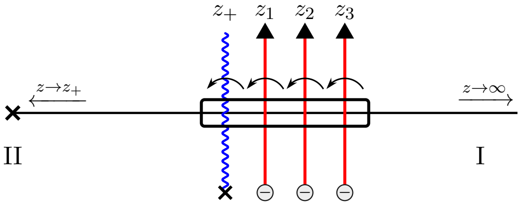

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$$

$$
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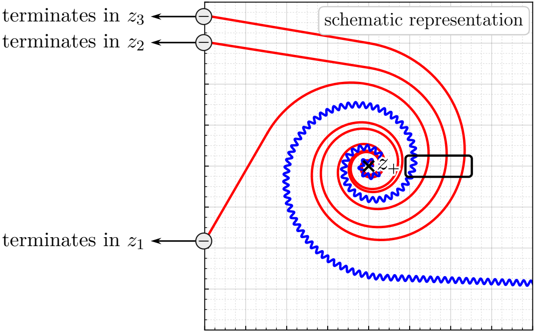
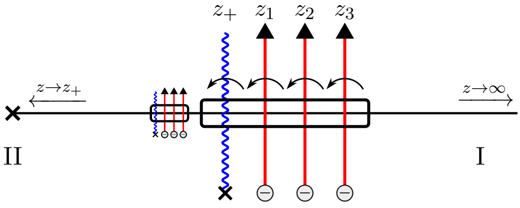
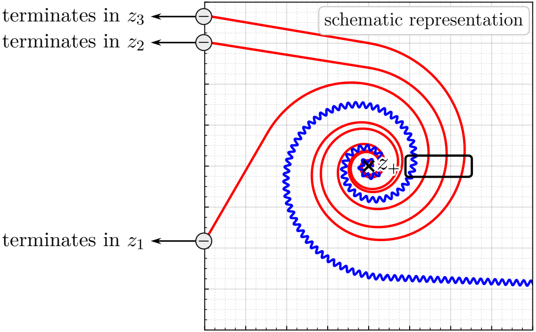


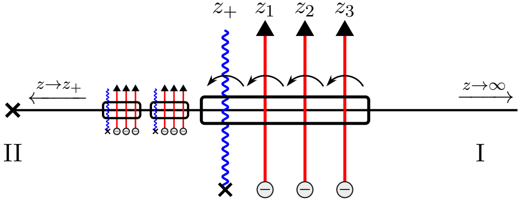


$$\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}^{(z_3)}_{\mathrm{II}} = \begin{pmatrix}\mathcal{V}_{31}^{1/2} & 0\\ 0 & \mathcal{V}_{31}^{-1/2}\end{pmatrix}\begin{pmatrix}\nu_+ & 0\\ 0 & \nu_-\end{pmatrix}\begin{pmatrix}1 & 0\\ i & 1\end{pmatrix}$$

$$\times \begin{pmatrix}\mathcal{V}_{12}^{1/2} & 0\\ 0 & \mathcal{V}_{12}^{-1/2}\end{pmatrix}\begin{pmatrix}1 & 0\\ i & 1\end{pmatrix}\begin{pmatrix}\mathcal{V}_{23}^{1/2} & 0\\ 0 & \mathcal{V}_{23}^{-1/2}\end{pmatrix}\begin{pmatrix}1 & 0\\ i & 1\end{pmatrix} \qquad \begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}^{(z_3)}_{\mathrm{I}}$$

Exact quantization condition





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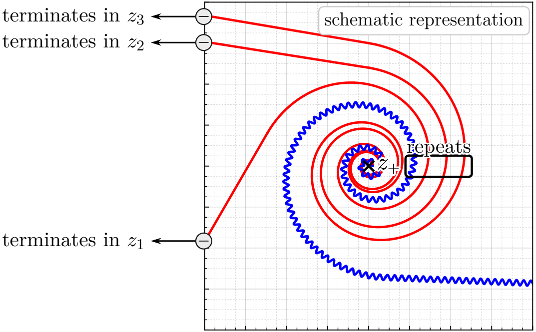

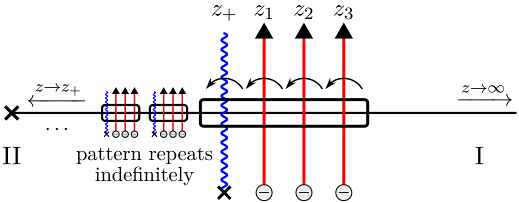

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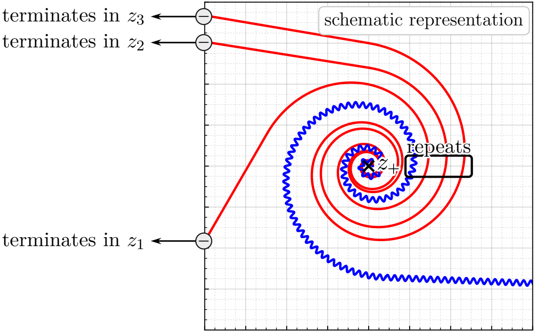
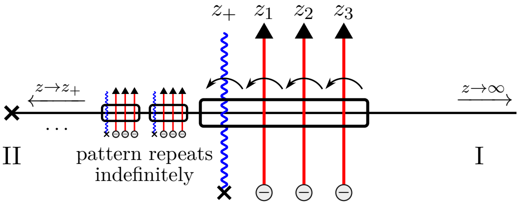
Exact quantization condition

Miyachi, Namba, Omiya & Oshita (2025), *Phys. Rev. D* 111(12)



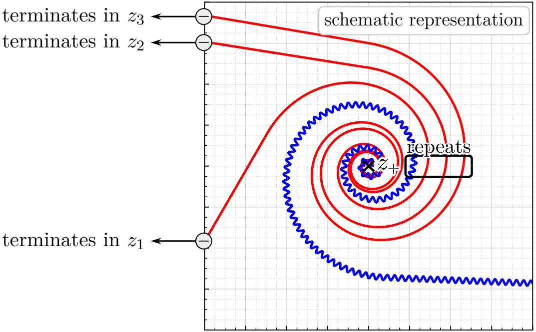


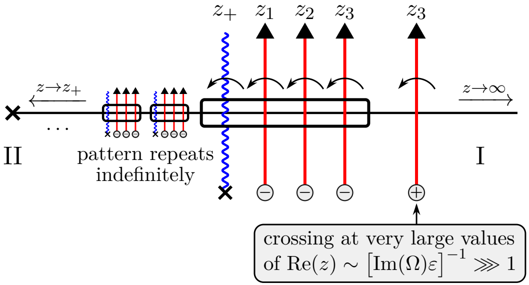
$$
\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}^{(z_3)}_{\mathrm{II}} = \lim_{N\to\infty}\left\{\begin{pmatrix}\mathcal{V}_{31}^{1/2} & 0\\ 0 & \mathcal{V}_{31}^{-1/2}\end{pmatrix}\begin{pmatrix}\nu_+ & 0\\ 0 & \nu_-\end{pmatrix}\begin{pmatrix}1 & 0\\ i & 1\end{pmatrix}\right.
$$

$$
\left.\times\begin{pmatrix}\mathcal{V}_{12}^{1/2} & 0\\ 0 & \mathcal{V}_{12}^{-1/2}\end{pmatrix}\begin{pmatrix}1 & 0\\ i & 1\end{pmatrix}\begin{pmatrix}\mathcal{V}_{23}^{1/2} & 0\\ 0 & \mathcal{V}_{23}^{-1/2}\end{pmatrix}\begin{pmatrix}1 & 0\\ i & 1\end{pmatrix}\right\}^N \qquad \begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}^{(z_3)}_{\mathrm{I}}
$$

Exact quantization condition

Miyachi, Namba, Omiya & Oshita (2025), *Phys. Rev. D 111* (12)





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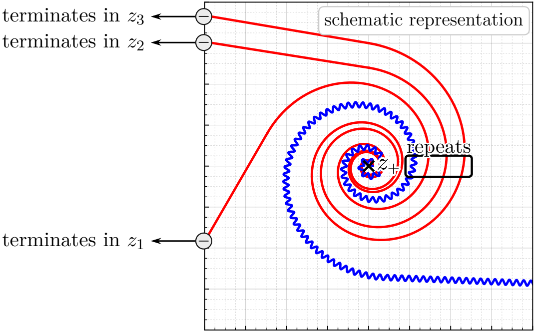
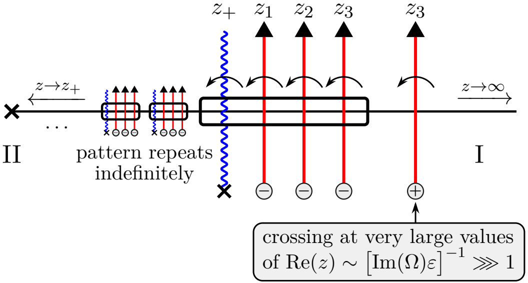
Exact quantization condition

Miyachi, Namba, Omiya & Oshita (2025), *Phys. Rev. D 111* (12)

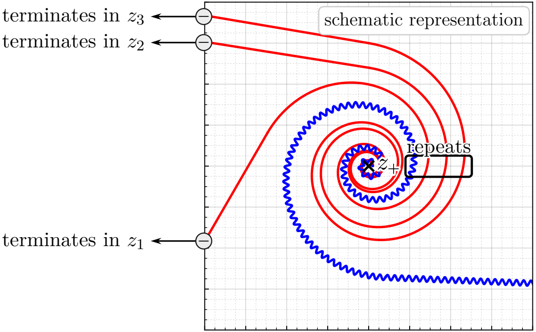


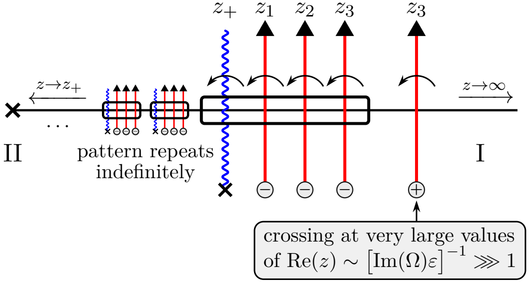


$$\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}^{(z_3)}_{\mathrm{II}} \propto \lim_{N\to\infty}\begin{pmatrix} 1 & i \\ \dfrac{i\mathcal{R}(\mathcal{R}^N-1)}{\mathcal{R}-1}\left[1+(1+\mathcal{V}_{12})\mathcal{V}_{23}\right] & \mathcal{R}^N - \dfrac{\mathcal{R}(\mathcal{R}^N-1)}{\mathcal{R}-1}\left[1+(1+\mathcal{V}_{12})\mathcal{V}_{23}\right]\end{pmatrix}\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}^{(z_3)}_{\mathrm{I}}$$

Exact quantization condition

Miyachi, Namba, Omiya & Oshita (2025), *Phys. Rev. D 111* (12)





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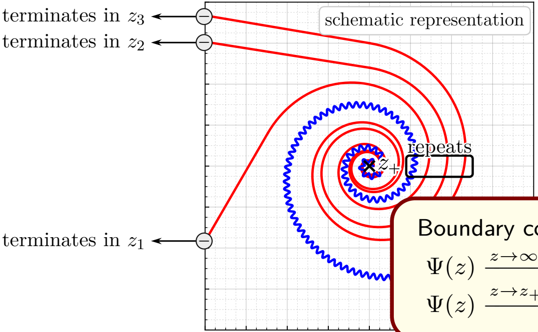
with $\mathcal{R} = \nu_-/\nu_+ = e^{-2i\pi\Omega(1+\varepsilon)^2}$

Exact quantization condition

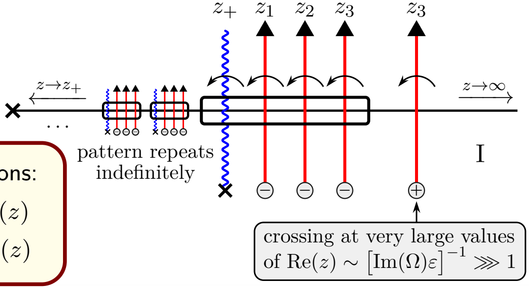
Miyachi, Namba, Omiya & Oshita (2025), *Phys. Rev. D 111* (12)



Boundary conditions:

$$\Psi(z) \xrightarrow{z\to\infty} \Psi_+(z)$$

$$\Psi(z) \xrightarrow{z\to z_+} \Psi_-(z)$$



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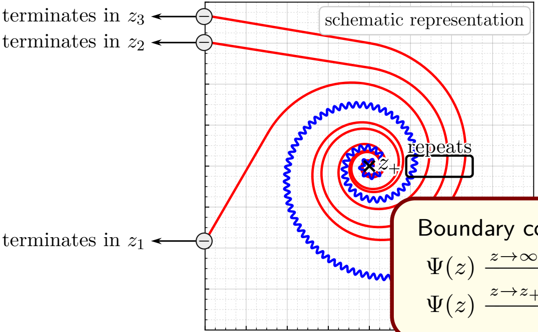
Miyachi, Namba, Omiya & Oshita (2025), *Phys. Rev. D 111*(12)



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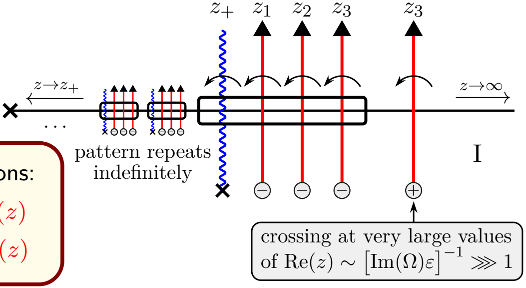
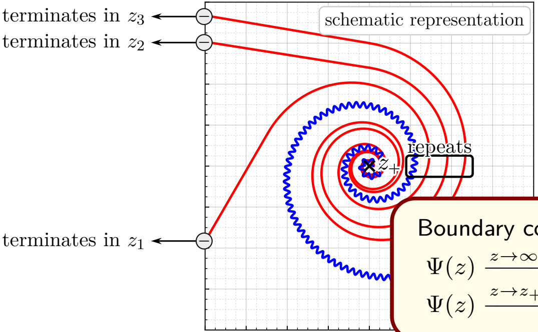


$$\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}^{(z_3)}_{\mathrm{II}} \propto \lim_{N\to\infty}\begin{pmatrix} 1 & i \\ \dfrac{i\mathcal{R}(\mathcal{R}^N-1)}{\mathcal{R}-1}\left[1+(1+\mathcal{V}_{12})\mathcal{V}_{23}\right] & \mathcal{R}^N - \dfrac{\mathcal{R}(\mathcal{R}^N-1)}{\mathcal{R}-1}\left[1+(1+\mathcal{V}_{12})\mathcal{V}_{23}\right]\end{pmatrix}\begin{pmatrix}\Psi_+\\ \Psi_-\end{pmatrix}^{(z_3)}_{\mathrm{I}}$$

with $\mathcal{R} = \nu_-/\nu_+ = e^{-2i\pi\Omega(1+\varepsilon)^2} \Longrightarrow$ Required connection condition: $\lim_{N\to\infty} \mathcal{M}_{22}/\mathcal{M}_{21} = 0.$

Exact quantization condition

Miyachi, Namba, Omiya & Oshita (2025), *Phys. Rev. D* *111*(12)



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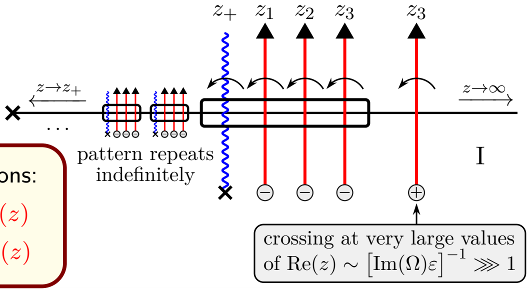

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with $\mathcal{R} = \nu_-/\nu_+ = e^{-2i\pi\Omega(1+\varepsilon)^2} \implies$ **Exact quantization condition:** $\boldsymbol{1+\mathcal{R}(1+\mathcal{V}_{12})\mathcal{V}_{23} = 0}$.

Simplifying the exact quantization condition — Voros symbols

Exact quantization condition: $1 + e^{-2i\pi\Omega(1+\varepsilon)^2}\left(1+\mathcal{V}_{12}\right)\mathcal{V}_{23} = 0$

Simplifying the exact quantization condition — Voros symbols

Exact quantization condition: $1 + \mathcal{V}_{12} = -e^{2i\pi\Omega(1+\varepsilon)^2}\mathcal{V}_{23}^{-1}$

Simplifying the exact quantization condition — Voros symbols

Exact quantization condition: $\boldsymbol{1 + \mathcal{V}_{12} = -e^{2i\pi\Omega(1+\varepsilon)^2}\mathcal{V}_{23}^{-1}}$

At first, Voros symbol $\mathcal{V}_{23}$:

$$\mathcal{V}_{23} \sim \varepsilon^{-2(2j+1)} \times \mathcal{O}(1)$$

Simplifying the exact quantization condition — Voros symbols

Exact quantization condition: $\boldsymbol{1 + \mathcal{V}_{12} = -e^{2i\pi\Omega(1+\varepsilon)^2}\mathcal{V}_{23}^{-1} = \mathcal{O}\left(\varepsilon^{4j+2}\right)}$

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$\Longrightarrow$ term on the RHS can be neglected
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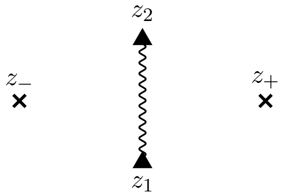

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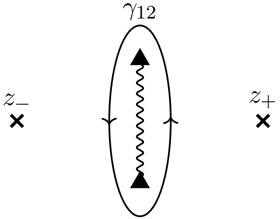

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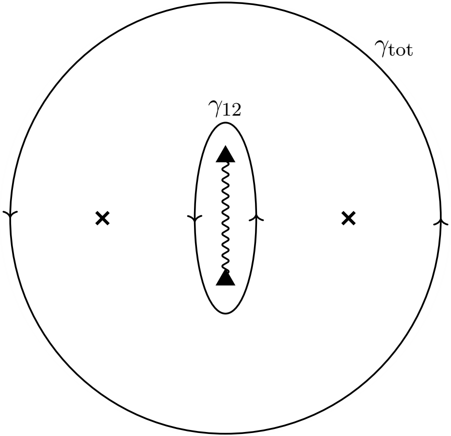

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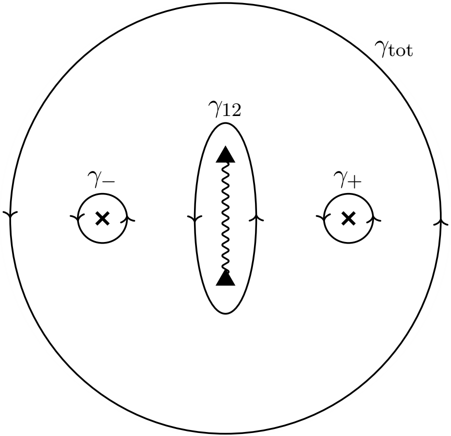

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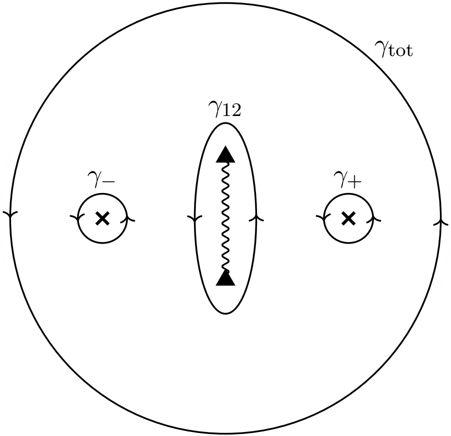
Note

We **analytically** compute $\mathcal{V}_{12}$ to **all orders** in η without the need to ever expand in the formal WKB parameter.

At last, Voros symbol $\mathcal{V}_{12}$:



Analytical results

Analytical results

Previously known analytical expression:

$$\omega_n^{(\mathrm{ZDM})} \sim -i\,\frac{r_+ - r_-}{2r_+^2}\,\big(n + j + 1\big) \qquad n \in \mathbb{N}_0\,.$$

Analytical results

Hod (2010), *Phys. Lett. A 374*
Kim, Myung & Park (2012), *Eur. Phys. J. C 73*

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Our result:

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Our result:

$$\Omega_n = \big(n+j+1\big)\Big\{1 \qquad \Big\}$$

Nils Wagner

Analytical results

Hod (2010), *Phys. Lett. A 374*
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$$\Omega_n = \big(n+j+1\big)\left\{1 - \left[1 - \frac{6(n+j+1)}{2j+1}\right]\varepsilon^2\right.$$

Analytical results

Hod (2010), *Phys. Lett. A 374*
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Analytical results

Hod (2010), *Phys. Lett. A 374*
Kim, Myung & Park (2012), *Eur. Phys. J. C 73*

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$$\omega_n^{(\mathrm{ZDM})} \sim -i\,\frac{r_+ - r_-}{2r_+^2}\,(n+j+1) = -\frac{i\varepsilon}{M}\,\underbrace{(n+j+1)\Big[1+\mathcal{O}(\varepsilon)\Big]}_{\Omega_n}\,, \qquad n\in\mathbb{N}_0\,.$$

Our result:

$$\Omega_n = (n+j+1)\Bigg\{1-\left[1-\frac{6(n+j+1)}{2j+1}\right]\varepsilon^2 + \Bigg[1-\frac{n+j+1}{2(2j+1)}\left(33-\frac{1}{4j(j+1)-3}\right)+\frac{72(n+j+1)^2}{(2j+1)^2} - \frac{2(n+j+1)^3}{(2j+1)^3}\left(35+\frac{4}{j(j+1)}+\frac{4}{4j(j+1)-3}\right)\Bigg]\varepsilon^4+\mathcal{O}\big(\varepsilon^6\big)\Bigg\}$$

Not applicable to the $j=0$ sector! The case $j=0$ differs qualitatively and needs to be treated separately.

Comparison with numerical computations

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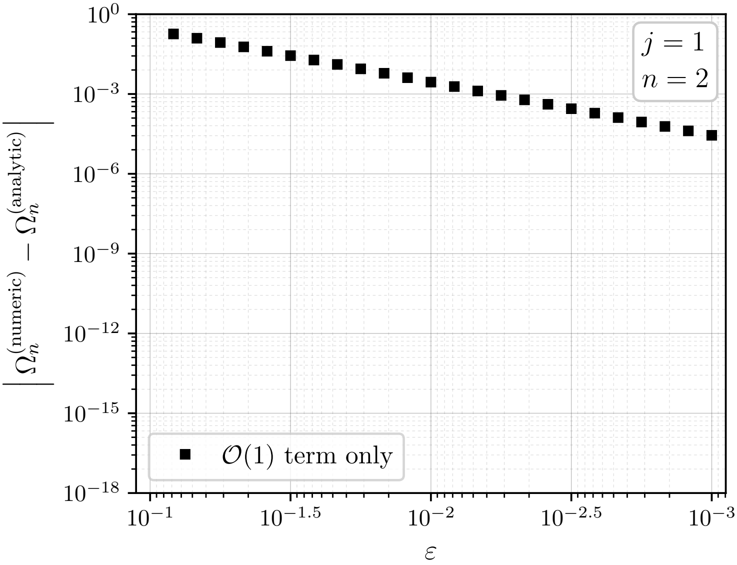

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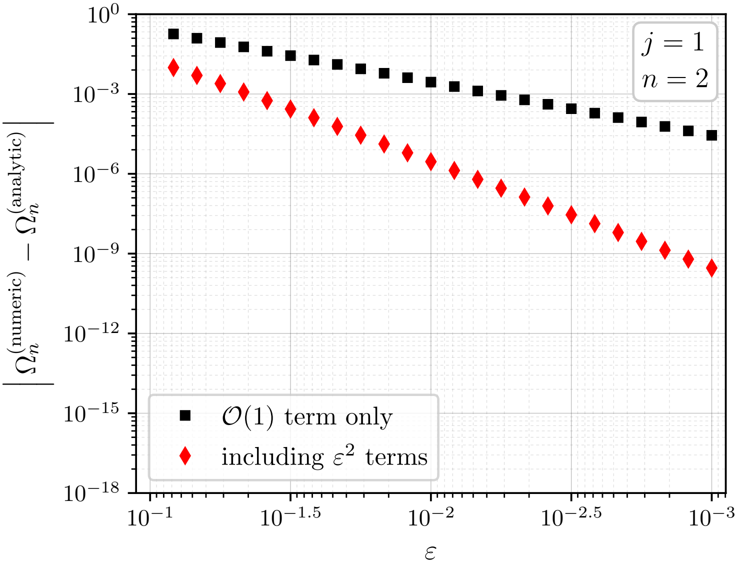

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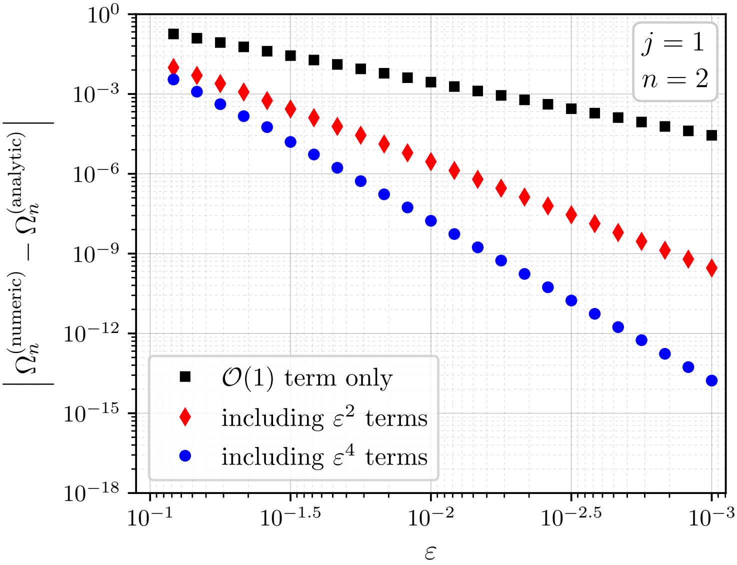

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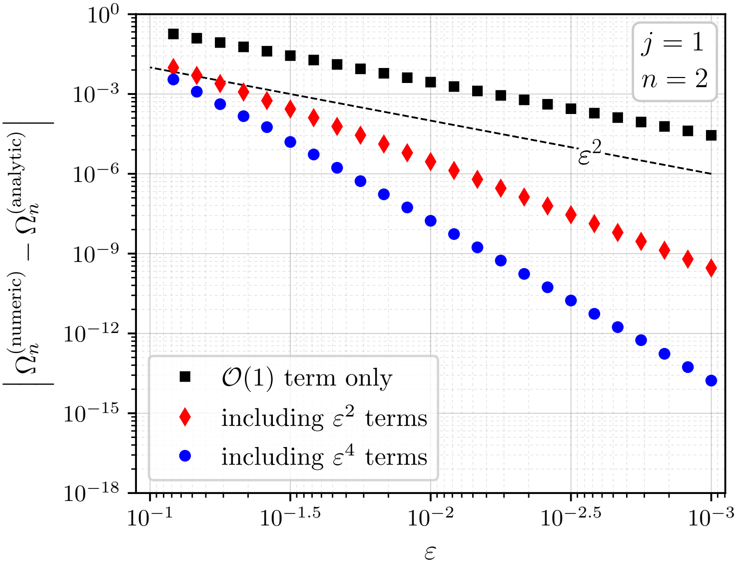

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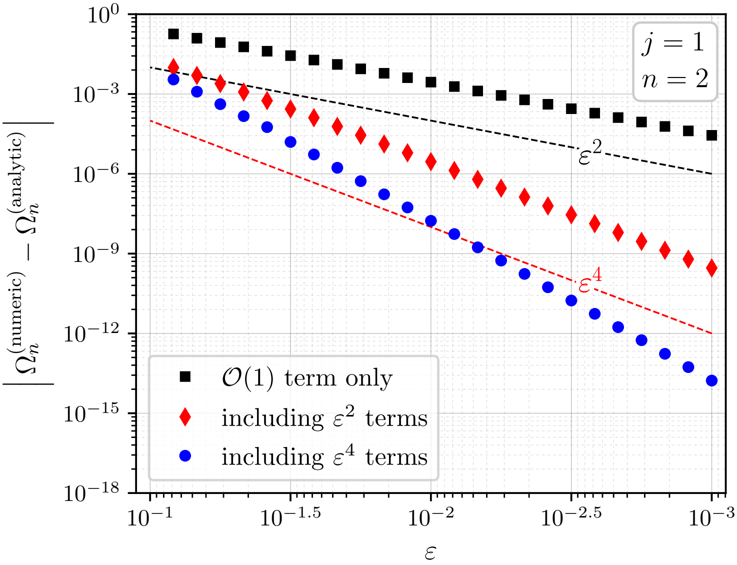

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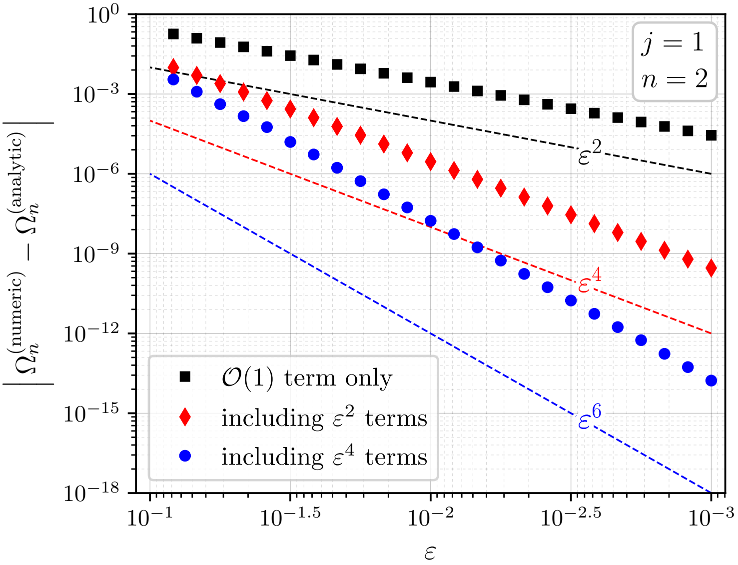

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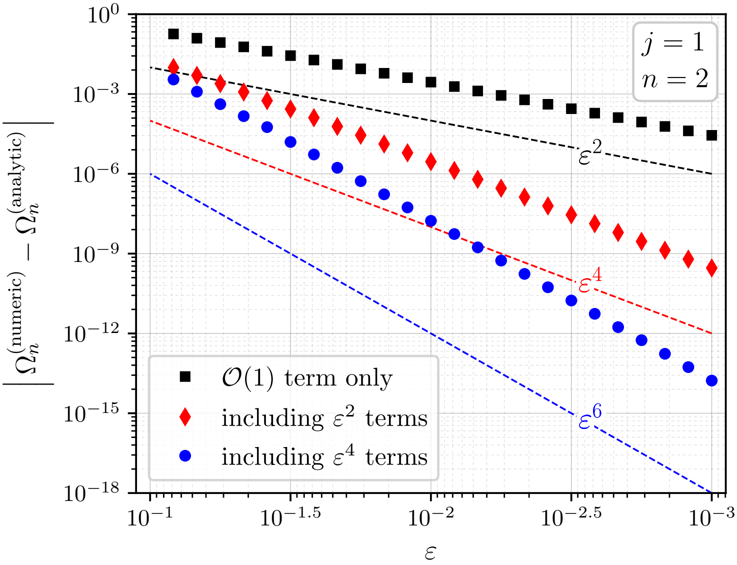

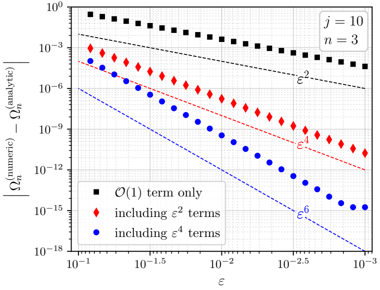

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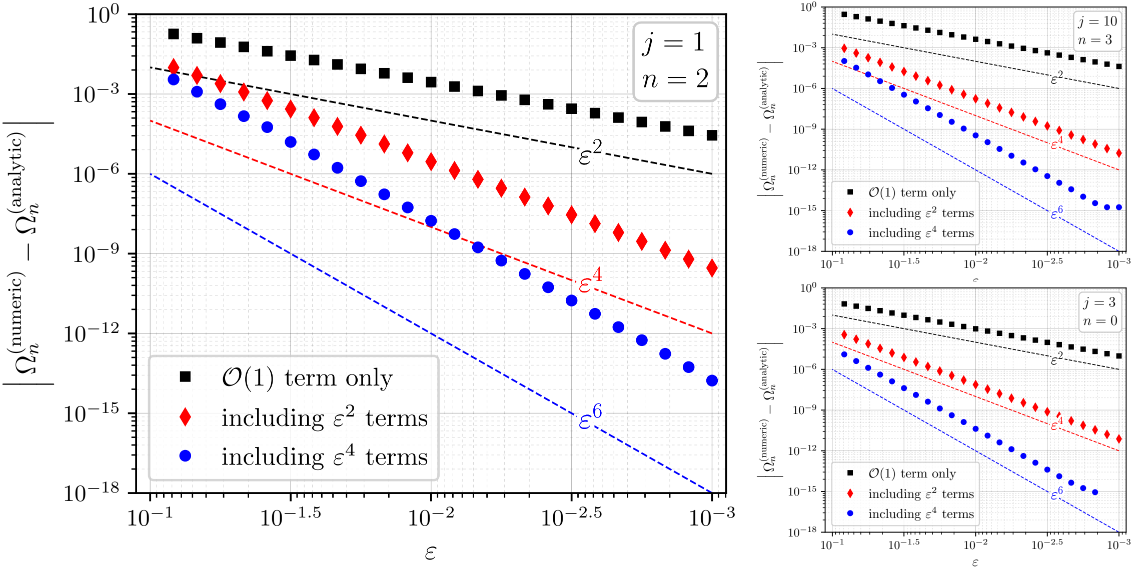
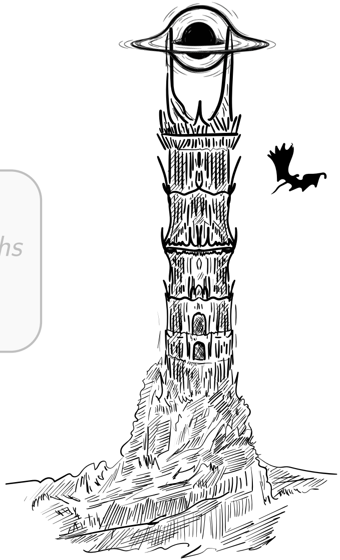


Outline of the talk

1. **What are "zero-damped modes"?**
   - *Motivation: Black hole mergers*
   - *QNMs & ZDMs*
   - *Scalar perturbations in a RN background*
2. **Exact WKB in a Nutshell**
   - *Formal WKB solutions & Stokes graphs*
   - *Connection formulae*
   - *The harmonic oscillator reloaded*
3. **Applying EWKB to the RN case**
   - *Exact ZDM quantization condition*
   - *Determining the Voros symbols*
   - *ZDM spectrum at NNLO*
4. **Summary**



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Thanks for your attention!