Gravitational waves from critical dynamics

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Effective Theories for Nonperturbative physics, Durham
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Papers and collaborators for this talk

Gravitational waves from critical dynamics

- Part I: Foundations (arXiv:2609.XXXX) $\leftarrow$ mostly this today
- Part II: Models (arXiv:26XX.XXXX)
- Part III: Cosmological signals (arXiv:26XX.XXXX)

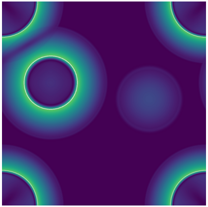
Thomas Martin

Paul Saffin

Classifying phase transitions

First order

- jump in $\frac{\mathrm{d}f}{\mathrm{d}T}$ (enthalpy)
- locally stable phases coexist
- correlations $\propto \frac{e^{-\frac{r}{\xi}}}{r^{d-2}}$
- bubble nucleation



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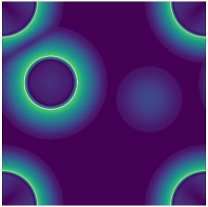
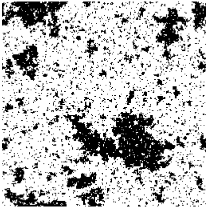
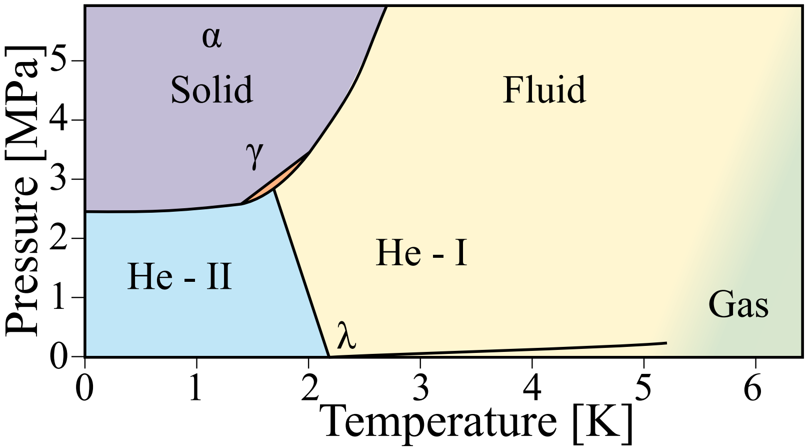


Second order

- jump in $\frac{\mathrm{d}^2 f}{\mathrm{d}T^2}$ (heat capacity)
- no stable phase coexistence
- correlations $\propto \frac{1}{r^{d-2+\eta}}$
- critical dynamics



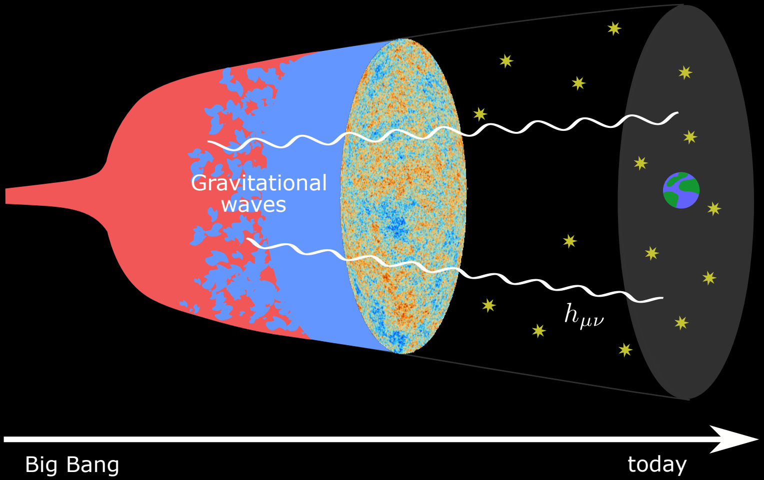
Phase diagram of Helium-4



What does a second-order phase transition look like?

Helium cooled through its lambda transition.

https://youtu.be/XpFNZFQIKak



Gravitational waves
$h_{\mu\nu}$
Big Bang
today

Dynamical universality

Static universality

- correlation length diverges at T_c,

$$\xi \propto |T - T_c|^{-\nu}$$

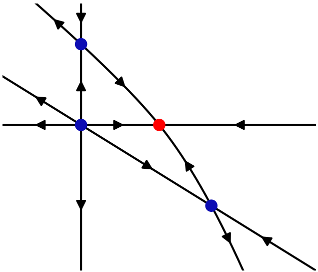
- long-distance physics governed by scale-invariant IR fixed points,

$$\langle \phi(x)\phi(0)\rangle \propto |x|^{2-d-\eta}$$

- quantities scale with ξ near T_c, e.g.

$$C \propto \xi^{\frac{\alpha}{\nu}}$$

- many UV models flow to the same IR fixed point



Dynamics fragments static universality

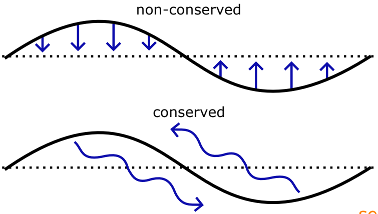
- multiple dynamical equations

$$\partial_t \phi = -\gamma \frac{\delta S_3}{\delta \phi} + \zeta \,, \qquad \partial_t \phi = \tilde{\gamma} \nabla^2 \frac{\delta S_3}{\delta \phi} + \tilde{\zeta} \,,$$

  have same static equilibrium

$$P(\phi) \propto \mathrm{e}^{-S_3}$$

- plus, order parameter can couple to hydrodynamic modes



see Liu & Glorioso '18

Dynamical universality

- correlation time diverges with correlation length

$$\tau \propto \xi^z$$

- joint fixed point of statics and dynamics (universality!)
- ~ 10 dynamical universality classes classified

Model	Designation	System	Dimension order of parameter	Non-conserved fields	Conserved fields	Non-vanishing Poisson bracket
Relaxational	A	Kinetic Ising anisotropic magnets	n	ψ	None	None
	B	Kinetic Ising uniaxial ferromagnet	n	None	ψ	None
	C	Anisotropic magnets structural transition	n	ψ	m	None
Fluid	H	Gas–liquid binary fluid	1	None	ψ j	$\{\psi, \mathbf{j}\}$
Symmetric planar magnet	E	Easy-plane magnet, $h_z = 0$	2	ψ	m	$\{\psi, m\}$
Asymmetric planar magnet	F	Easy-plane magnet, $h_z \neq 0$ superfluid helium	2	ψ	m	$\{\psi, m\}$
Isotropic antiferromagnet	G	Heisenberg antiferromagnet	3	ψ	m	$\{\psi, \mathbf{m}\}$
Isotropic ferromagnet	J	Heisenberg ferromagnet	3	None	ψ	$\{\psi, \psi\}$

Hohenberg & Halperin '77, Folk & Moser '06

The scaling hypothesis

At the transition, there is an anisotropic scale symmetry,

$$x_i \to \lambda x_i \,, \qquad t \to \lambda^z t \,,$$

which has the Ward identity

$$\left(zt\frac{\partial}{\partial t} + x_i \frac{\partial}{\partial x_i} + \Delta \right) \mathcal{O}(t,x) = 0$$

implying observables are scaling functions

$$\implies \mathcal{O}(t,x) = |x|^{-\Delta} \cdot \mathcal{O}\left(\frac{t}{|x|^z}, 1 \right)$$

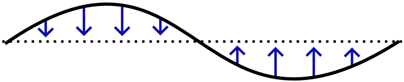
Model A

- simplest model of critical dynamics

$$\partial_t \phi = -\gamma \frac{\delta S_3}{\delta \phi} + \zeta$$
$$= -\gamma \left(-\nabla^2 \phi + m^2 \phi + \frac{\lambda}{3!} \phi^3 \right) + \zeta$$

- $z \approx 2.03$, and relativistic version reduces to it

Ohnishi & Kunihiro '05



Sourcing gravitational waves

Gravitational waves

GWs sourced by transverse-traceless energy-momentum tensor,

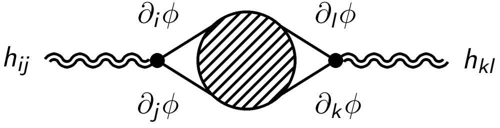
$$\ddot{h}_{ij}(\mathbf{x}, t) - \nabla^2 h_{ij}(\mathbf{x}, t) = (16\pi G)\, T_{ij}^{\mathrm{TT}}(\mathbf{x}, t)$$

which for Model A is

$$T_{ij}^{\mathrm{TT}} = \Lambda_{ijkl}\, T \partial_i \phi \partial_j \phi$$

so that GW power is sourced by

$$\langle \dot{h}^2 \rangle \sim \Lambda_{ijkl}\, T^2 \, \langle \partial_i \phi(t,x) \partial_j \phi(t,x) \partial_k \phi(0,0) \partial_l \phi(0,0) \rangle$$



Scaling source

GW source has scaling dimension

$$[\langle T_{ij} T_{kl} \rangle] = d + z$$

so reduces to a scaling function at the fixed point,

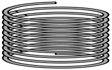
$$\Lambda_{ijkl} \langle T_{ij} T_{kl} \rangle = k^{d+z} F\left(\frac{\omega}{k^z}\right)$$

with extra arguments, e.g. the reduced temperature, away from T_c.

Strong coupling

In $d = 3$, effective coupling of $k\xi \sim 1$ modes diverges with ξ,

$$\lambda_{\text{eff}} = \frac{\lambda\xi}{\sqrt{1 + (k\xi)^2}} \to \infty$$



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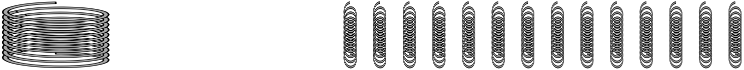


Strong-coupling methods needed:

- ϵ-expansion $\leftarrow$ our pick — Wilson & Fisher '72
- lattice simulations — e.g. Schwitzer, Schlichting & von Smekal '20
- functional renormalisation group — ERG 2026, in Sussex right now!
- large-N expansion — e.g. Natsuume & Okamura '10
- holography — e.g. Maeda, Natsuume & Okamura '08

Path integral representation

With a few standard tricks we can formulate a path integral for

$$E^a[\phi, \zeta] = \partial_t \phi^a - F^a[\{\phi\}] - \zeta^a = 0$$

Taking the noise partition function,

$$Z = \int \mathcal{D}\zeta \; e^{-\frac{1}{2}\int_{x,t} \zeta^a (\sigma^{-1})^{ab} \zeta^b}$$

and inserting a "1",

$$1 = \int \mathcal{D}\phi \; \delta\big(E^a[\phi, \zeta]\big) \det\left[\frac{\delta E}{\delta \phi}\right]$$

then using a couple of standard integral relations, gives

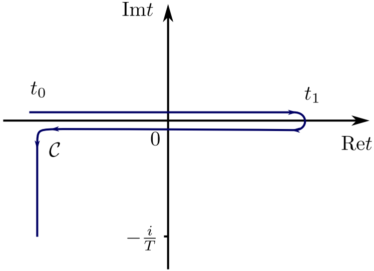
$$Z = \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} \; \mathrm{e}^{i S_{\mathrm{MSR}}}$$

where

$$S_{\mathrm{MSR}} = \int_{x,t} \left[\tilde{\phi}^a (\partial_t \phi^a - F^a[\phi]) + \frac{i}{2} \tilde{\phi}^a \sigma^{ab} \tilde{\phi}^b \right] .$$

Martin, Siggia & Rose '73, Janssen '76, De Dominicis '76

Aside: EFT on the Schwinger-Keldysh contour



The MSR path integral is the classical limit of the Schwinger-Keldysh/closed-time path/in-in path integral, with

$$\phi = \frac{1}{2}(\phi_1 + \phi_2) \qquad \tilde{\phi} = \phi_1 - \phi_2$$

e.g. Greiner & Müller '97, Crossley, Glorioso & Liu '15

Path integral for Model A plus gravitons

$$\left\langle \dot{h}_{ij}\dot{h}_{kl} \right\rangle = \int \mathcal{D}\phi \mathcal{D}\tilde{\phi} \mathcal{D}h \mathcal{D}\tilde{h} \, \left(\dot{h}_{ij}\dot{h}_{kl} \right) \, \mathrm{e}^{i \int_{t,x} \mathscr{L}(\phi, \tilde{\phi}, h, \tilde{h})}$$

where $\mathscr{L}$ splits up as

$$\mathscr{L} = \mathscr{L}_{\mathsf{A}} + \mathscr{L}_h - \tilde{h}_{ij} T_{ij}^{\mathsf{TT}} \,,$$

and the order parameter part is

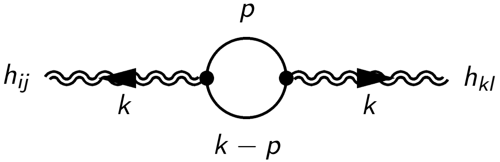
$$\mathscr{L}_{\mathsf{A}} = \tilde{\phi}\Big[\dot{\phi} + \gamma\Big(- \nabla^2 + m^2 + \frac{\lambda}{3!}\phi^2 \Big)\phi + i\gamma\tilde{\phi}\Big]$$

and the graviton part is

$$\mathscr{L}_h = \frac{1}{16\pi G} \tilde{h}_{ij} \Lambda_{ijkl} \left[\ddot{h}_{kl} + \Gamma_T \dot{h}_{kl} - c_T^2 \nabla^2 h_{kl} + i\beta \tilde{h}_{kl} \right]$$

see e.g. Salcedo et al '25, Colas '25

Lowest order diagram



This leading non-zero diagram gives

$$\mu^{3-d}\int_{\omega}\int_{p}\frac{8\gamma^2T^2\Lambda_{ijkl}p_i(k-p)_jp_k(k-p)_l}{(\omega^2+\gamma^2(p^2+m^2)^2)((k^0-\omega)^2+\gamma^2((k-p)^2+m^2)^2)}$$

which can be integrated with Feynman and Schwinger parameters.

ϵ-expansion results

- work in $d = 4 - \epsilon$ *spatial* dimensions
- fixed point at $\lambda = O(\epsilon)$ Wilson & Fisher '72
- divergent $\frac{1}{\epsilon}$ poles cancel against h_{ij} counterterms

  e.g. Arnold & Yaffe '93
- renormalised Model A result is

$$\left.\frac{\mathrm{d}^2\rho_{\mathrm{GW}}}{\mathrm{d}\ln(k)\mathrm{d}t}\right|_{\mathrm{crit.}} \approx \frac{5\,T^2\left(\frac{2}{5}+\gamma_E\right)}{32\pi^4 M_{\mathrm{Pl}}^2\gamma\xi_0}\left(\frac{k^3}{4\xi}+\frac{5}{96}\xi k^5\right)[1+O(\epsilon)]\;,$$

where we have dropped some complicated logs for simplicity.

Comparison to hydrodynamic fluctuations

Hydrodynamic fluctuations source GWs, proportional to the shear viscosity η, Hawking '66, Laine & Ghiglieri '15

$$\left.\frac{\mathrm{d}^2\rho_{\mathsf{GW}}}{\mathrm{d}\ln(k)\mathrm{d}t}\right|_{\text{hydro.}} = \frac{16\eta T k^3}{\pi M_{\mathrm{Pl}}^2}\,.$$

Model A gives:

- small correction to k^3 viscosity term, suppressed as $\propto \xi^{-1}$
- large new k^5 term, enhanced as $\propto \xi^{1}$

Part II: Models (teaser)

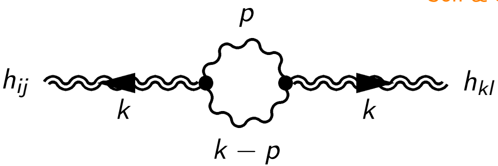
Models B and H

- Model B ($z \approx 4$) describes a conserved order parameter,

$$\partial_t \phi = \tilde{\gamma} \nabla^2 \frac{\delta S_3}{\delta \phi} + \tilde{\zeta} \, .$$

- Model H ($z \approx 3.1$) couples to hydrodynamic modes, e.g. QCD

Son & Stephanov '04

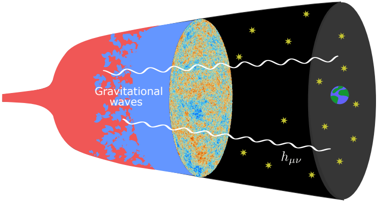


- For these models with larger z:
  - $\langle T_{ij} T_{kl} \rangle$ loop integral becomes IR dominated
  - GW signal is IR enhanced (and shear viscosity $\eta \propto \xi^{z-d}$)

(Siggia, Hohenberg, Halperin '76)

Part III: Cosmological signals (teaser)

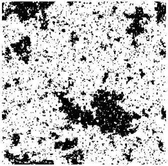
A slow cosmological quench



- Finite time quench: $t_Q \sim \frac{1}{H} \sim \frac{M_{Pl}}{T^2}$ $\leftarrow$ Planck enhanced
- Kibble-Zurek $\Longrightarrow$ $\xi_{max} \propto t_Q^{\frac{\nu}{1+\nu z}}$ Kibble '76, Zurek '85
- Out of equilibrium dynamics on largest scales

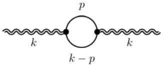
Summing up

Critical dynamics for second-order phase transitions:

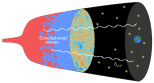


- dynamical universal classes
- strongly coupled at transition
- scaling GW source: $\langle TT \rangle = k^{d+z} F(\frac{k^z}{\omega})$

Tackling the calculation:



- ϵ-expansion suitable
- result depends on dynamical universality class
- Model A gives:



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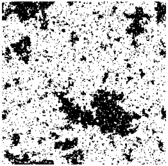

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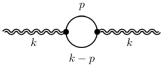

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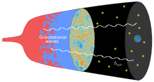

Thanks for listening!