Thermal nucleation in perturbation theory

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27 Aug 2026

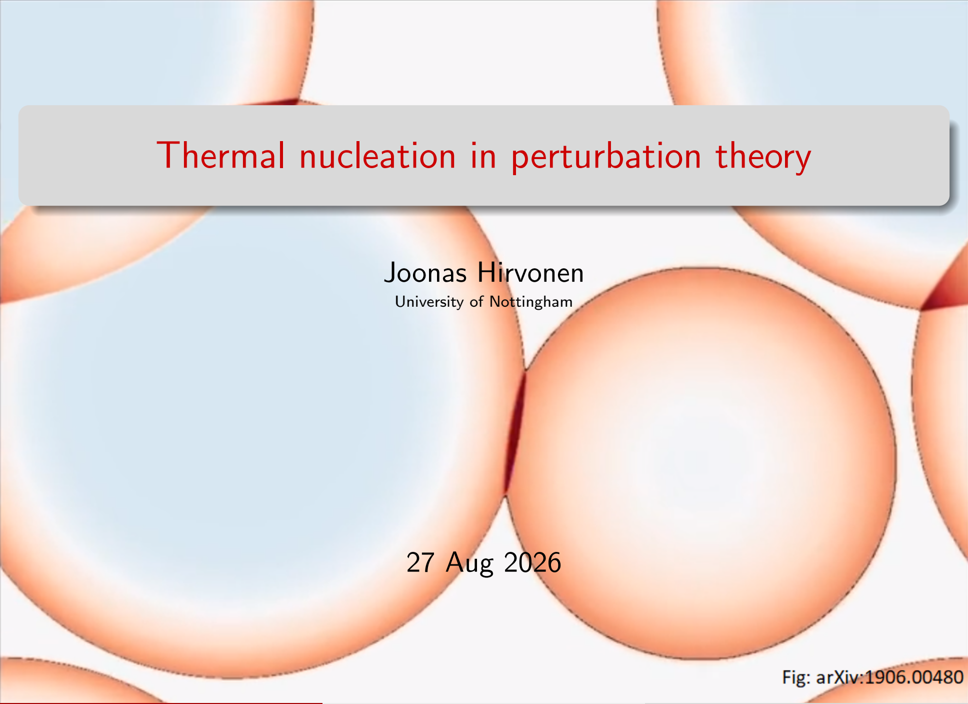


Fig: arXiv:1906.00480

Overview of the talk

0. Introduction
1. Wigner function approach
   - Real-time quantum foundations
   - Potential for going further
2. Equilibrium effective field theory approach
   - Very useful for calculations
   - See Maciej's and Oliver's talks
3. Kinetic theory approach
   - Backs up equilibrium EFTs
   - Dynamical prefactor

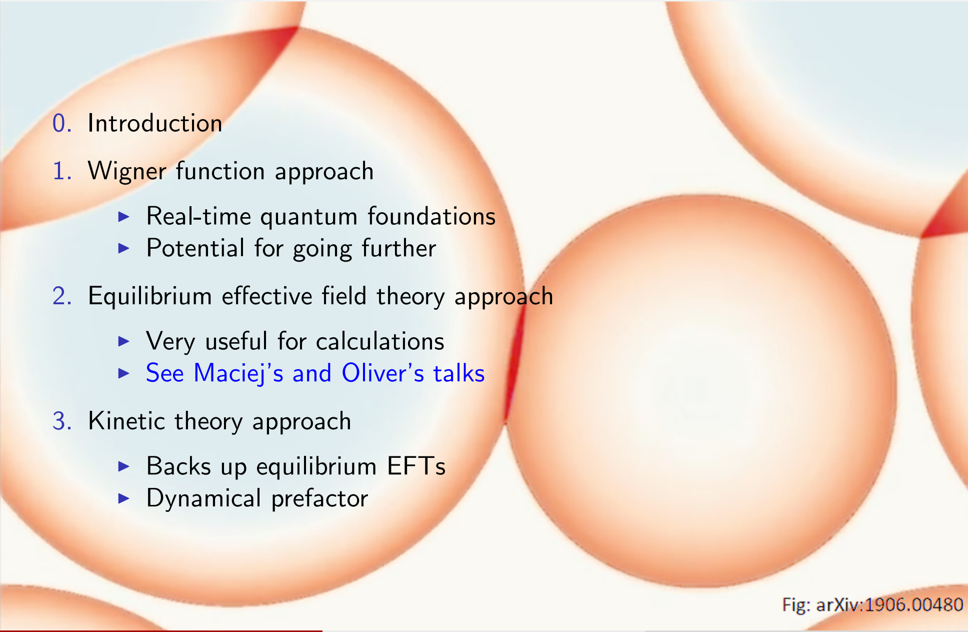


Fig: arXiv:1906.00480

Nucleation

- System trapped in ϕ_1
- Escaping locally with bubble nucleation to ϕ_2

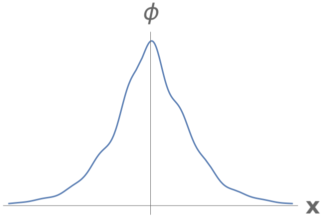


Figure: Cross section of a bubble

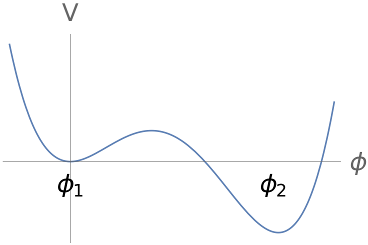


Figure: Potential with a transition

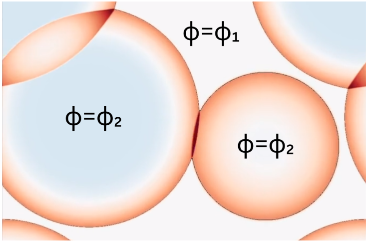


Figure: Growing bubbles [Cutting et al. '20]

Cosmological phase transitions

- Plasma shocks from bubbles

  $\rightarrow$ Gravitational waves

- Possibly observable with e.g. LISA
  - BSM signal

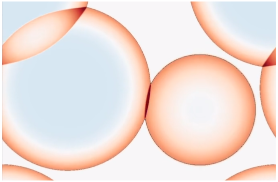


Figure: Growing bubbles (arXiv:1906.00480)

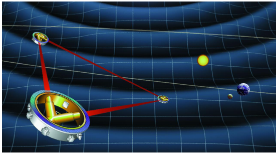


Figure: Gravitational waves and LISA (ESA)

Nucleation rates and GWs

- Nucleation rate, Γ, changes in time
- Allows to take place
- Duration

$$\Delta t \sim \left(\frac{\mathrm{d}}{\mathrm{d}t} \ln \Gamma \right)^{-1}$$

- Transition temperature

$$\Delta t^4 \, \Gamma \sim 1 \Rightarrow T$$

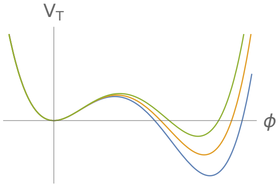


Figure: T-dependent potential

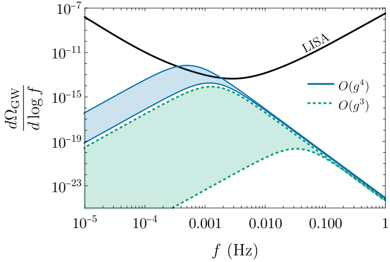


Figure: GW uncertainties [Gould, Tenkanen '21]

Nucleation in laboratories

- Controlled environments
- Testing our theories
- Analogs of the early universe

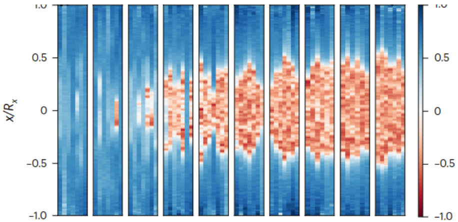


Figure: Nucleation events in ferromagnetic superfluids [Zenesini & al. '23]

Thermal nucleation in QCD

- Hypothetical first order transition in
  - heavy-ion collisions
  - neutron star mergers
- Can the nucleation dynamics be understood further?

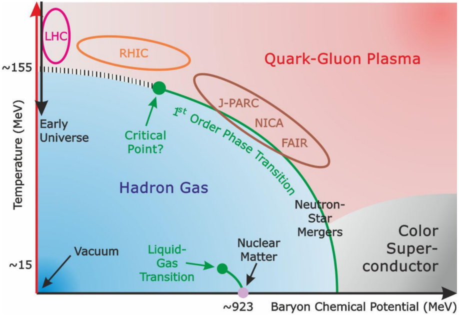


Figure: QCD phase diagram with experiments [A. Steidl, CRC-TR-211]

Nucleation and Wigner functions [Hirvonen '26]

- Defining nucleation rate
  - Quantum foundation
- Non-perturbative formula
  - Flux of Wigner function

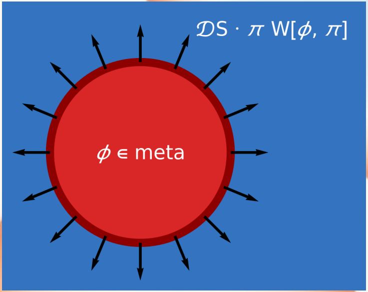


Figure: Flux of Wigner function out of metastable phase

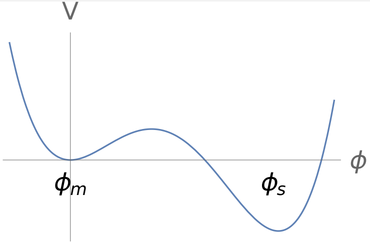
$$\Gamma \propto \int_{\partial \text{meta}} \mathcal{D}S \cdot \pi W[\phi, \pi]$$

Fig: arXiv:1906.00480

Nucleation rate

- System fluctuating around ϕ_m
  - Metastable phase
- Nucleation rate
  - Probability P_{meta} leaking

$$\Gamma = -P_{\text{meta}}^{-1} \frac{\mathrm{d}}{\mathrm{d}t} P_{\text{meta}}$$



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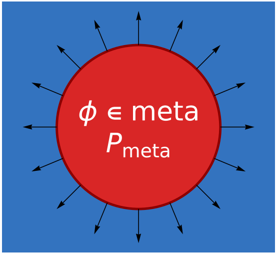


Figure: Metastable phase in configuration space

Statistical description of QFTs

- State: density matrix, $\hat{\rho}$
  - Field basis, $\rho[\phi, \phi']$
- Probability density:
  - Diagonal, $\rho[\phi, \phi]$
  - Yields P_{meta}

$$\rho[\phi, \phi'] = \langle\phi|\, \hat{\rho}\, |\phi'\rangle$$

$$P_{\text{meta}} = \int_{\phi\in\text{meta}} \mathcal{D}\phi\, \rho[\phi, \phi]$$

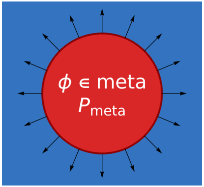


Figure: Metastable phase in configuration space

Wigner function, W

- Transformation
  - $\phi_{\mathsf{a}} \quad (= \phi - \phi')$

    $\longrightarrow \quad \pi \quad (= \partial_t \phi_{\mathsf{r}})$
- Captures probabilities

$$W[\phi_{\mathsf{r}}, \pi]$$

$$= \frac{1}{\mathcal{N}} \int \mathcal{D}\phi_{\mathsf{a}} \, e^{-i\pi\cdot\phi_{\mathsf{a}}} \rho\left[\phi_{\mathsf{r}} + \frac{\phi_{\mathsf{a}}}{2}, \phi_{\mathsf{r}} - \frac{\phi_{\mathsf{a}}}{2}\right]$$

$$\rho[\phi_{\mathsf{r}}, \phi_{\mathsf{r}}] = \int \mathcal{D}\pi \, W[\phi_{\mathsf{r}}, \pi]$$

$$P_{\mathsf{meta}} = \int_{\phi_{\mathsf{r}} \in \mathsf{meta}} \mathcal{D}\phi_{\mathsf{r}} \mathcal{D}\pi \, W[\phi_{\mathsf{r}}, \pi]$$

Continuity equation

- Time evolution: von Neumann
- Integrating over both sides
- Structure
  - Flow given by π
  - *Local* in ϕ_r

$$\partial_t \hat{\rho} = i[\hat{\rho}, \hat{H}]$$

$$\frac{1}{\mathcal{N}} \int \mathcal{D}\pi \mathcal{D}\phi_\mathrm{a}\, e^{-i\pi\cdot\phi_\mathrm{a}} \bullet$$

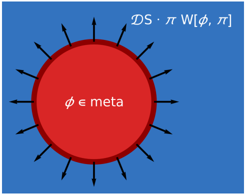
$$\partial_t \rho[\phi_\mathrm{r}, \phi_\mathrm{r}] = -\int \mathcal{D}\pi\, \pi \cdot \frac{\delta}{\delta\phi_\mathrm{r}} W[\phi_\mathrm{r}, \pi]$$

$$\int_{\phi_\mathrm{r}\in\mathrm{meta}} \mathcal{D}\phi_\mathrm{r} \bullet$$

Nucleation rate with Wigner functions

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{\mathsf{meta}} = -\int \mathcal{D}\pi \int_{\partial(\phi_{\mathsf{r}}\in\mathsf{meta})} \mathcal{D}S \cdot \pi W[\phi_{\mathsf{r}}, \pi]$$

- Decrease of probability
  - Flux out of the boundary
- Γ contains all nucleation
  - Vacuum decay (i.e. quantum tunneling)
  - Thermal, over-the-barrier escape

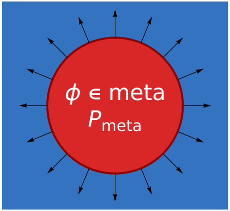


$$\boxed{\Gamma = \frac{\int \mathcal{D}\pi \int_{\partial(\phi_{\mathsf{r}}\in\mathsf{meta})} \mathcal{D}S \cdot \pi W[\phi_{\mathsf{r}}, \pi]}{\int_{\phi_{\mathsf{r}}\in\mathsf{meta}} \mathcal{D}\phi_{\mathsf{r}}\mathcal{D}\pi W[\phi_{\mathsf{r}}, \pi]}}$$

Nucleation from classical to quantum (class. e.g. [Langer '69])

- Rate:

$$\Gamma = -P_{\text{meta}}^{-1} \frac{\mathrm{d}}{\mathrm{d}t} P_{\text{meta}}$$



	Classical	Quantum
Phase space function	Probability distribution, $\rho[\phi, \pi]$	Wigner function, $W[\phi, \pi]$
P_{meta}	$\int \mathcal{D}\pi \int_{\text{meta}} \mathcal{D}\phi\, \rho[\phi, \pi]$	$\int \mathcal{D}\pi \int_{\text{meta}} \mathcal{D}\phi\, W[\phi, \pi]$
$\frac{\mathrm{d}}{\mathrm{d}t} P_{\text{meta}}$	$\int \mathcal{D}\pi \int_{\partial\text{meta}} \mathcal{D}S \cdot \pi\, \rho[\phi, \pi]$	$\int \mathcal{D}\pi \int_{\partial\text{meta}} \mathcal{D}S \cdot \pi\, W[\phi, \pi]$

- Allows to build on classical nucleation theory

Quantum field nucleating over the barrier [Hirvonen '26]

- Over-the-barrier nucleation
  - $O(d) \times S^1$ symmetric bounce
  - (Survives to high T)
  - from Schwinger–Keldysh
- Nucleating Wigner function
- Rate

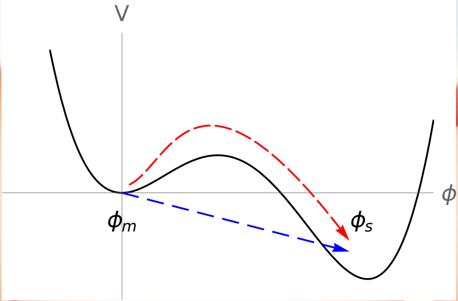


Figure: Two decay channels: quantum tunneling and over-the-barrier escape

Fig: arXiv:1906.00480

Schwinger-Keldysh path integral

- Time dependent expectation values given an initial state $\hat{\rho}(t_i)$
  - Fields "doubled"

$$\langle \hat{\phi}(t_f) \rangle = \int^{\phi_2(t_f)=\phi_1(t_f)} \mathcal{D}\phi_1 \mathcal{D}\phi_2 \, \rho[\phi_1(t_i), \phi_2(t_i)] \, e^{iS(\phi_1) - iS(\phi_2)} \, \phi_1(t_f)$$

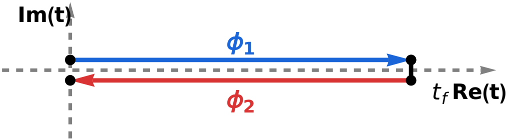


Figure: Schwinger-Keldysh contour

Equation of motion

- Transitioning DoF: $\phi_{\mathrm{r}} = \frac{\phi_1+\phi_2}{2}$
  - (the continuity equation)
- Expanding in $\phi_{\mathrm{a}} = \phi_1 - \phi_2$

  $\rightarrow$ Semi-classical EoM
  - Loop expansion around a saddle
  - Low-T limit

$$\int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \exp\{i(S[\phi_1] - S[\phi_2])\}$$

$$= \int \mathcal{D}\phi_{\mathrm{r}} \mathcal{D}\phi_{\mathrm{a}} \exp\left\{ i \int \mathrm{d}^D x [-\phi_{\mathrm{a}}(\Box\phi_{\mathrm{r}} + V'(\phi_{\mathrm{r}})] + \mathcal{O}(\phi_{\mathrm{a}}^3) \right\}$$

$$\propto \int \mathcal{D}\phi_{\mathrm{r}} \prod_{0<t<t_f,\mathbf{x}} \delta(\Box\phi_{\mathrm{r}} + V'(\phi_{\mathrm{r}}))$$

$$\left(S[\phi] = \int \mathrm{d}^D x \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) \right] \right)$$

Critical bubble and nucleation dynamics

- Thermal bath: frame
- Static saddle: critical bubble, ϕ_{cb}
  - Gatekeeper to nucleation
- Linear EoM around ϕ_{cb}
  - One unstable direction, ϕ_-, with neg. eigenvalue $\lambda_- < 0$

$$\partial_t^2 \phi_{\mathrm{r}} - \nabla^2 \phi_{\mathrm{r}} + V'(\phi_{\mathrm{r}}) = 0$$

$$-\nabla^2 \phi_{\mathrm{cb}} + V'(\phi_{\mathrm{cb}}) = 0$$

$$\left(\partial_t^2 - \nabla^2 + V''(\phi_{\mathrm{cb}})\right)\delta\phi_{\mathrm{r}} = 0$$

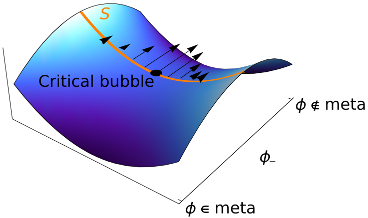


Figure: Critical bubble and negative eigenmode

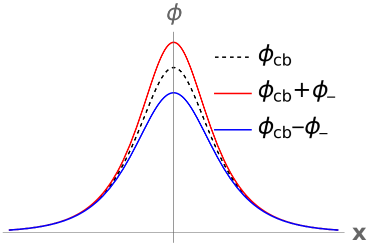


Figure: Critical bubble and negative eigenmode

Steps for the rate

1. Equilibrium density matrix around ϕ_{cb}
2. Transform to Wigner function
3. Separate the part originating from meta
4. Integrate the flux

- Caveat:
  - Assumes equilibrium flow from meta (Andrey's talk)

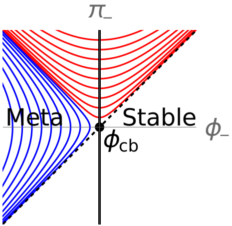
$$\rho_{\text{cb}}[\phi, \phi'] \propto \int_{\delta\phi_{\text{E}}(0)=\phi}^{\delta\phi_{\text{E}}(\beta)=\phi'} \mathcal{D}\delta\phi_{\text{E}}\, e^{-S_{\text{E}}[\phi_{\text{cb}}+\delta\phi_{\text{E}}]}$$

$$\downarrow$$

$$W_{\text{cb}}[\delta\phi_{\text{r}}, \pi]$$

$$\downarrow$$

$$W_{\text{nucl}}[\delta\phi_{\text{r}}, \pi] = \theta(\pi_- - \sqrt{|\lambda_-|}\phi_-)\, W_{\text{cb}}[\delta\phi_{\text{r}}, \pi]$$



$$W_{\mathsf{nucl}}[\delta\phi_{\mathsf{r}}, \pi] = \theta(\pi_- - \sqrt{|\lambda_-|}\phi_-)\, W_{\mathsf{cb}}[\delta\phi_{\mathsf{r}}, \pi]$$

$$\rho_{\mathsf{nucl}}[\phi, \phi'] = \frac{1 - \mathrm{erf}\left[\sqrt{|\lambda_-|\tan\left(\frac{\beta|\lambda_-|}{2}\right)}\left(\phi_- - \frac{i}{2}\cot\left(\frac{\beta|\lambda_-|}{2}\right)\phi_{\mathsf{a}-}\right)\right]}{2}\, \rho_{\mathsf{cb}}[\phi, \phi']$$

Steps for the rate

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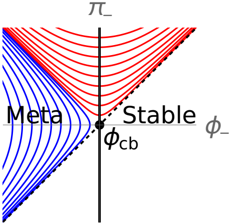
$$\rho_{\mathrm{cb}}[\phi, \phi']$$
$$\propto \int_{\delta\phi_{\mathrm{E}}(0)=\phi}^{\delta\phi_{\mathrm{E}}(\beta)=\phi'} \mathcal{D}\delta\phi_{\mathrm{E}}\, e^{-S_{\mathrm{E}}[\phi_{\mathrm{cb}}+\delta\phi_{\mathrm{E}}]}$$
$$\downarrow$$
$$W_{\mathrm{cb}}[\delta\phi_{\mathrm{r}}, \pi]$$
$$\downarrow$$
$$W_{\mathrm{nucl}}[\delta\phi_{\mathrm{r}}, \pi]$$
$$= \theta(\pi_{-} - \sqrt{|\lambda_{-}|}\phi_{-})\, W_{\mathrm{cb}}[\delta\phi_{\mathrm{r}}, \pi]$$



Low temperature limit

- Wigner function factorizes
- Negative eigenmode part divergent at

$$T = \frac{\sqrt{|\lambda_-|}}{\pi}$$

- Nonperturbative in ϕ_a

$$\begin{aligned}
W_{\text{nucl}}[\phi_{\text{r}}, \pi] = {} & e^{-\beta \Delta E_{\text{cb}}} \prod_j \sinh\left(\frac{\beta\sqrt{\lambda_j^{\text{meta}}}}{2}\right) \\
& \times \frac{\theta(\pi_- - \sqrt{|\lambda_-|}\phi_-)}{\pi \cos\left(\frac{\beta\sqrt{|\lambda_-|}}{2}\right)} e^{-\tan\left(\frac{\beta\sqrt{|\lambda_-|}}{2}\right)\left(-\sqrt{|\lambda_-|}\phi_-^2 + \frac{\pi_-^2}{\sqrt{|\lambda_-|}}\right)} \\
& \times \prod_{j_0} \frac{1}{\pi} e^{-\frac{\beta \pi_{j_0}^2}{2}} \\
& \times \prod_{j_+} \frac{1}{\pi \cosh\left(\frac{\beta\sqrt{\lambda_{j_+}}}{2}\right)} e^{-\tanh\left(\frac{\beta\sqrt{\lambda_{j_+}}}{2}\right)\left(\sqrt{\lambda_{j_+}}\phi_{\text{r}j_+}^2 + \frac{\pi_{j_+}^2}{\sqrt{\lambda_{j_+}}}\right)} .
\end{aligned}$$

Result

- Exp. suppression from ϕ_{cb}
- Equilibrium *quantum* fluctuations
  - Periodic Euclidean time, $\tau \in [0, \beta)$
  - Only the positive modes
- Equiv. to finite-dimensional QM [Affleck '80]
  - Differs from [Linde '80]

$$\Gamma = \frac{V}{2\pi}\left(\frac{S_{\text{E,cb}}}{2\pi}\right)^{\frac{d}{2}}\left(\frac{\det\left(-\partial_\mu^2 + V''(\phi_{\text{meta}})\right)}{\det^{+}\left(-\partial_\mu^2 + V''(\phi_{\text{cb}})\right)}\right)^{\frac{1}{2}} e^{-S_{\text{E,cb}}}$$

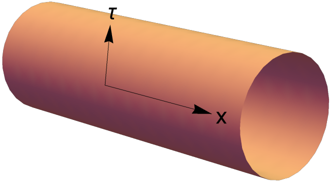


Figure: Periodic Euclidean spacetime of the operators

Time-dependent rate in cosmology

- Rate decreases exponentially: $e^{-S_{\mathrm{E,cb}}} = e^{-\beta E_{\mathrm{cb}}}$
  - Larger in the past
- Largest rates
  - Require thermal resummations
  - Critical bubble strongly affected
  - Beyond the linearized EoM

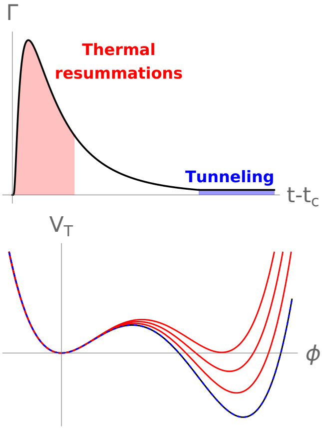
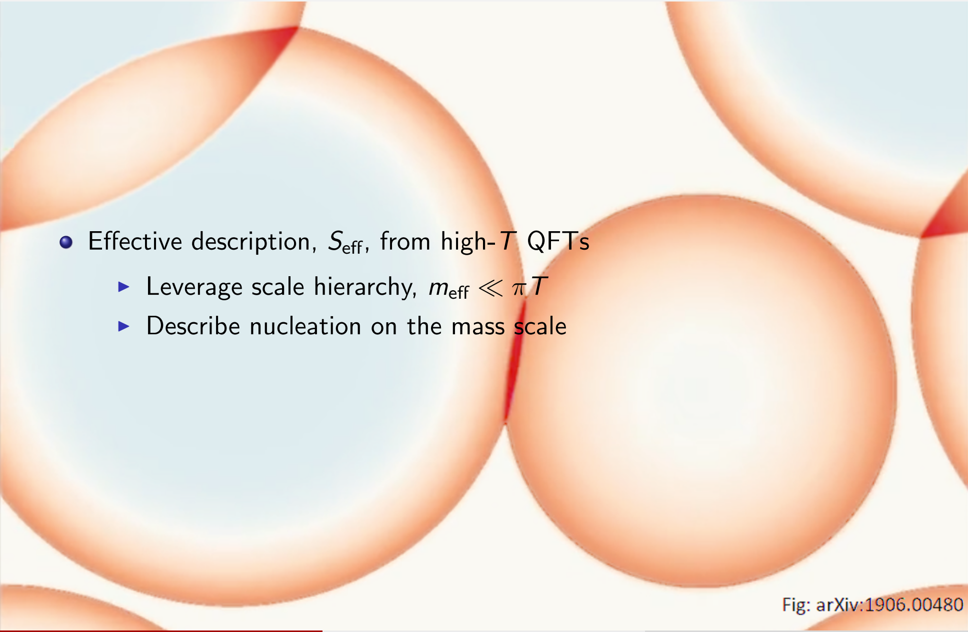


Figure: Time-dependent rate and potential

Effective field theory approach [Gould & Hirvonen '21]



- Effective description, S_{eff}, from high-T QFTs
  - Leverage scale hierarchy, $m_{\mathrm{eff}} \ll \pi T$
  - Describe nucleation on the mass scale

Fig: arXiv:1906.00480

Fluctuation scales

$$\Gamma = \frac{V}{2\pi}\left(\frac{S_{\mathrm{E,cb}}}{2\pi}\right)^{\frac{d}{2}}\left(\frac{\det\left(-\partial_\mu^2 + V''(\phi_{\mathrm{meta}})\right)}{\det^{+}(-\partial_\mu^2 + V''(\phi_{\mathrm{cb}}))}\right)^{\frac{1}{2}} e^{-S_{\mathrm{E,cb}}}$$

- Two scales in the determinants
  - Mass scale, $\sqrt{V''}$
  - Thermal scale, $T = \beta^{-1}$
- Thermal corrections grow at high-T
  - $T \gg \sqrt{V''}$
- Nucleation on the mass scale!
  - Integrate out πT scale

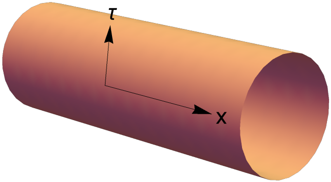


Figure: Periodic Euclidean spacetime

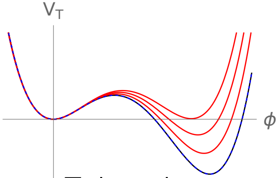
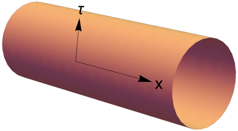


Figure: T-dependent potential

High-temperature dimensional reduction
[Kajantie et al. '95, Braaten & Nieto '95]



$$Z = \int \mathcal{D}\Phi \, e^{-S_{\mathrm{E}}}$$

$$= \int \mathcal{D}\Phi_0^{\mathrm{IR}} \, \mathcal{D}\Phi_0^{\mathrm{UV}} \, \mathcal{D}\Phi_{n\neq 0} \, e^{-S_{\mathrm{E}}}$$

$$= \int \mathcal{D}\Phi_0^{\mathrm{IR}} \, e^{-S_{\mathrm{eff}}[\Phi_0^{\mathrm{IR}}]} \, ,$$

- Equilibrium EFT for the mass scale
  - Describes nucleating bubbles
- Integrating out thermal scale
  [Hirvonen '22]
  - Usual 1PI action...
  - ...but expand in light masses
    [Burgess '20, Hirvonen '22]

$$S_{\mathrm{eff}}[\Phi_0^{\mathrm{IR}}]$$

$$= -\ln \int \mathcal{D}\Phi_0^{\mathrm{UV}} \, \mathcal{D}\Phi_{n\neq 0} \, e^{-S_{\mathrm{E}}[\Phi_0^{\mathrm{IR}} + \Phi_0^{\mathrm{UV}} + \Phi_{n\neq 0}]}$$

$$\frac{1}{P^2 + m_{\mathrm{nucl}}^2} \to \frac{1}{P^2} - \frac{m_{\mathrm{nucl}}^2}{P^4} + \dots$$

Nucleation rate

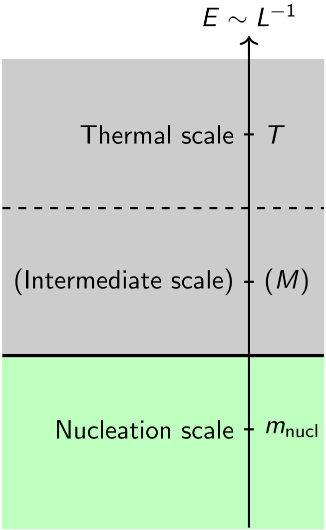
- Effective action, S_{eff}
  - Critical bubble
  - Rate
- Determinants with `BubbleDet` [Ekstedt, Gould, Hirvonen '23]
- No
  - double counting
  - imaginary parts
  - diverging derivative expansions

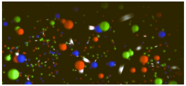
$$\frac{\delta S_{\text{eff}}}{\delta \phi}[\phi_{\text{cb}}] = 0$$

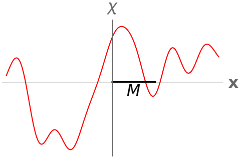
$$\Gamma = \frac{V}{2\pi}\left(\frac{S_{\text{eff,cb}}}{2\pi}\right)^{\frac{3}{2}} \underbrace{\left(\frac{\det\left(-\nabla^2 + V''_{\text{eff,meta}}\right)}{\det^{+}\left(-\nabla^2 + V''_{\text{eff,cb}}\right)}\right)^{\frac{1}{2}}}_{E\,\sim\,m_{\text{eff}}} e^{-\overbrace{S_{\text{eff,cb}}}^{E\sim\pi T}}$$

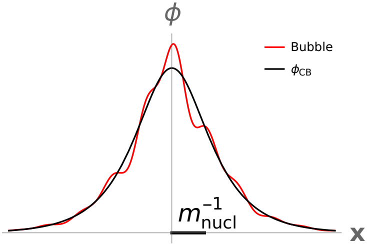
Scales in high-temperature nucleation

- Particles
- Fields
  - Approx. classical









Kinetic theory approach [Hirvonen '24]

$$\Box\phi + V_T'(\phi) = -\sum_a \frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E}\delta f_a$$

$$\left(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla} + \mathbf{F}_a\cdot\partial_{\mathbf{p}}\right) f_a = 0$$

- Real-time, off-equilibrium description
  - Infrared *classical* field, $\phi = \phi_{\mathrm{r}}^{\mathrm{IR}}$
  - Thermal particles, f_a
  - Equivalent to contours at leading order
- Equilibrium quite OK!

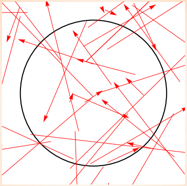


Figure: Bubble in a particle shower

Fig: arXiv:1906.00480

Effective Liouville equation

- EoMs into Liouville equation
- Apply classical nucleation theory

$$\Box\phi + V_T'(\phi) = -\sum_a \frac{\mathrm{d}m_a^2}{\mathrm{d}\phi}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3 2E}\delta f_a$$

$$\left(\partial_t + \mathbf{v}\cdot\boldsymbol{\nabla} + \mathbf{F}_a\cdot\partial_{\mathbf{p}}\right) f_a = 0$$

$$\downarrow$$

$$\frac{\partial}{\partial t}\rho[\phi,\pi,\delta f_a] = -\int \mathrm{d}^d\mathbf{x}\Big(\dot\phi\frac{\delta}{\delta\phi} + \dot\pi\frac{\delta}{\delta\pi} + \sum_a\int \mathrm{d}^d\mathbf{p}\,\delta\dot{f}_a\frac{\delta}{\delta\delta f_a}\Big)\rho[\phi,\pi,\delta f_a]$$

Same potential

- Matching ϕ equilibrium statistics
  - Kinetic theory, linear order in δf_a
  - Dimensionally reduced EFT from imaginary time
- Applies to wall velocities too

$$\rho_{\text{eq}}^{\text{kin}}[\phi] = \rho_{\text{eq}}^{\text{EFT}}[\phi]$$
$$\Downarrow$$
$$V_T(\phi) = V_{\text{eff}}(\phi)$$

Rate

$$\Gamma_{\text{kin}} = \frac{\kappa}{\sqrt{|\lambda_-|}}\Gamma_{\text{EFT}}$$

$$\phi = \phi_{\text{CB}} + \delta\phi \ \exp(\kappa t)$$

$$f_a = f_a^{\text{eq}} + \delta f_a \ \exp(\kappa t)$$

- Dynamical prefactor
- Exponential growth rate, κ
  - Modified by off-eq. particles
  - Approximation from `BubbleDet`: $\kappa \approx \sqrt{|\lambda_-|}$

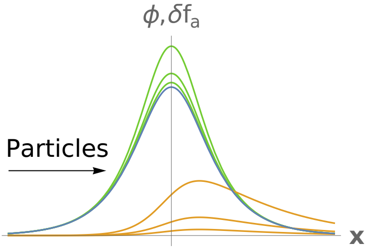
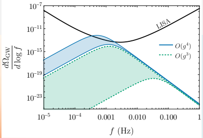
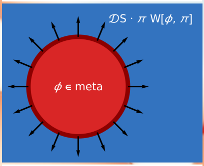


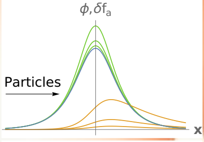
Figure: Exponentially growing bubble, pushes particles out of equilibrium

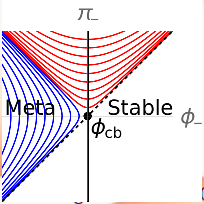
Summary

- Wigner function approach
  - Quantum basis
  - Generalization to high-T
  - Beyond classical field and collisionless
- EFTs useful in high-T
  - See Maciej's and Oliver's talks
- Kinetic theory
  - Backs equilibrium EFTs
  - When is κ important?
- Equilibrium flow assumption Andrey's talk
  - When does it hold?
  - 3+1 dim, high-T, cosmological phase transitions









Thanks for listening!

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