

Thermal partition function from the S-matrix reloaded

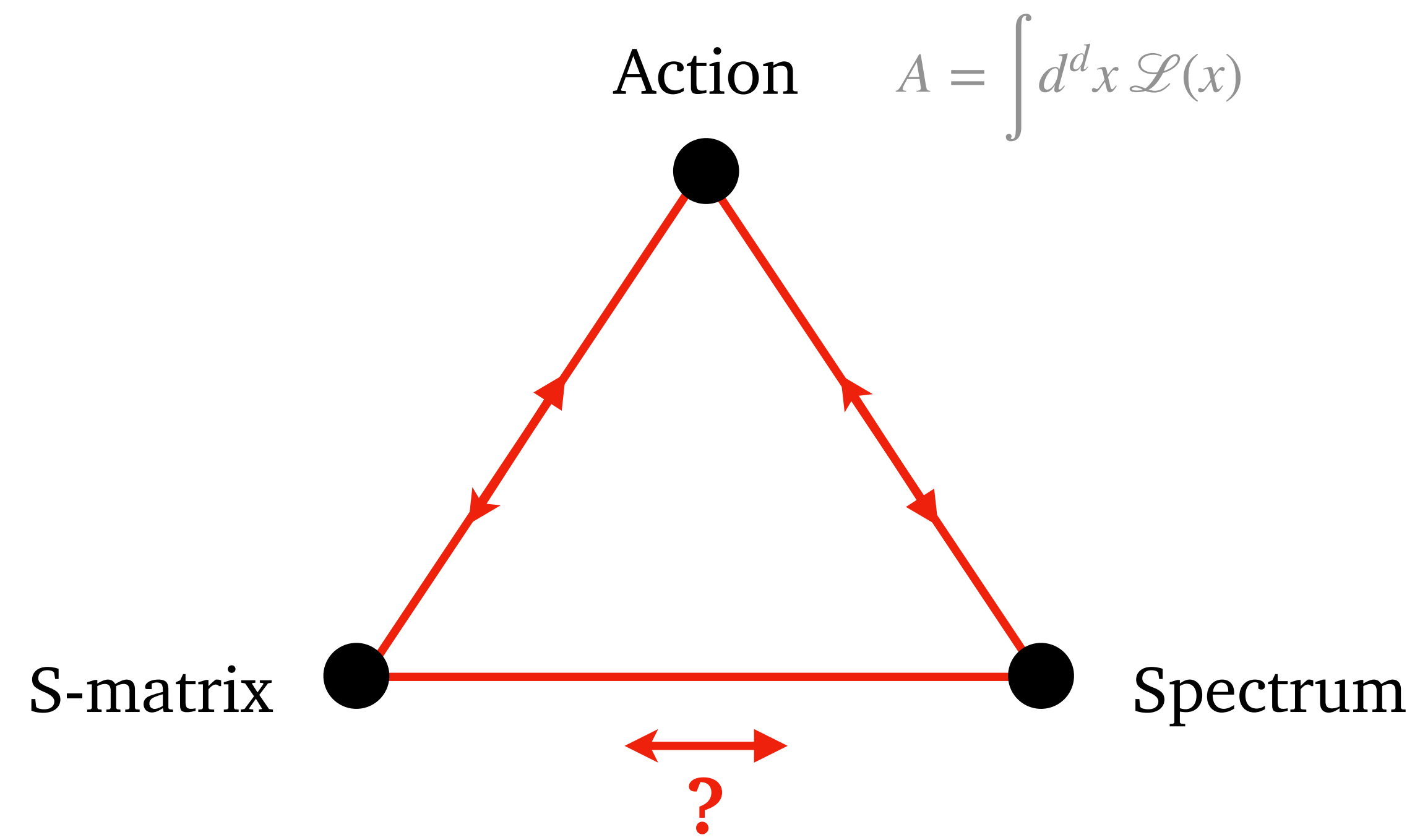
Joan Elias Miró, ICTP Trieste

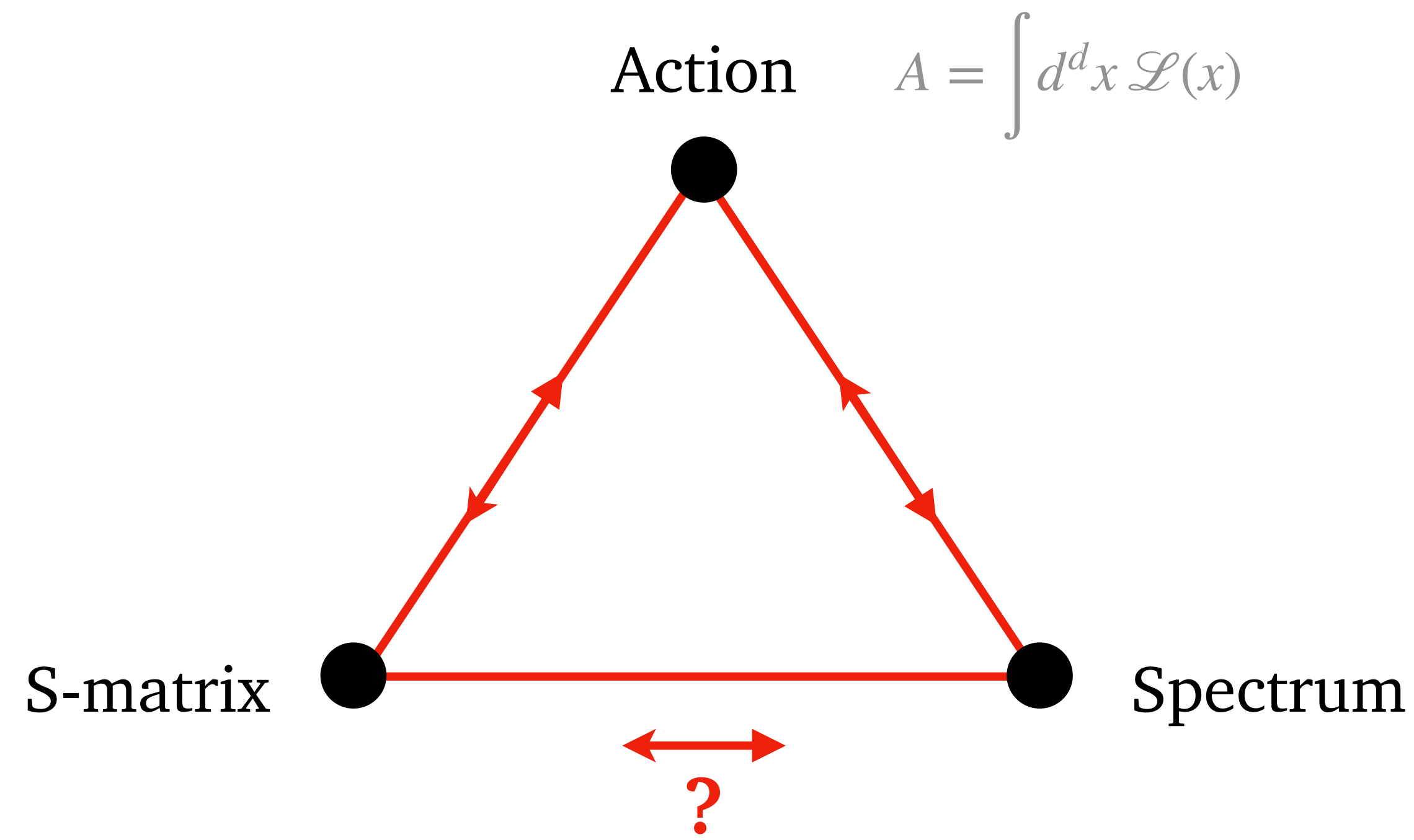
EFTs for Nonperturbative Physics — September 2, 2026

Based on work

- [hep-th/2408.06729](#) with P. Baratella and E. Gendy
- [hep-th/2512.16466](#) with P. Baratella

See also the recent work by D. Schubring [hep-th/2408.00729](#)





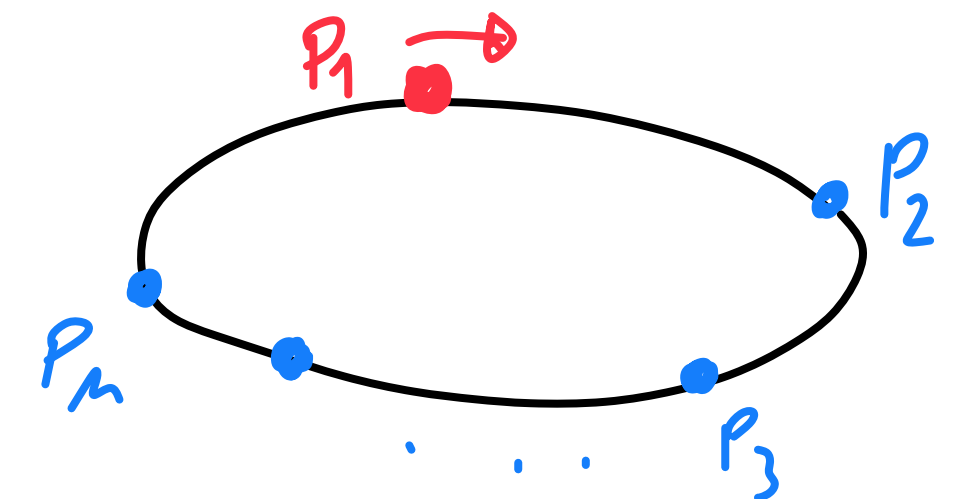
In the elastic regime ABA / Lüscher's Formula, and TBA

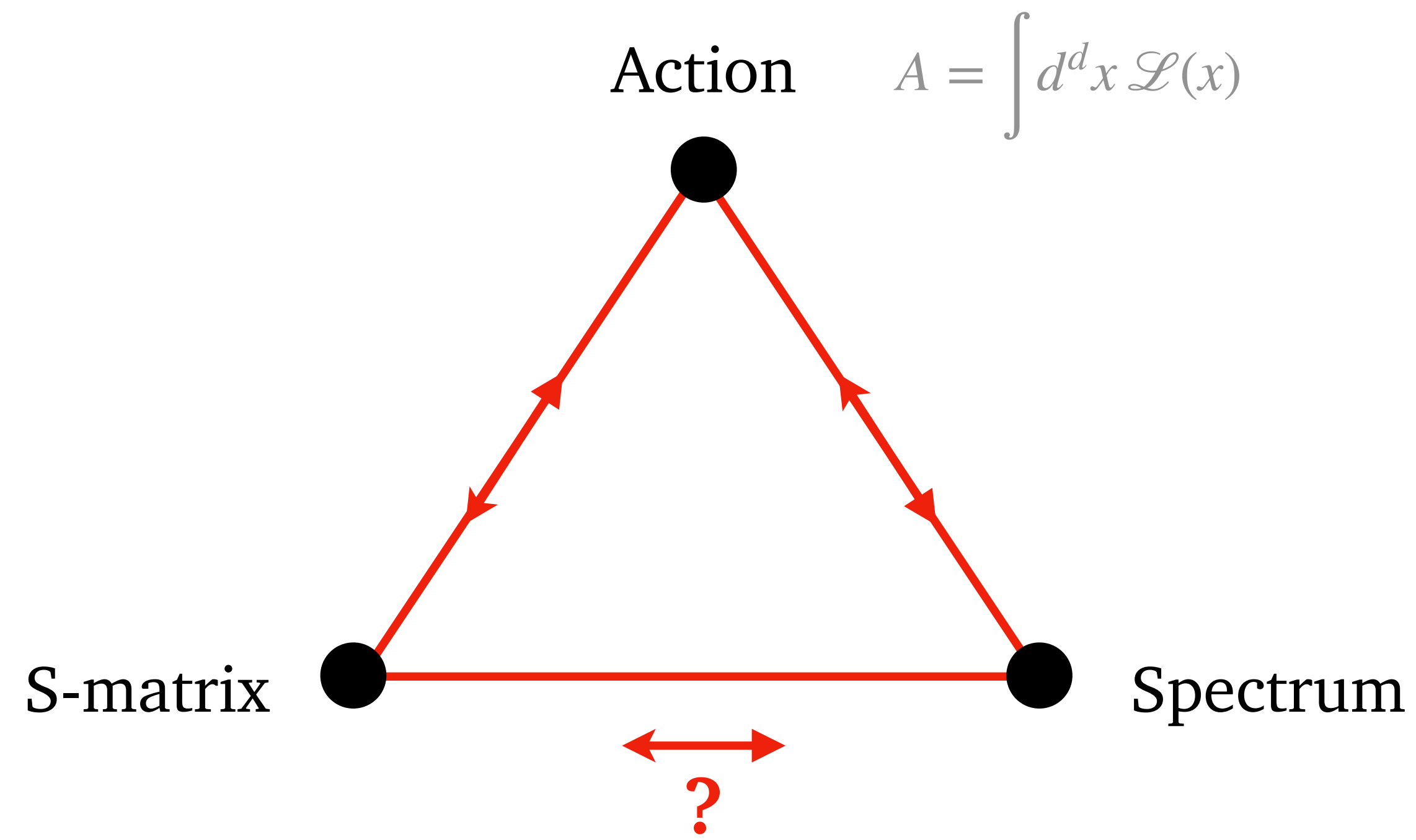
1.- Consider $x_1 \ll x_2$ (mutatis mutandis $x_1 \gg x_2$)

$$\psi(x_1 \gg x_2) = e^{iP_1 x_1 + iP_2 x_2} + e^{iP_1 x_2 + iP_2 x_1} S_{12}(P_1, P_2)$$

2.- Place the theory at finite volume w/ $x_1 \ll x_2 \ll x_1 + R$

$$\psi(x_1, x_2) = \psi(x_1 + R, x_2) \Rightarrow e^{iP_1 R} S_{12}(P_1, P_2) = 1$$

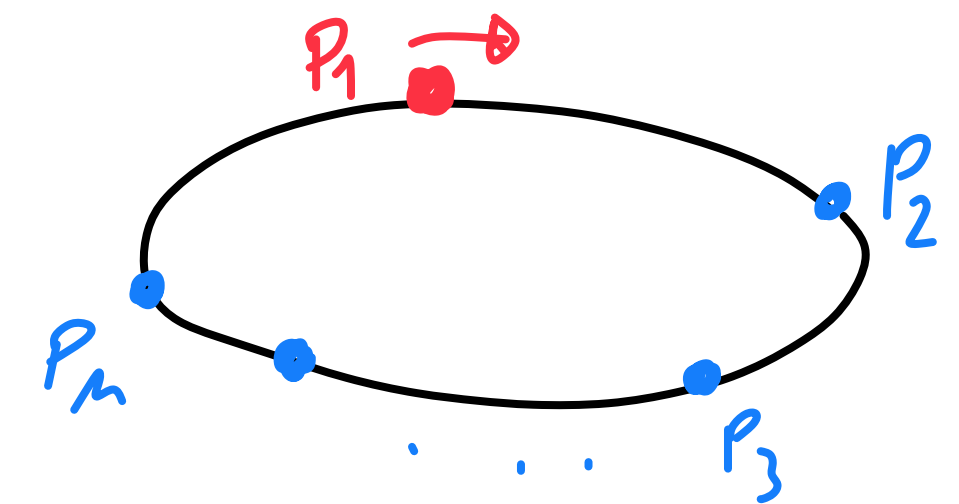




In the elastic regime ABA / Lüscher's Formula, and TBA

Same logic for multiple rescatterings

$$p_i R + \sum_j 2 \delta_{ij} (p_i, p_j) = 2\pi N_i$$



Action $A = \int d^d x \mathcal{L}(x)$

$S(s) = 1 + i\frac{s}{4}\ell_s^2 - \frac{s^2}{32}\ell_s^4 + i(\gamma_3 - \frac{1}{384})s^3\ell_s^6 + \dots$

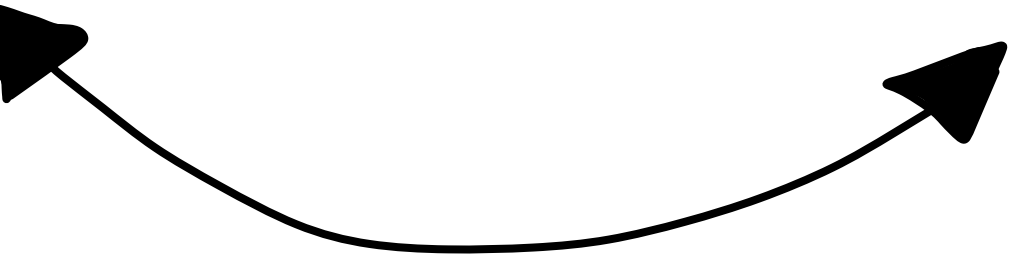
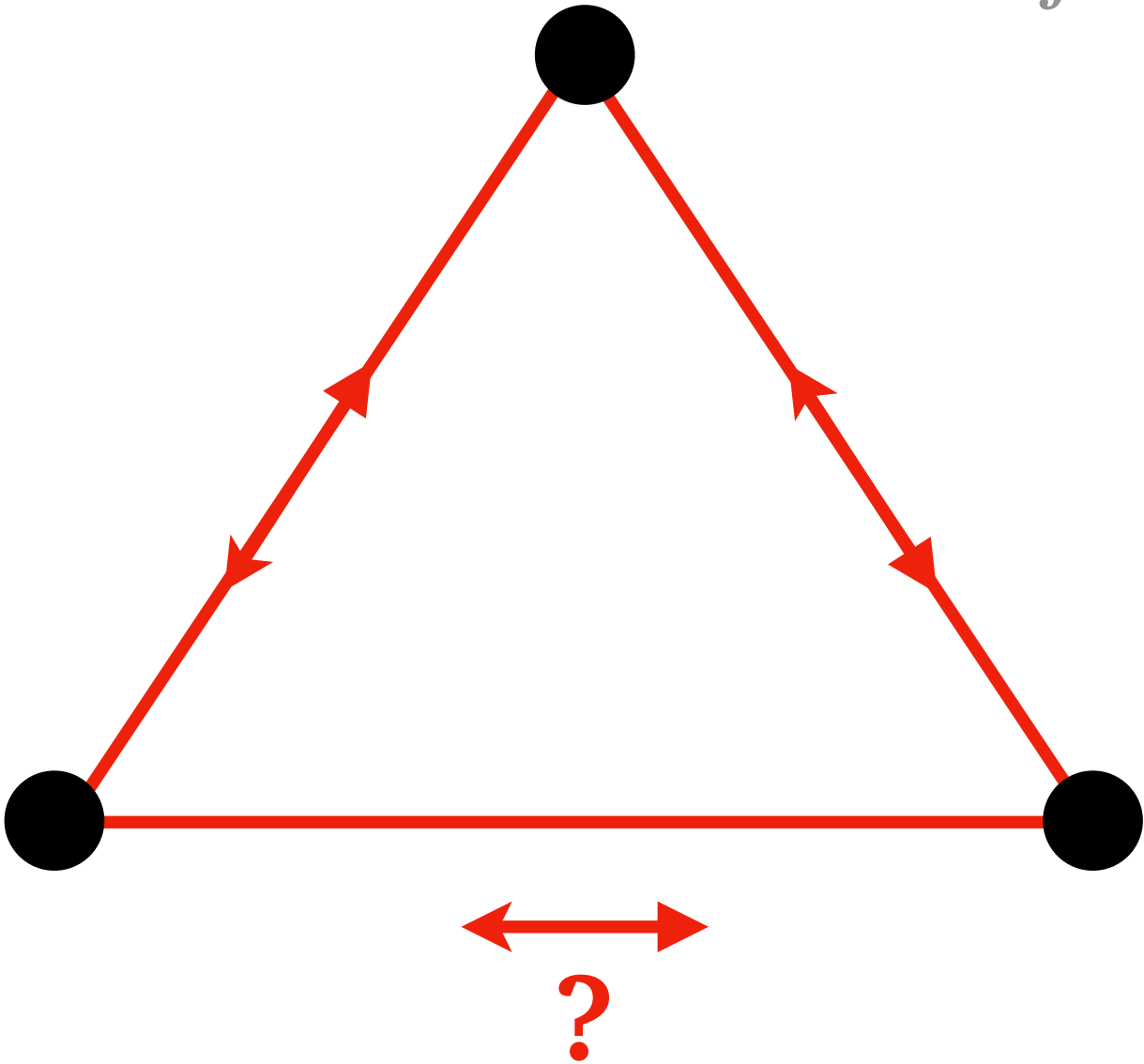
S-matrix

$E_0(R) = \frac{R}{\ell_s^2} \sqrt{1 - \frac{\pi\ell_s^2}{3R^2}} - \frac{32\pi^6}{225} \frac{\gamma_3\ell_s^6}{R^7} + O(R^{-9})$

Spectrum

$= + \text{X} + \text{loop} + (\text{X} + \text{loop}) + \dots$

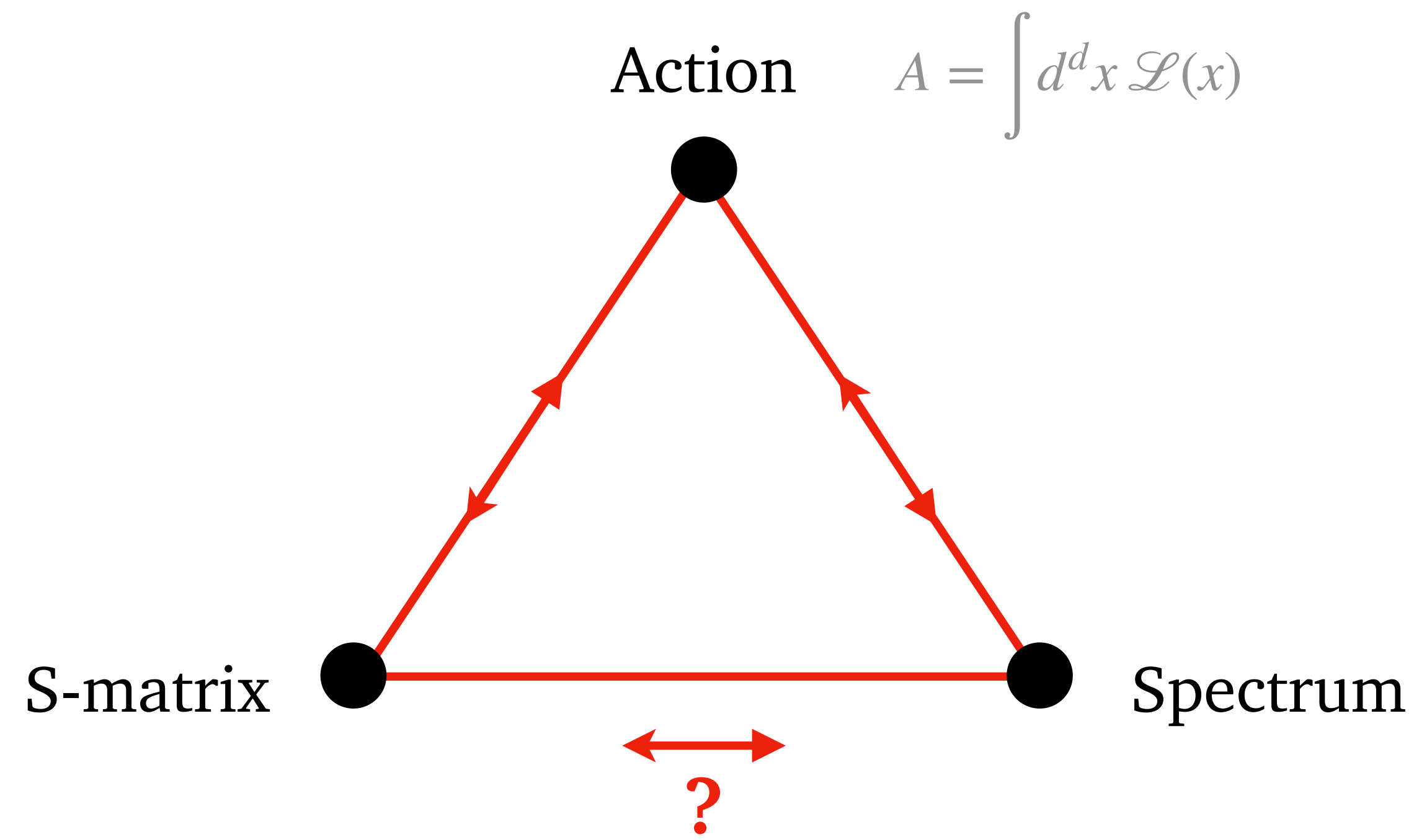
$\bigcirc + \infty + \infty\infty + \dots$



some sort of tracing involved.

$\text{loop} + \text{X} + \text{loop}$

Intuition indeed realized!



DMB equation — Dashen, Ma, Bernstein [Phys.Rev. 197 (1969) 345-350]

$$Z(\beta) = Z_0(\beta) - \frac{\beta}{2\pi i} \int_0^\infty dE e^{-\beta E} \text{Tr} \log S(E)$$

Can a vacuum S-matrix reproduce thermal physics beyond the orders where the forward limit is harmless?

yes, apparent forward singularities are resolved by the energy-dependent DMB operator, while genuine IR sensitivity reorganizes into screening/thermal masses.

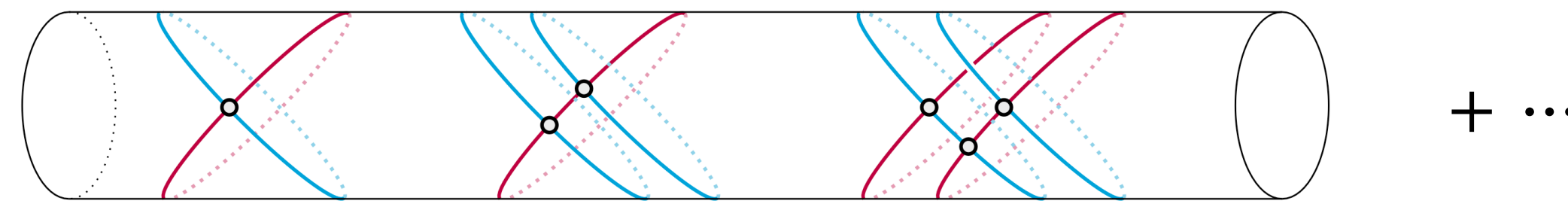
$$Z(\beta) = Z_0(\beta) - \frac{\beta}{2\pi i} \int_0^\infty dE e^{-\beta E} \text{Tr} \log S(E + i\epsilon)$$

1 Purely imaginary $\text{Tr} \log S = \log \det S$

2 Fully (disconnected) S-matrix $\langle \alpha | S(E = E_\alpha) | \beta \rangle = \mathbb{I} + 2\pi i \delta(E_\alpha - E_\beta) T_{\alpha\beta}$

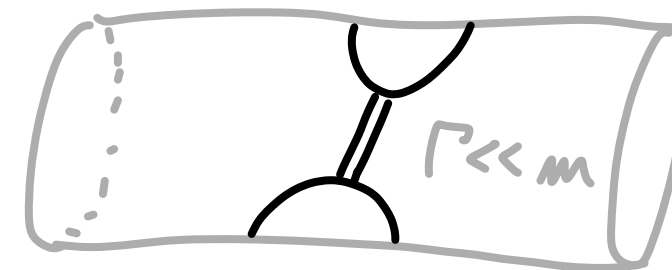
3 Note that the particle density is given by $\rho(\epsilon) = \rho_0(\epsilon) + \frac{1}{2\pi i} \partial_\epsilon \text{Tr} \log S(\epsilon)$

4 $S(E)$ is the vacuum S-matrix, Trace + integral with Boltzmann weights generates thermal physics. Pictorially:



5 Thermal partition function. Derivation formally valid on any $S_1 \times \Sigma$. We will take R^d .

6 Long lived particles count as degrees of freedom

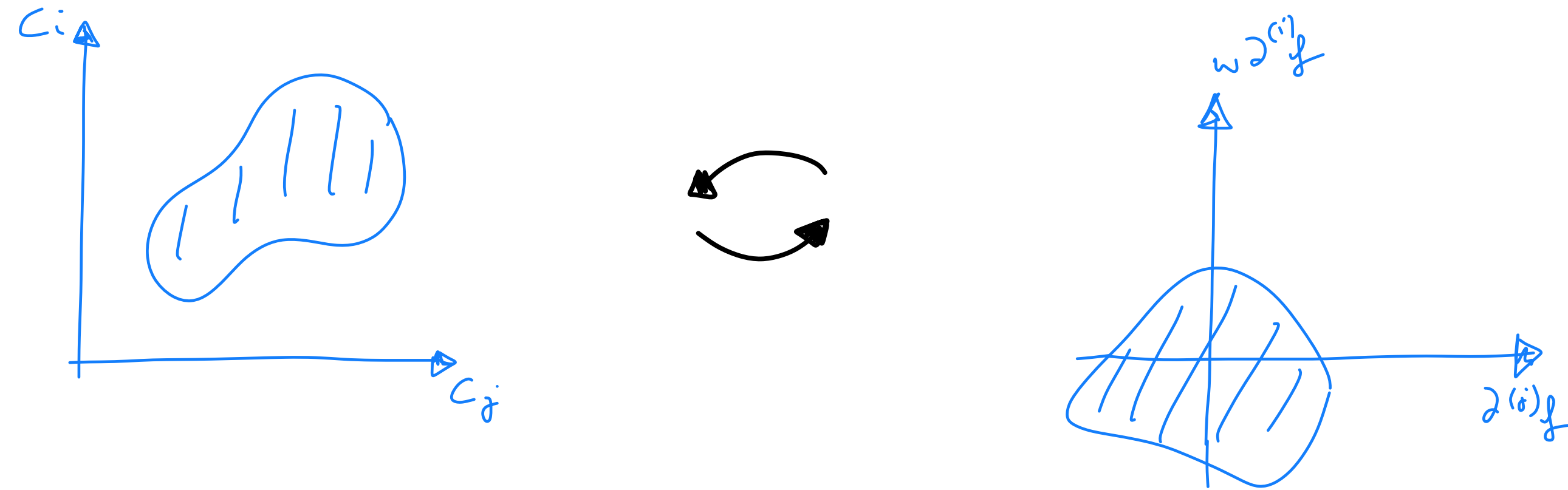


7 DMB mostly used in non-relativistic context, at tree-level with 2-particle states — leading order.

↳ Why now?

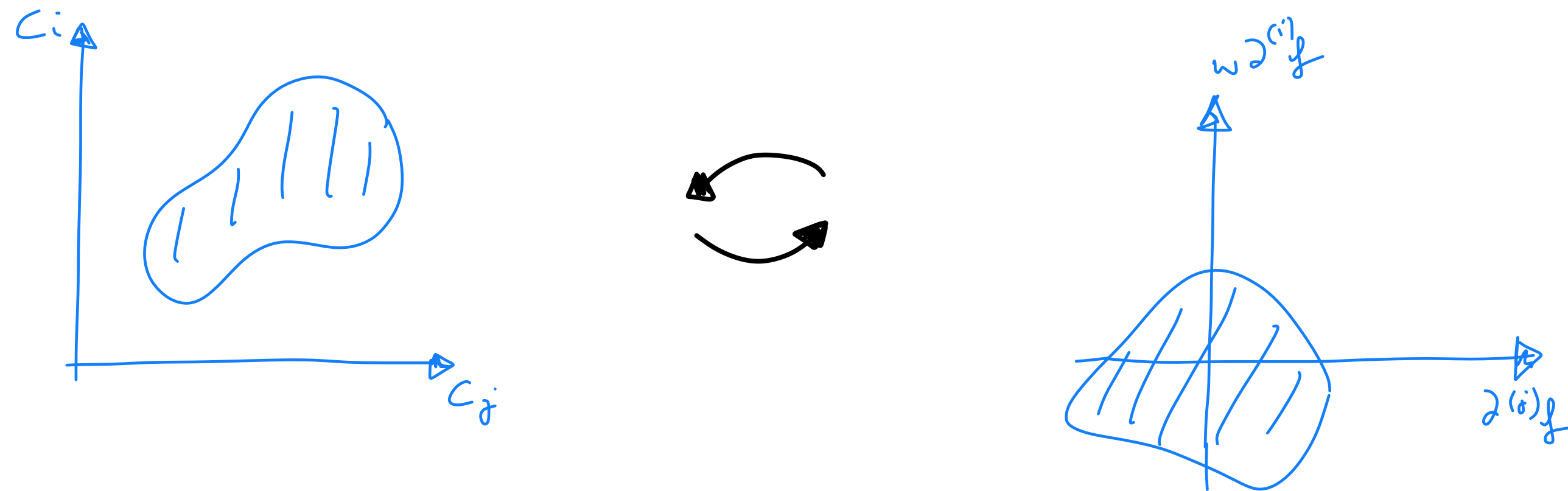
* Many new results constraining the space of EFT's, typically in terms of S-matrix w/o reference to $\mathcal{L}$.

How can we transfer this info to other physical observables?



* Many new results constraining the space of EFT's, typically in terms of S-matrix w/o reference to $\mathcal{L}$.

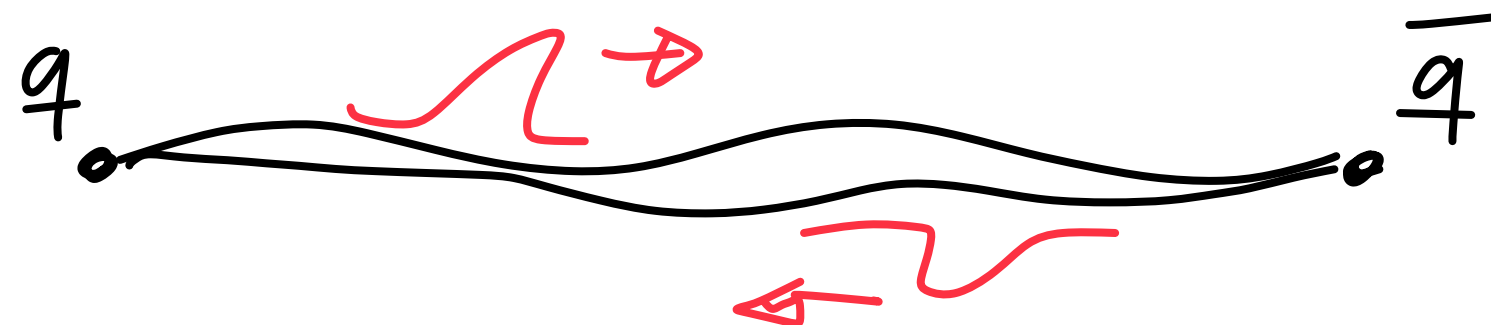
How can we transfer this info to other physical observables?



↪ e.g. "Ghosh condensation" or $P(X = (\partial\phi)^2)$ theories.

on $X = \text{ste}$ background $c_s = \frac{P'(X)}{P'(X) + 2X P''(X)} < 1 \Rightarrow P(X) \text{ convex}.$

↪ Bounds on Flux Tubes



$$V(r) = \sigma R + \frac{\pi(D-2)}{R} + \dots = \sqrt{R^2 - \frac{\pi}{3}(D-2)} + \frac{\delta(D)}{R^7} + O(1/R^9)$$

$\delta(D)$ is bounded
by S-matrix bootstrap
 $S(s) = e^{i[s + \text{weff.} \times \delta(D) \times s^3 + O(s^5)]}$

✱ With the development of on-shell S-matrix methods, we have access to precise Matrix elements with arbitrary number of particles. Bad convergence of QCD- New calculation scheme (no ghost, no gauge redundancies). Alternative treatment of the QCD equation of state based on an IR safe S-matrix?

✱ The method shines when the path integral formulation is not available, yet we may have access to the S-matrix.

Plan of the talk



1.- Introduction



2.- Formal derivation of DMB

3.- Applications & recent developments

2.- Formal derivation

straight forward to account for.
But I'll omit from now on.

$$Z(\mu, \beta) = \text{Tr} e^{-\beta(H - \mu Q)}$$

$$= \int_0^\infty dE e^{-\beta E} \text{Tr} \left[\frac{1}{E - H + i0^+} - \frac{1}{E - H - i0^+} \right]$$

$\equiv G^* - G$. Recall that $\lim_{\epsilon \rightarrow 0} \frac{i\pi}{x + i\epsilon} = \delta(x)$

Next we need to relate this kernel to the S-matrix

$$\text{Define } G_0(E) \equiv \frac{1}{E - H_0 + i0^+} \quad \& \quad G(E) \equiv \frac{1}{E - H + i0^+}$$

2.- Formal derivation

$$1.- \quad S(E) = G_0^* (G^*)^{-1} G G_0^{-1}$$

From Lippman - Schwinger

$$\psi_{\pm} = \phi + G_0(E \pm i\epsilon) V \psi_{\pm} \quad \leadsto \quad \psi^{\pm} = \Omega^{\pm} \phi \quad \text{w/ } (\Omega^{\pm})^{-1} = (1 - G_0 V)^{-1}$$

$$\text{Then } S(E) = (\Omega^{-})^{\dagger} \Omega^{+}$$

$$2.- \quad \text{Tr } S^{-1} \partial_E S = \text{Tr} [-G_0^* + G^* - G + G_0]$$

↑ simple algebra

$$\Rightarrow Z(\beta) = \frac{1}{2\pi i} \int_0^{\infty} dE \, e^{-\beta E} \left\{ \text{Tr} [S^{-1}(E) \partial_E S(E)] - \text{Tr} [G_0(E) - G_0^*(E)] \right\}$$

$$= Z_0(\beta) - \beta \int_0^{\infty} \frac{dE}{2\pi i} e^{-\beta E} \text{Tr} \log S(E) \equiv e^{-\beta F} //$$

Free energy density $f = \lim_{V \rightarrow \infty} \frac{F}{V}$. Upon taking the 'log' we get an extensive quantity, i.e. a single power of V . $\log Z = f \cdot V \times \beta$.

$$F(\beta) = F_0(\beta) - \frac{1}{2\pi i} \int_0^\infty dE \, e^{-\beta E} \text{Tr}_\kappa \log S(E)$$

connected.

... from the free energy $F = E - TS$ thermodynamic quantities follow:

$$\hookrightarrow \text{e.g.} \quad w = \frac{p}{\ell} ; \quad p = -f ; \quad C = \frac{\partial(\beta \ell)}{\partial \beta} ; \quad S = -\partial_T F.$$

$$\hookrightarrow \text{or derived quantities such as } C_v = T \frac{\partial^2 p}{\partial T^2} ; \quad C_s^q = \frac{\partial \ell}{\partial p}, \text{ etc.}$$

Plan of the talk

- ✓ 1.- Introduction
- ✓ 2.- Formal derivation of DMB
- ➔ 3.- Applications & recent developments

3.- Applications & recent developments

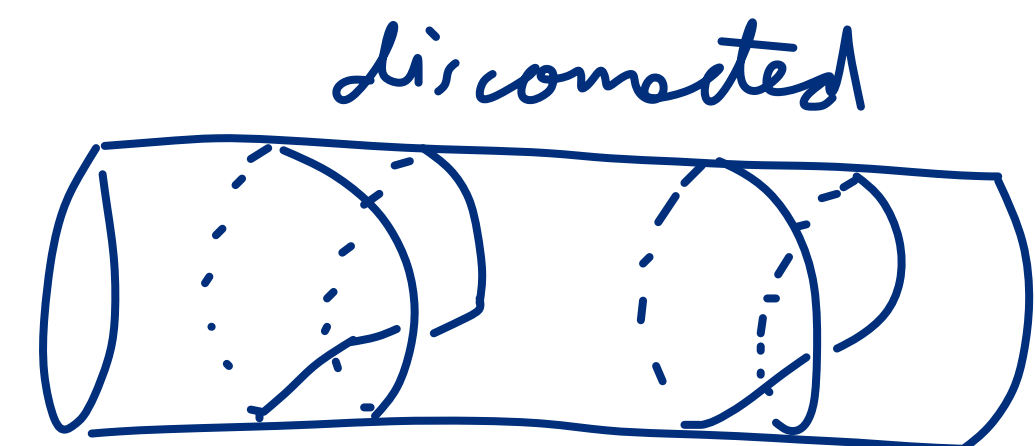
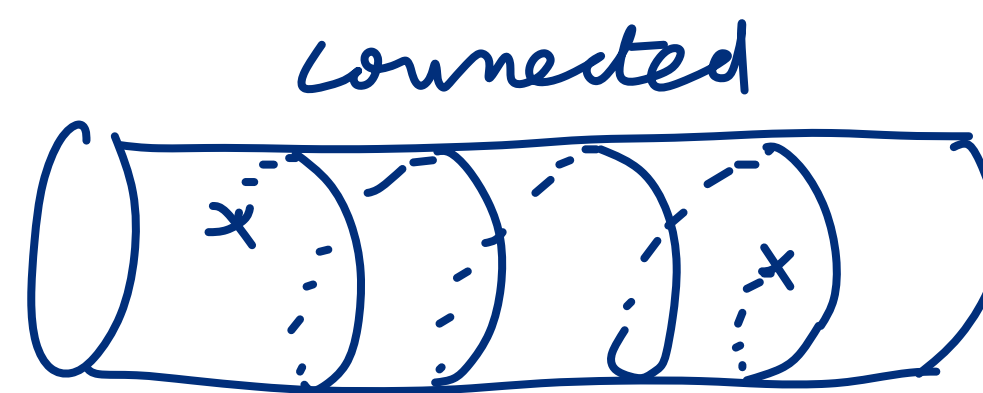
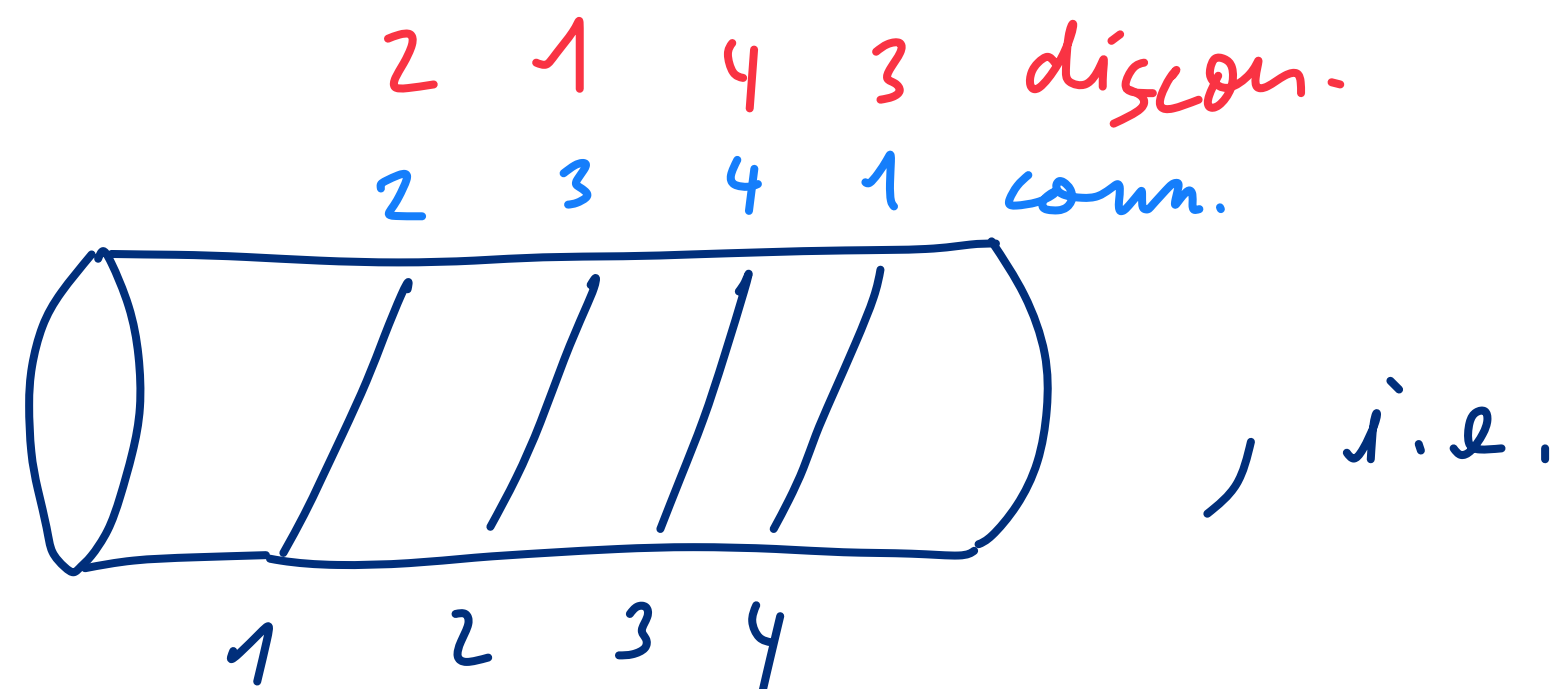
* Simplest things first: free theory

$$-\beta F_0 = \text{Tr}_c \bar{e}^{\beta H_0} = \sum_{N=0}^{\infty} \frac{1}{N!} \int d^3 k_1 \dots d^3 k_N \langle k_1, \dots, k_N | \bar{e}^{\beta H_0} | k_1 \dots k_N \rangle_c$$

↳ Next we use that H_0 is diagonal and $\langle k_1 \dots k_n | k'_1 \dots k'_n \rangle = \delta(k_1 - k'_1) \dots \delta(k_n - k'_n) \pm \text{perms.}$

↳ when we identify $k_i = k'_i$ through the trace we may get many $\delta(0)$'s, up to $\delta(0)^N$

↳ Here " c " enters crucially, leaving terms with a single $\delta(0)$. (no $\delta(0)$ for $N=0$).



For a given connected history there are $(N-1)!$ equivalent ones, due to cyclic perms.

All in all

$$- \beta F_0 = 1 + \delta(0) \int d^3k \sum_{N=1}^{\infty} \frac{1}{N} \bar{e}^{N\beta E(k)} = 1 - \delta(0) \int d^3k \log(1 - \bar{e}^{\beta E(k)})$$

$\delta(0) \rightarrow V/(2\pi)^3$ and therefore

$$f_0 = \frac{T}{2\pi^3} \int_0^{\infty} dk k^2 \ln(1 - \bar{e}^{\beta k}) = -\frac{\pi^2 T^4}{90} \times 2(N_c^2 - 1)$$

polarizations $\times$ # gluons

aside, to be compared with text book calculation:

$$\text{---} + \text{---} = -\pi^2 \frac{T^4}{90} (N_c^2 - 1) [-2 + 4]$$

* Interactions at leading order:

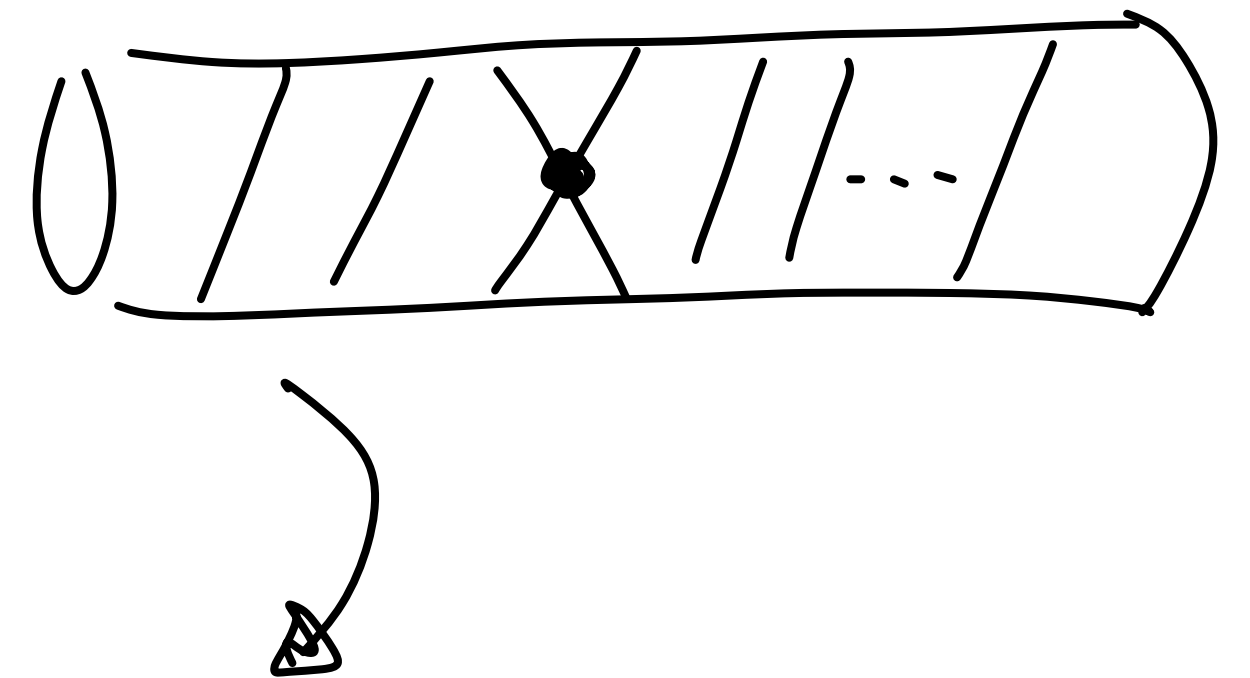
$$F - F_0 = \int dE \, e^{-\beta E} \text{Tr}_e [T \delta(E - H)] + O(T^2)$$

a bit of algebra $\rightarrow$

$$= \underbrace{(2\pi)^3 \delta(0)}_{V} \sum_{N=2}^{\infty} \frac{1}{N!} \int \prod_{i=1}^N d^3 \tilde{k}_i \, n(k_i) \langle k_1 \dots k_N | T | k_1 \dots k_N \rangle + O(T^2)$$

w/ $d^3 \tilde{k}_i = \frac{d^3 k_i}{(2\pi)^3 2\pi \epsilon_i}$ & $n(k_i) = \frac{1}{e^{\beta \epsilon_i} - 1}$ the number density.

$f \equiv \frac{F}{V}$, free energy density.



* QCD at $O(\alpha_s)$

We need the $2 \rightarrow 2$ S-matrix element:

all incoming: $\mathcal{M}(1^-, 2^+, 3^+, 4^-) = 2g_s^2 \langle 14 \rangle^2 [23]^2 \left(\frac{f^{abe} f^{cde}}{s_{12} s_{14}} + \frac{f^{ace} f^{bde}}{s_{13} s_{14}} \right)$

contains all dynamical info for $++ \rightarrow ++$, $-- \rightarrow --$ & $+- \rightarrow +-$

Forward limit very simple for all of them:

in forward $\mathcal{M} = -2g_s^2 f^{abc} f^{abc} = -2g_s^2 N_c (N_c^2 - 1)$

Therefore

$$\mathcal{I} - \mathcal{I}_0 = 2g_s^2 \left(\overset{++}{\frac{1}{2}} + \overset{--}{\frac{1}{2}} + \overset{+-}{1} \right) N_c (N_c^2 - 1) \left[\int_0^\infty \frac{4\pi K^2 dK}{(2\pi)^3 2K(e^{\beta K} - 1)} \right]^2 + O(g_s^4)$$

$$= \alpha_s N_c (N_c^2 - 1) \frac{\pi T^4}{36}$$



↑
state of the
art is $g_s^4 +$
corrections.

To be compared w/ text book computation, e.g. Ch. 4.5 of Laine & Vuorinen,
 "Basics of Thermal Field Th."



$$= \frac{g^2}{4} 3 N_c \oint_1 \frac{\delta^{ac} g_{\mu\nu}}{k^2} \oint_1 \frac{1}{k^2} = 3g^2 N_c (N_c^2 - 1) \frac{T^4}{(12)^2}$$



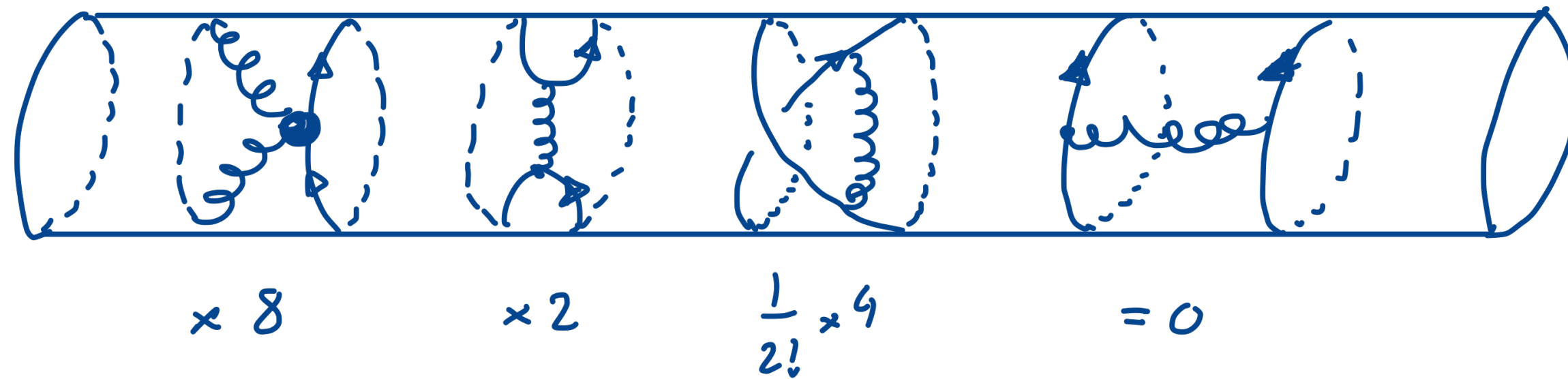
$$= - \frac{3g^2 N_c}{12} \oint_k \frac{\delta^{aa'}}{k^2} \oint_p \frac{k^2 + (k-p)^2 + p^2}{p^2 (k-p)^2} = - \frac{9}{4} g^2 N_c (N_c^2 - 1) \left(\frac{T^2}{12} \right)^2$$



$$= - \frac{1}{2} (-g^2 N_c) \oint \frac{\delta^{aa}}{k^2} \oint \frac{p^2 - k \cdot p}{p^2 (k-p)^2} = \frac{g^2}{4} N_c (N_c^2 - 1) \left(\frac{T^2}{12} \right)^2$$

obscures UV finiteness, gauge invariance, Thermal v.s. vacuum loops.

* Adding QCD matter + comments



We need two amplitudes:

$$\lim_{\text{forward}} \mathcal{M}(1g_-, 3g_+, 3\bar{u}_i, 4\bar{u}_i) = \lim_{\text{forward}} \mathcal{M}(1u_i, 3\bar{u}_+, 2\bar{u}_i, 4u_i) = -g^2(N_c^2 - 1)$$

$$-\Delta\mathcal{L}_{\text{quarks}} = 4 \int d\nu_g d\nu_q \left[\mathcal{M}(g_- u_+ \rightarrow g_- u_+) + \mathcal{M}(g_+ u_+ \rightarrow g_+ u_+) \right] +$$

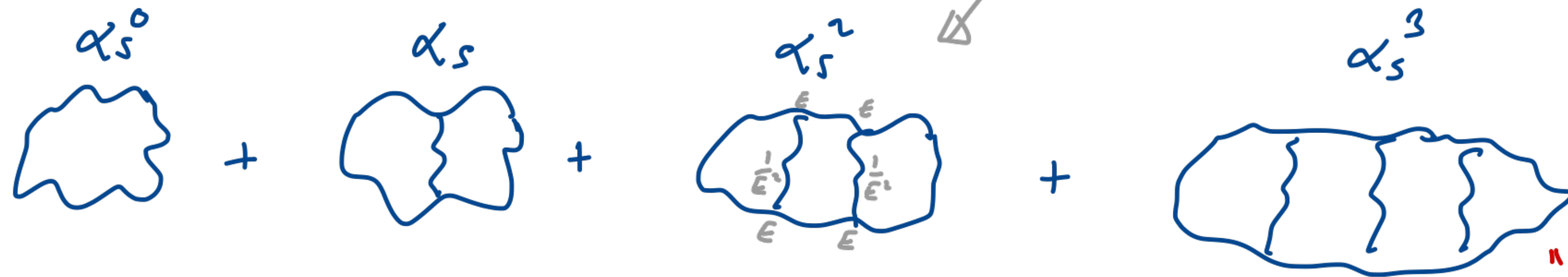
$$+ 2 \int d\nu_u d\nu_{\bar{u}} \mathcal{M}(\bar{u}_+ u_- \rightarrow \bar{u}_+ u_-) + 4 \int \frac{d\nu_1 d\nu_2}{2} \mathcal{M}[u_- u_- \rightarrow u_- u_-]$$

$$= \alpha_s (N_c^2 - 1) \frac{\pi T^4}{36} \frac{5}{4} N_f.$$

$$\int d\nu_{g/q} \equiv \int \frac{d^3k}{(2\pi)^3 2k (e^{\beta k} \mp 1)} = T^2 \begin{cases} \frac{1}{24} & \text{boson} \\ \frac{1}{48} & \text{fermion} \end{cases}$$

* QCD state of the net: @ fix order.

$$\left[\int d^3k \right]^3 \left(\frac{1}{E^2} \right)^6 E^4 = \int d^6 E \frac{1}{E^8} \text{ IR div.}$$



"Linde Problem": $g^6 T^4 \log(T/\lambda_{IR})$

$$\text{Diagram} \sim \text{Diagram} = T \int d^3k \frac{(g^2 T^2)^2}{(k^2 + g^2 T^2)^2} \sim \frac{g^4 T^5}{g T} = g^3 T^4 \propto \alpha_s^{3/2}$$

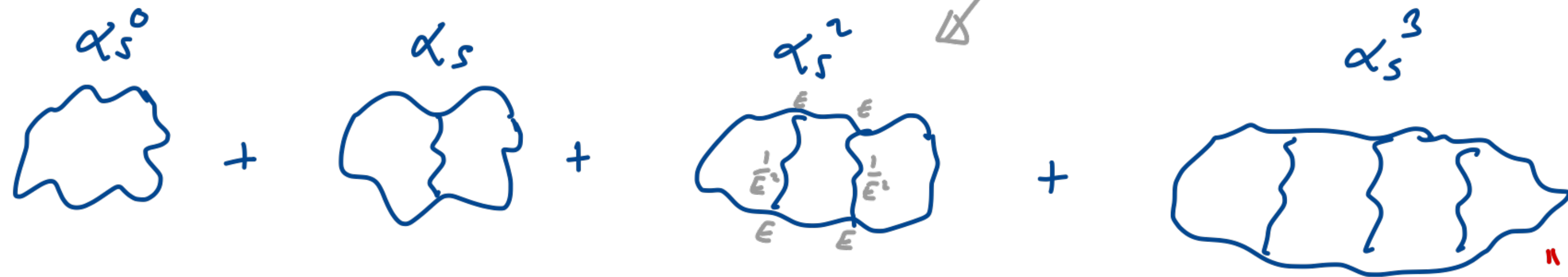
$\uparrow$
 $\Pi_{00} \propto g^2 T^2$

↳ Debye mass, electric component, generates non-integer " α_s " powers.

collective phenomena due to modes $k \sim gT \ll T$

* QCD state of the net: @ fix order.

$$\left[\int d^3k \right]^3 \left(\frac{1}{E^2} \right)^6 \epsilon^4 = \int d^6 \epsilon \frac{1}{\epsilon^8} \text{ IR div.}$$



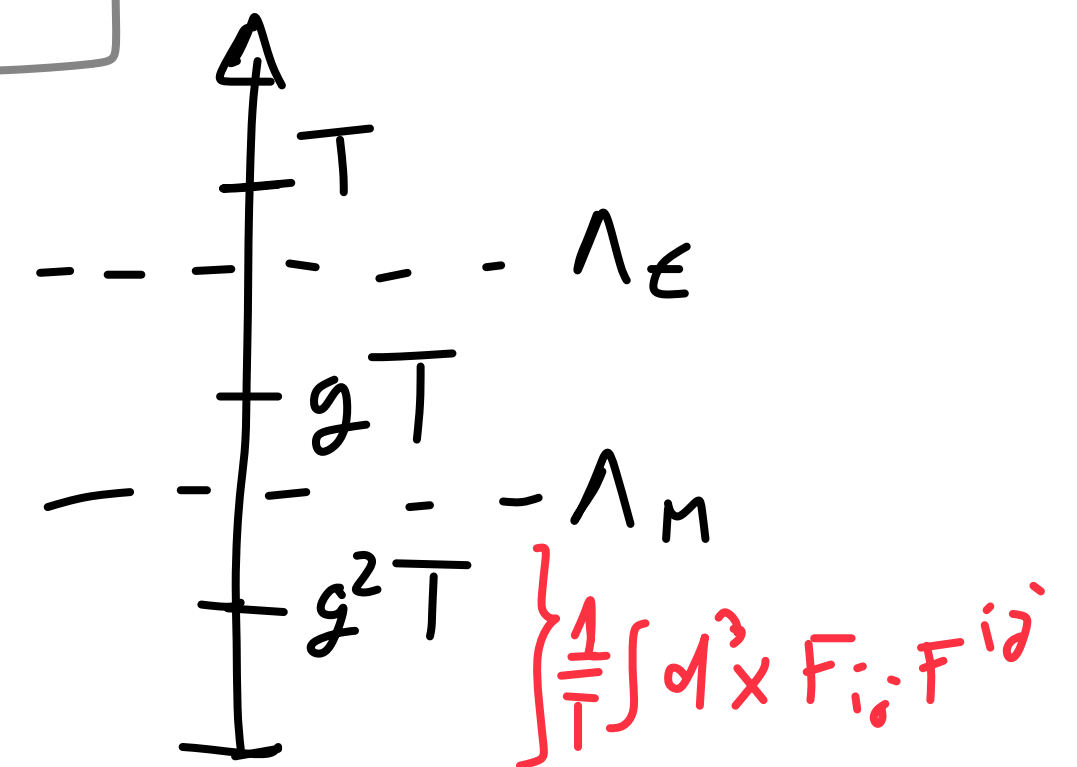
"Linde Problem": $g^6 T^4 \log(T/\Lambda_{IR})$

$$\text{Diagram} \sim \text{Diagram} = T \int d^3k \frac{(g^2 T)^2}{(k^2 + g^2 T^2)^2} \sim \frac{g^4 T^5}{g T} = g^3 T^4 \propto \alpha_s^{3/2}$$

$\uparrow$
 $\Pi_{00} \propto g^2 T^2$

↳ Debye mass, electric component, generates non-integer " α_s " powers.

collective phenomena due to modes $k \sim g T \ll T$



$$\Rightarrow g_m^2 = g^2 T$$

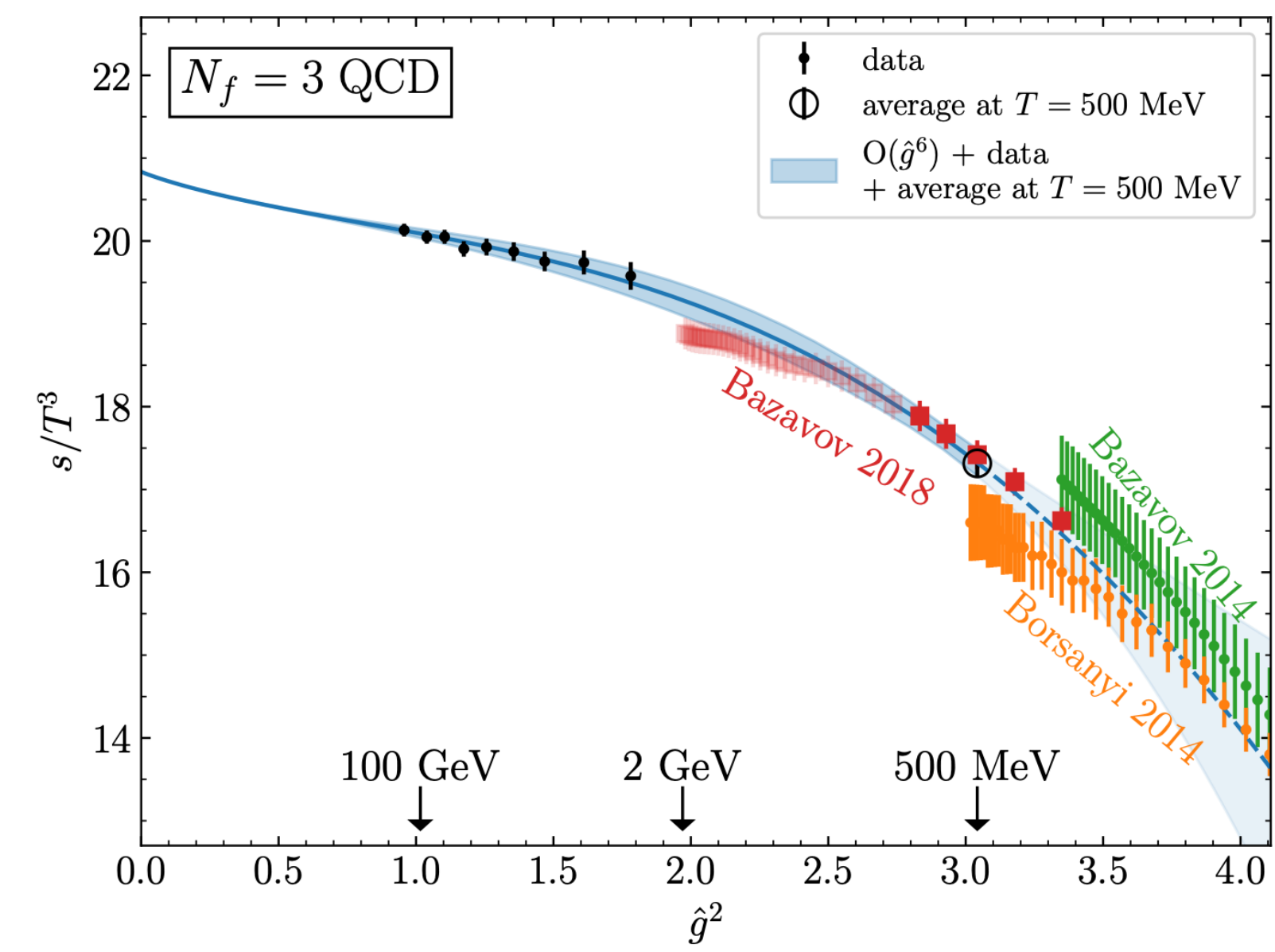
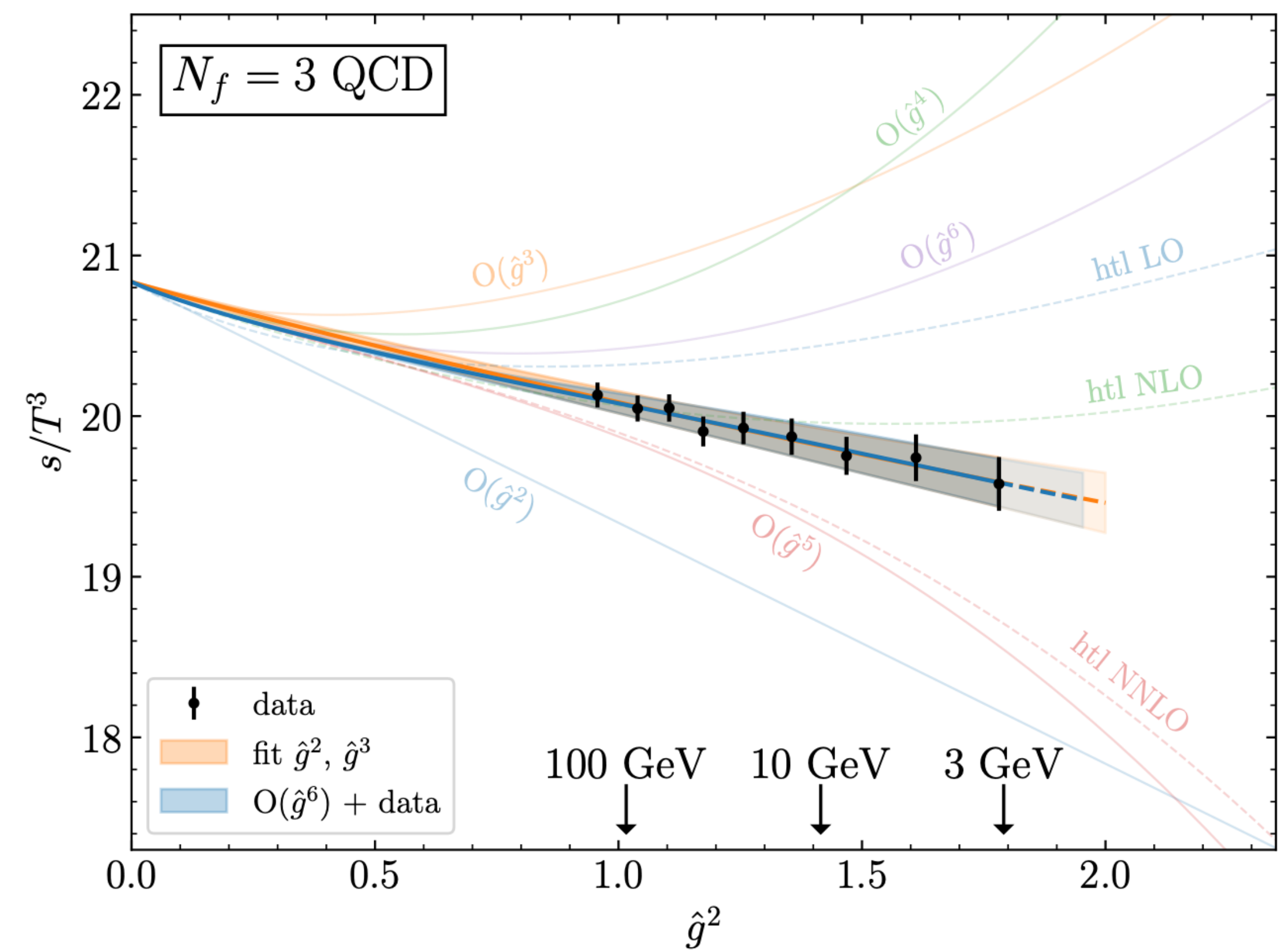
$$\Rightarrow \mathcal{L} = -\frac{\log Z}{V} \propto g_m^6$$

↳ Linde problem: softer modes $k \sim g^2 T \ll T$ are strongly coupled.

The effective expansion parameter is $\epsilon = \frac{g^2 T}{\Pi_m} \begin{cases} \frac{g}{\Pi} ; \text{ for } m \sim g T \text{ Debye mass} \\ 1 ; \text{ for color-magnetic fields } m \lesssim g^2 T \end{cases}$

$$\{ \dots \} \sim g^{2(n-1)} \left(T \int_0^{g^2 T} d^3k \right)^n \frac{k^{2(n-1)}}{(k^2 + \Lambda_{IR})^{3(n-1)}}$$

Recent Lattice MC determinations of QCD EoS up to very high temperatures vs pQCD



from Bresciani, Dalla Brida, Giusti
and Pepe hep-lat/2501.11603

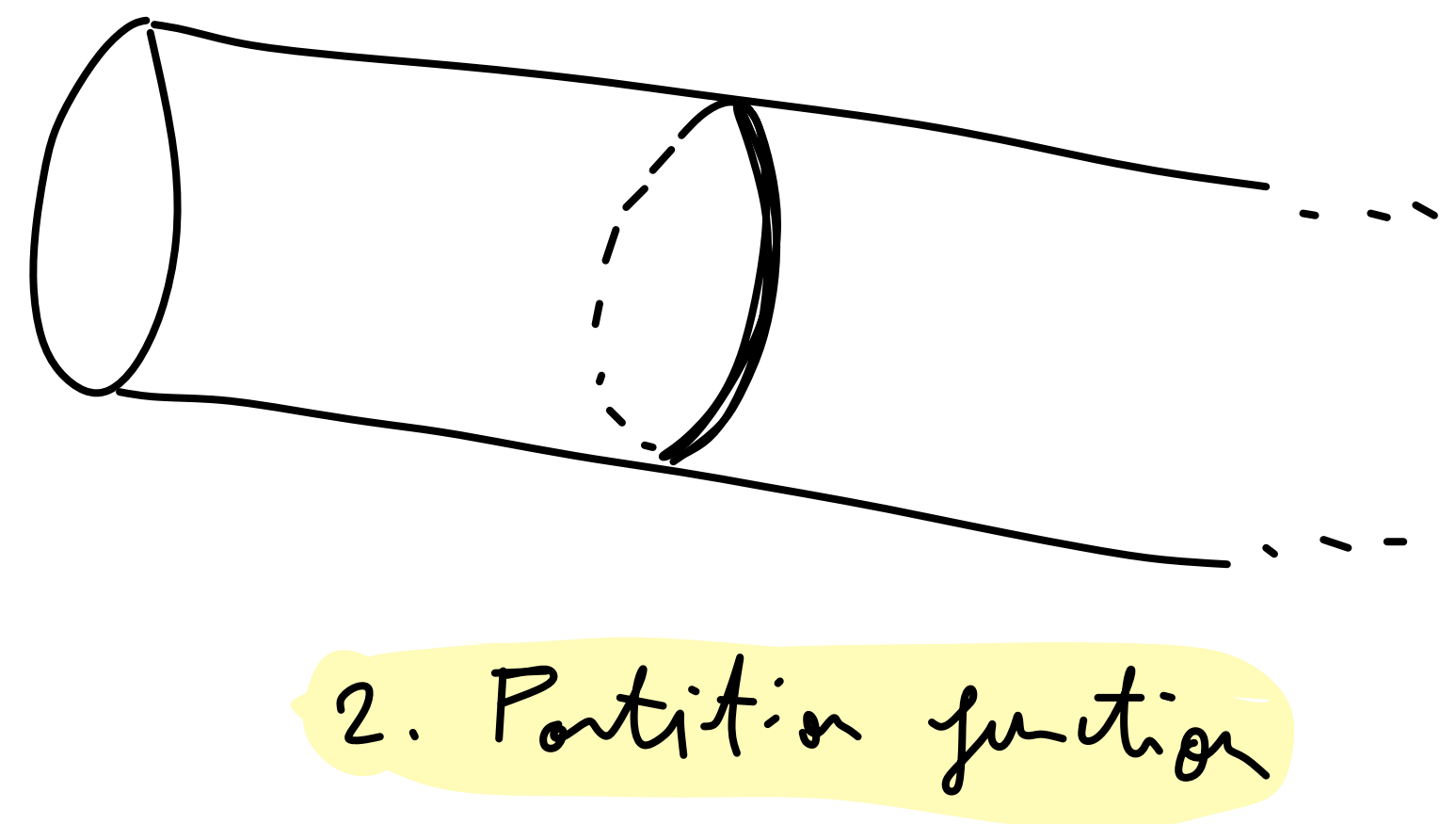
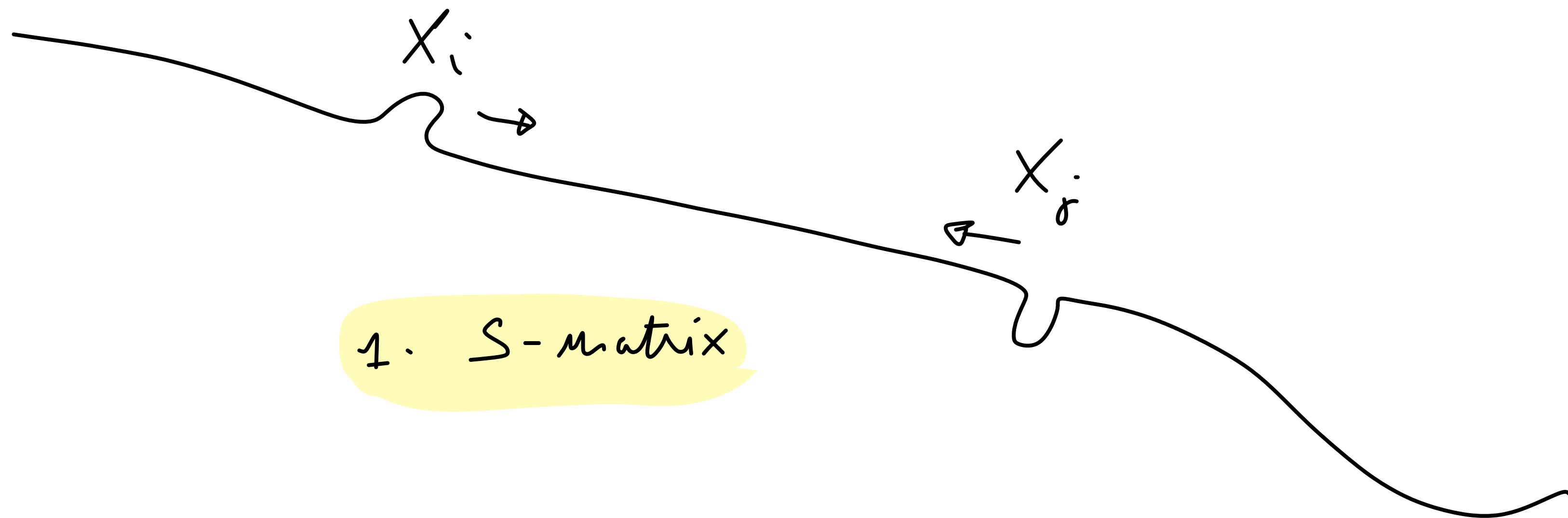
I will explain the DMB calculation up to the same order of htl NNLO.

- * Before addressing real QCD, we would like to test DMB to large orders.
- * In particular we should address the role of potential singularities arising from the forward limit of the scattering amplitude.
- * We need a model for which:
 - * The S-matrix & The partition function are known.
 - * No physical IR divs.
 - * Ideally a simple and interesting model.

* Simpler set up, Yang-Mills confining flux tubes:

Write down the most general 2d action realizing (non-linearly) the symmetry breaking pattern $ISO(D-1, 1) \mapsto ISO(1, 1) \times SO(D-2)$. There are " $D-2$ " goldstone bosons or transverse excitations. For $D=3$, there is an integrable realization of the system. (i.e. the Wilson coefficients of the action can be fixed to realize the following factorized S-matrix):

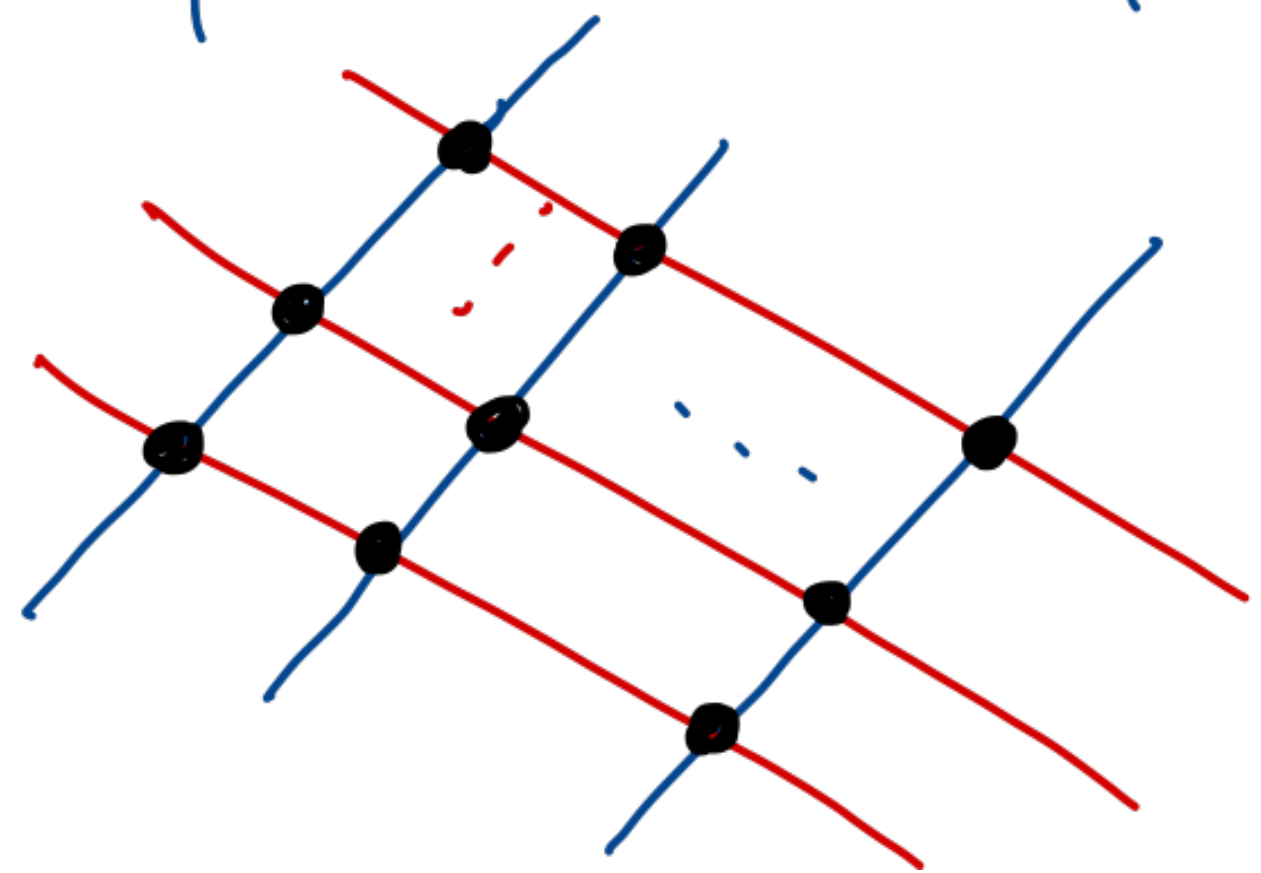
E.g. two interesting observables:



* Simpler set up, Yang-Mills confining flux tubes:

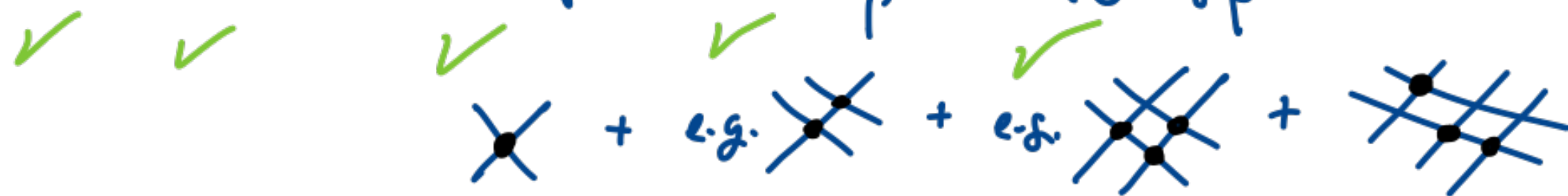
Write down the most general 2d action realizing (non-linearly) the symmetry breaking pattern $ISO(D-1, 1) \mapsto ISO(1, 1) \times SO(D-2)$. There are " $D-2$ " goldstone bosons or transverse excitations. For $D=3$, there is an integrable realization of the system. (i.e. the Wilson coefficients of the action can be fixed to realize the following factorized S-matrix):

$$S_{\alpha\beta} = e^{i \frac{l_s^2}{4} \sum_{i,j} s_{ij}^2} \langle \alpha | \beta \rangle$$



$$t(\beta) = \frac{1}{l_s^2} \sqrt{1 - \frac{l_s^2}{\beta^2} \frac{\pi}{3}}$$

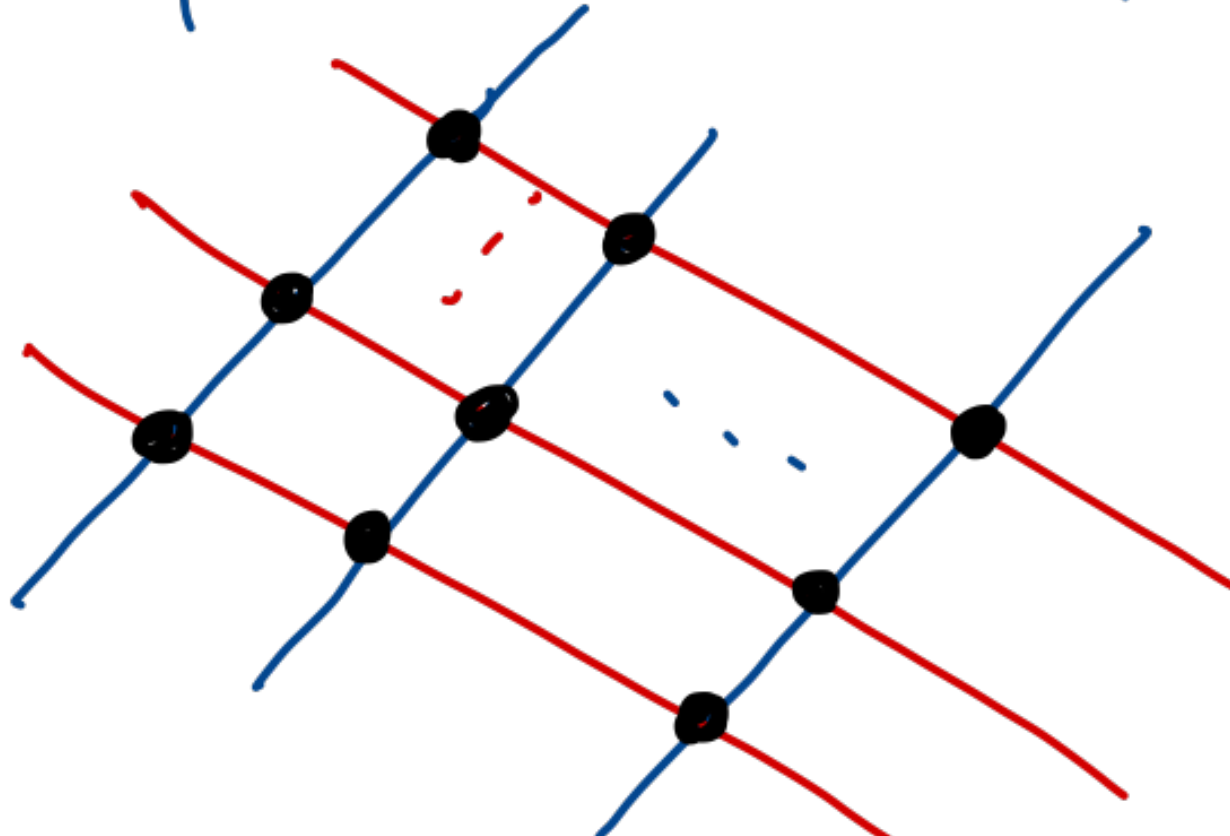
$$= \frac{1}{l_s^2} - \frac{\pi}{6\beta^2} - \frac{l_s^2 \pi^3}{22\beta^4} - \frac{l_s^4 \pi^3}{432\beta^6} - \frac{5 l_s^6 \pi^4}{10368\beta^8} + \dots$$



* Simple set up, Yang-Mills confining flux tubes:

Write down the most general 2d action realizing (non-linearly) the symmetry breaking pattern $ISO(D-1, 1) \mapsto ISO(1, 1) \times SO(D-2)$. There are " $D-2$ " goldstone bosons or transverse excitations. For $D=3$, there is an integrable realization of the system. (i.e. the Wilson coefficients of the action can be fixed to realize the following factorized S-matrix):

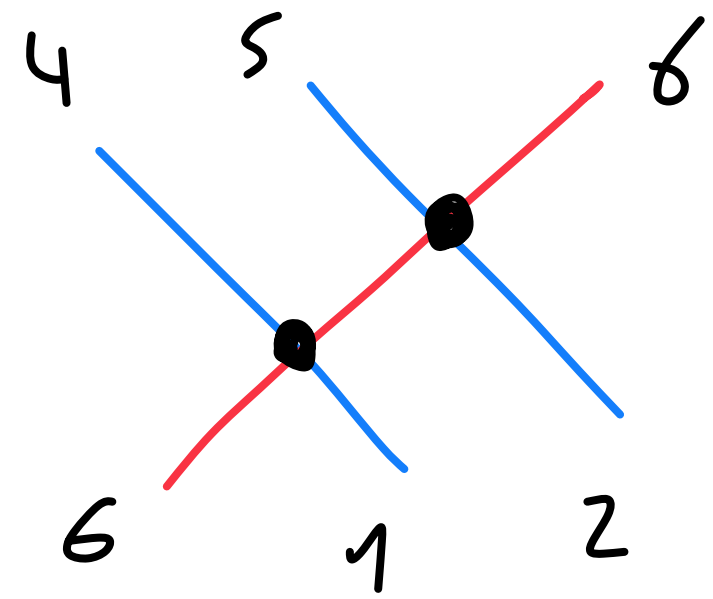
$$S_{\alpha\beta} = e^{i \frac{l_s^2}{4} \sum_{i,j} f_{ij}} \langle \alpha | \beta \rangle \longleftrightarrow f(\beta) = \frac{1}{l_s^2} \sqrt{1 - \frac{l_s^2}{\beta^2} \frac{\pi}{3}}$$

$$= \frac{1}{l_s^2} - \frac{\pi}{6\beta^2} - \frac{l_s^2 \pi^3}{22\beta^4} - \frac{l_s^4 \pi^3}{432\beta^6} - \frac{5 l_s^6 \pi^4}{10368\beta^8} + \dots$$



Clearly $\log S_{\alpha\beta} \propto l_s^2$ but $f(\beta)$ is non-linear in l_s^2 !

The paradox is resolved by careful treatment of $i\epsilon$ and the forward limit!

From $S_{\alpha\beta}$ we get the following singular $3 \rightarrow 3$ matrix element:



$$= \mathcal{M}(1_{\ell} 2_{\ell} 3_v \rightarrow 4_{\ell} 5_{\ell} 6_v) = 32\pi i \ell_s^4 E_1^2 E_2^2 E_3^2 \left(\delta(E_1 - E_4) + \delta(E_4 - E_5) \right)$$

in terms of Feynman Diags:

$$\mathcal{M}^{(sing)} = -\ell_s^4 (p_3 \cdot p_1 p_4^\mu + p_3 \cdot p_4 p_1^\mu) \frac{q_\mu q_\nu}{q^2 + i\epsilon} (p_6 \cdot p_2 p_5^\nu + p_6 \cdot p_5 p_2^\nu) + (1 \leftrightarrow 2, 4 \leftrightarrow 5).$$

$$\frac{q_\mu q_\nu}{q^2 + i\epsilon} = \delta_\mu^0 \delta_\nu^0 + \frac{1}{2|\vec{q}|} \left(\frac{q_\mu q_\nu}{q_0 - |\vec{q}| + i\epsilon} + \frac{\bar{q}_\mu \bar{q}_\nu}{-q_0 - |\vec{q}| + i\epsilon} \right)$$

$$\begin{aligned} q_\mu &= (|\vec{q}|, \vec{q}) \\ \bar{q}_\mu &= (|\vec{q}|, -\vec{q}) \end{aligned}$$

Next evaluate the diagram on the support of

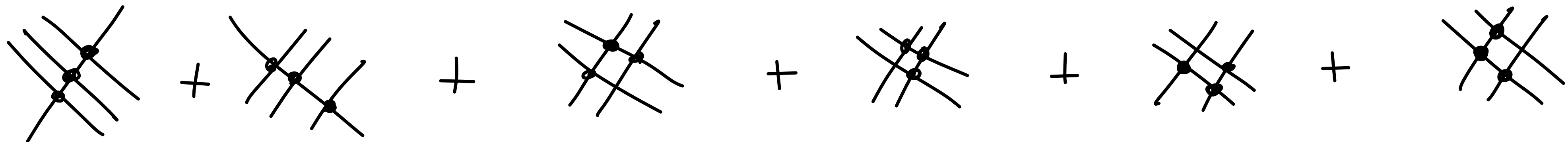
$$\delta(E-E_\alpha) \operatorname{Re} \frac{1}{E-E_\alpha+i\epsilon} = -\frac{1}{2} \partial_E \delta(E-E_\alpha)$$

All in all

$$f_{\text{NLO}} = f_{\text{Ler}} + f_{\text{rre}} = -\frac{l_s^4 \pi^3}{432 \beta^6}$$

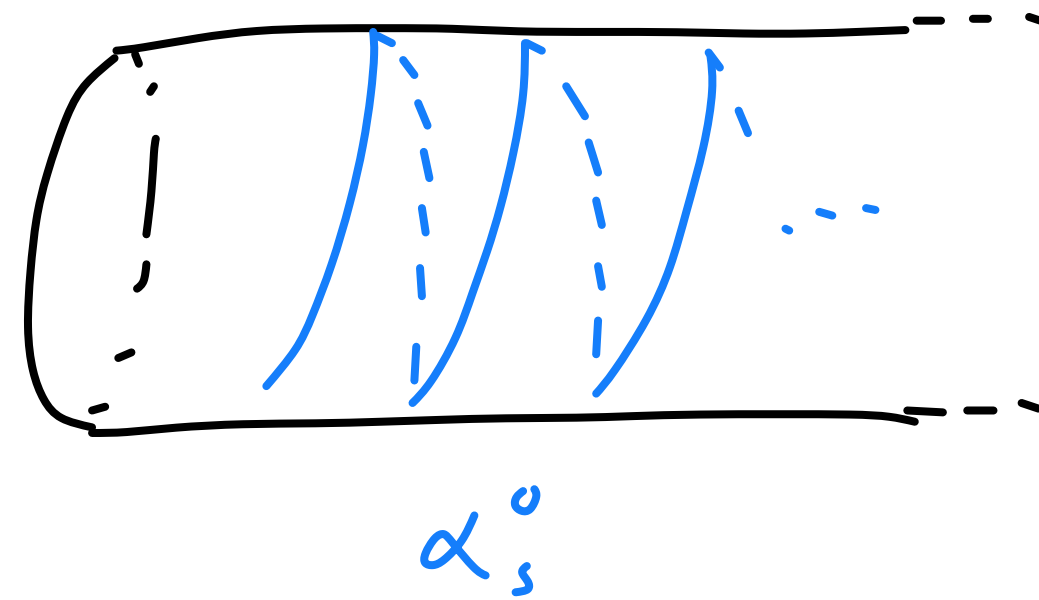
In agreement with $f(\beta) = \frac{1}{l_s^2} \sqrt{1 - \frac{l_s^2 \pi}{3 \beta^2}}$

In the paper we pushed the method to the next order:

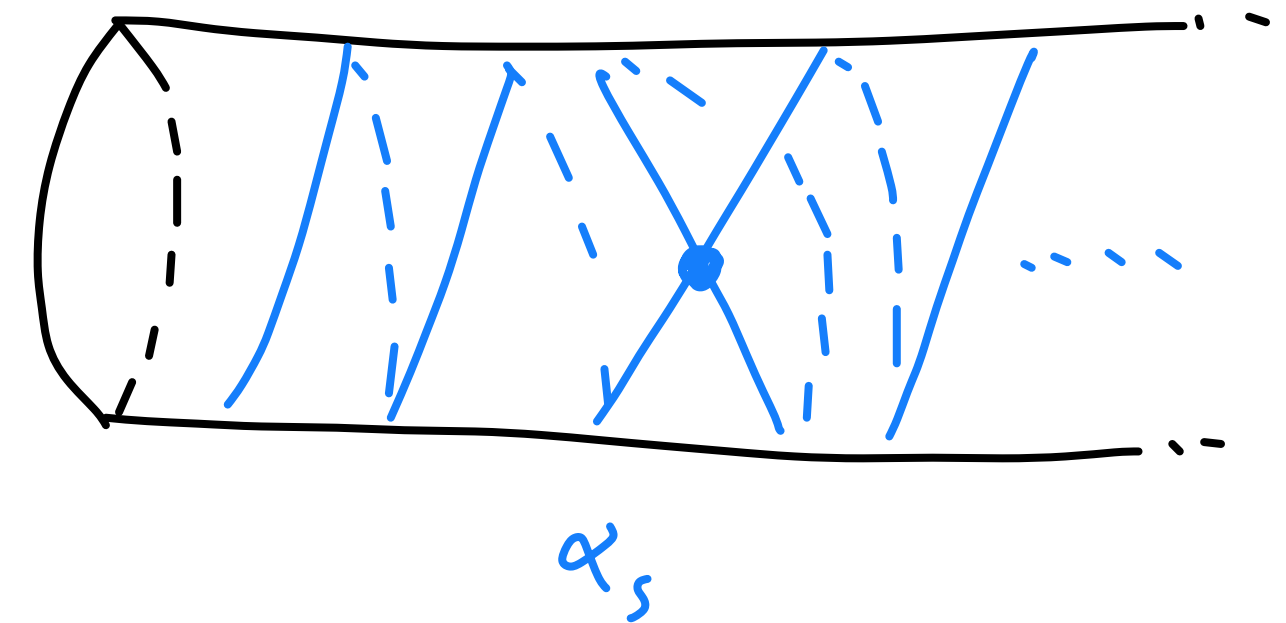


Coming back to perturbative QCD...

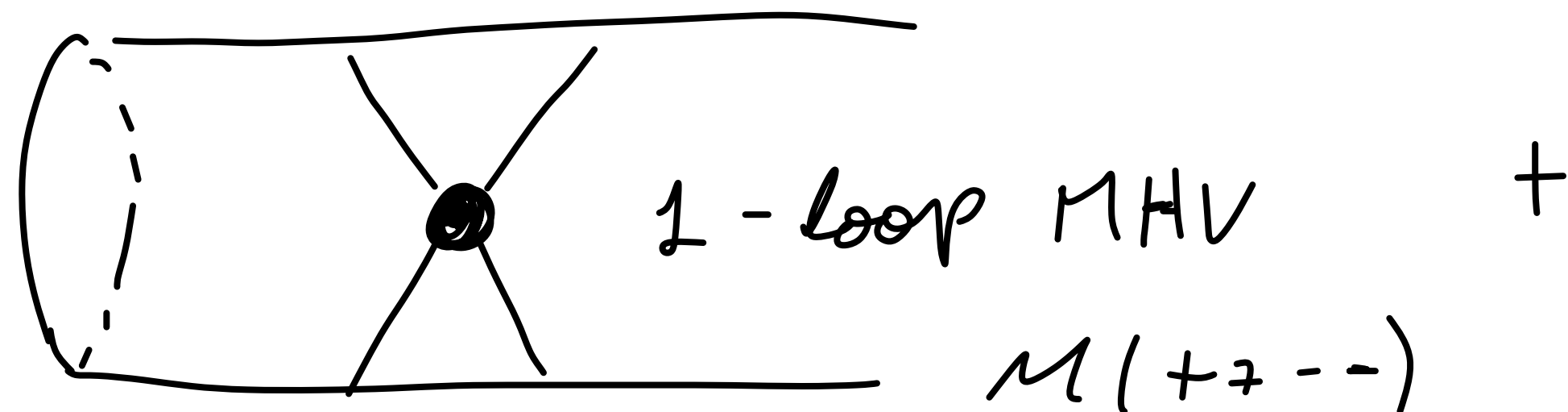
* So far I have shown



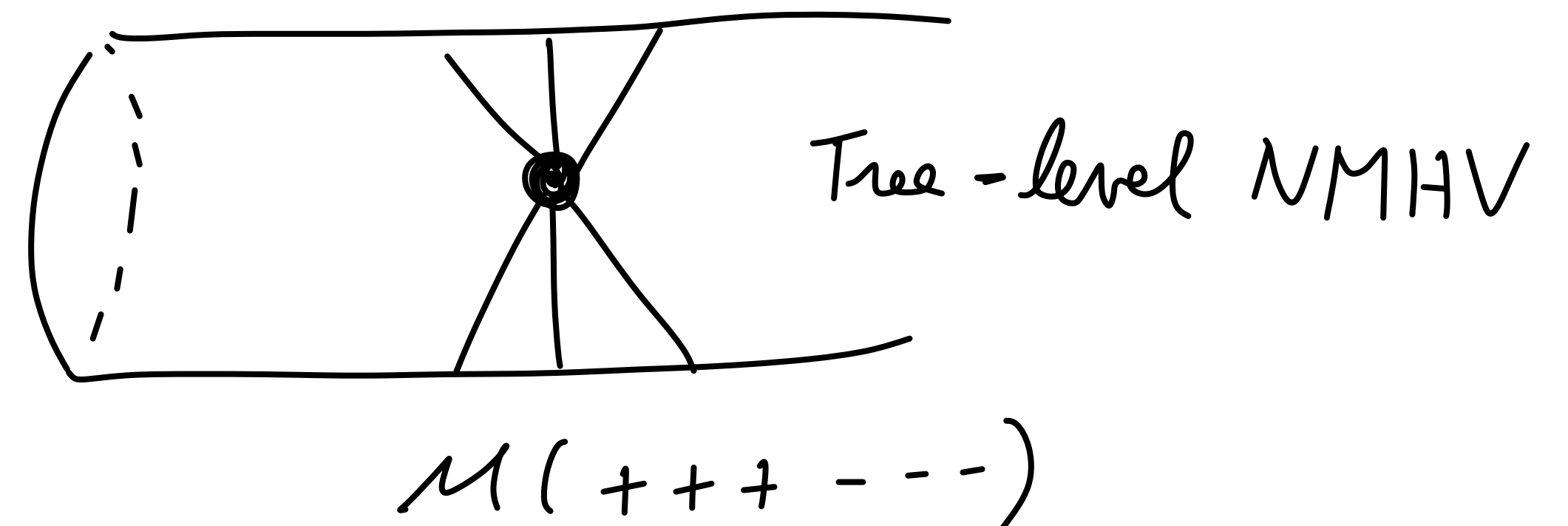
+



* At order α_s^2 we need to compute



+



↳ Simple, well known amplitudes.

* How does screening arise in this formalism?

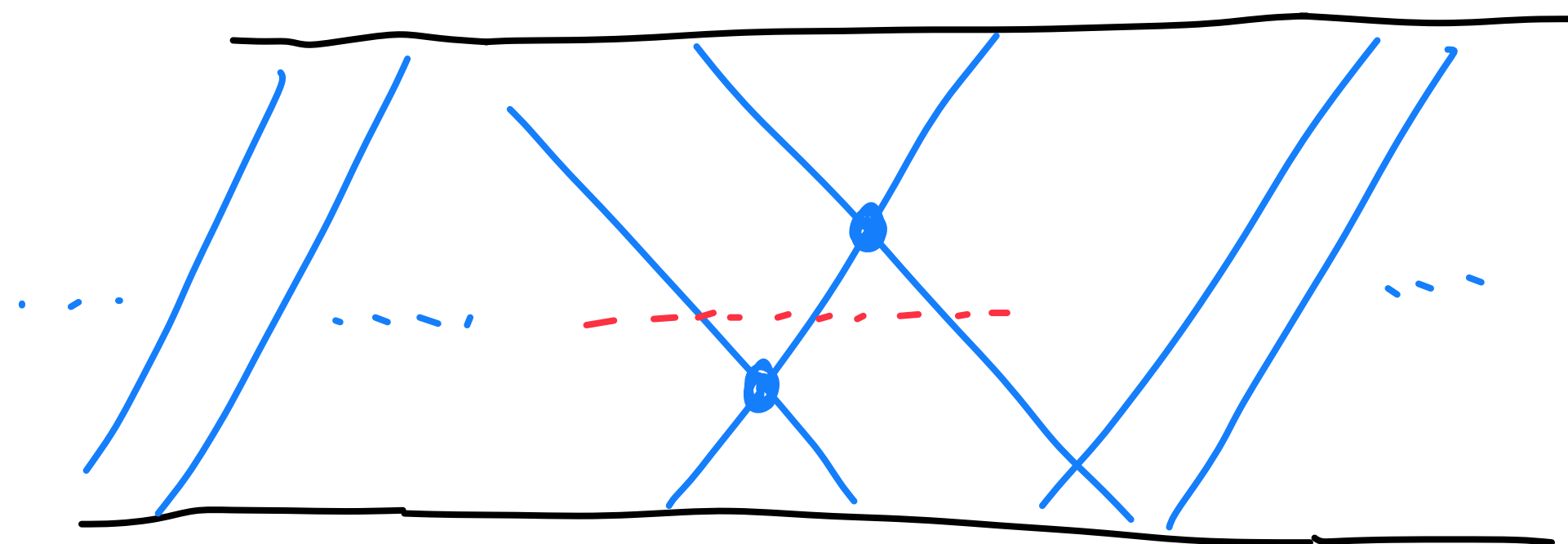
↳ Next I focus on the most singular α_s^2 diagrams to show the mechanism.

* Recall that

$$\langle 3_{\sigma}^{A'} 4_{\tau}^{B'} | T | 1_{\sigma}^A 2_{\tau}^B \rangle = (2\pi)^3 \delta^{(3)}(\vec{k}_3 + \vec{k}_4 - \vec{k}_1 - \vec{k}_2) 2g_s^2 \sum_c f^{A'B'C} f^{ABC} + \text{singular}$$

* Next we focus on the most singular diagrams.

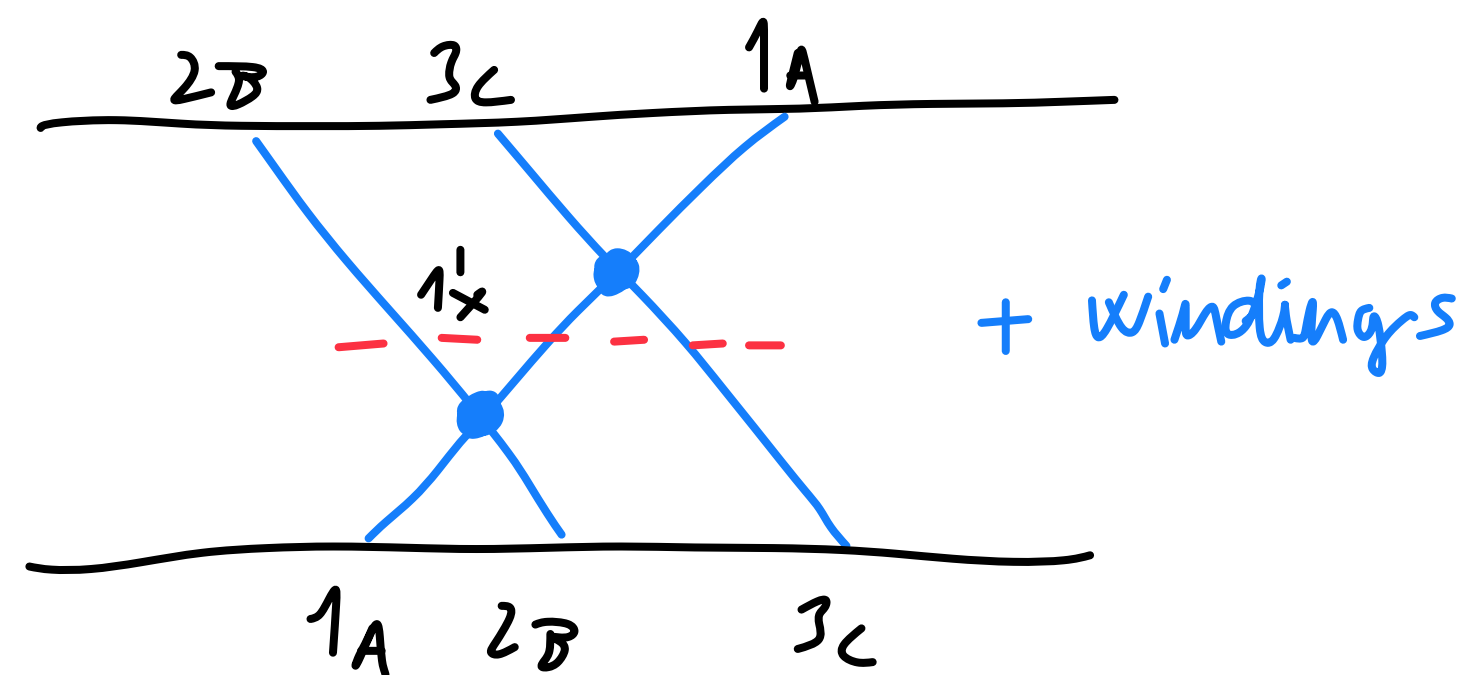
They contain the maximal number of on-shell internal propagators.



α_s^2

↪ $\frac{s}{\pi + i\epsilon}$
vanishes in the
forward limit $f_{AA\zeta\bar{\zeta}}^{\text{for}}$

* The diagram is given by



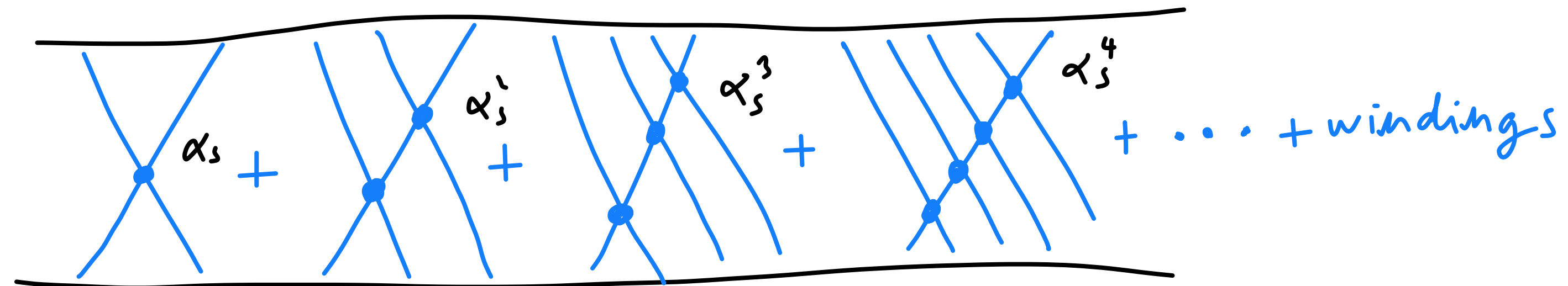
$$= \cancel{\times} \left(\int \frac{d^3k}{(2\pi)^3} \frac{M(E_k)}{2E_k} \right)^2 \left(\frac{1}{2} \int \frac{d^3k}{(2\pi)^3} \frac{M'(E_k)}{4E_k^2} \right)$$

$\uparrow$
 $[L^3 (2g_s)^2 N_\sigma^3 N_c^2 (N_c^2 - 1)]$

IR divergent

This IR divergence can be cured via a Debye mass. But in this formalism it seems unnatural to add a thermal mass $m_g \sim g_s T$. We are only trying over the vacuum S-matrix! .

* summing over all the "comb" diagrams.



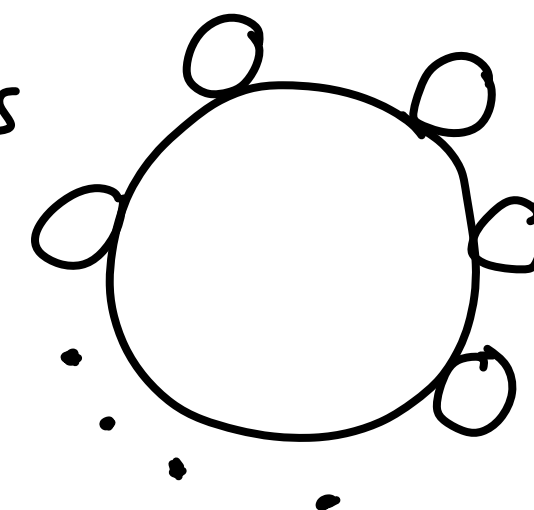
* It turns out

$$\sum_{n=0}^{\infty} f_n^{\text{comb}} = N_\sigma (N_c^2 - 1) \int \frac{d^3 k}{(2\pi)^3} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{I_T N_\sigma N_c 2g_s^4}{2E_k} \right)^n \partial^n \mu^{(-1)}$$

$$= \beta^{-1} N_\sigma (N_c^2 - 1) \int \frac{d^3 k}{(2\pi)^3} \ln \left(1 - e^{-\beta \sqrt{K^2 + g^2 \frac{N_c}{3\beta^2}}} \right)$$

↳ Finite result. this is nothing but Daisy QCD diagrams

↳ Expected behaviour IR div for $K \sim gT$ modes.



Summary: Can a vacuum S-matrix reproduce thermal physics beyond the orders where the forward limit is harmless?

yes, apparent forward singularities are resolved by the energy-dependent DMB operator, while genuine IR sensitivity reorganizes into screening/thermal masses.

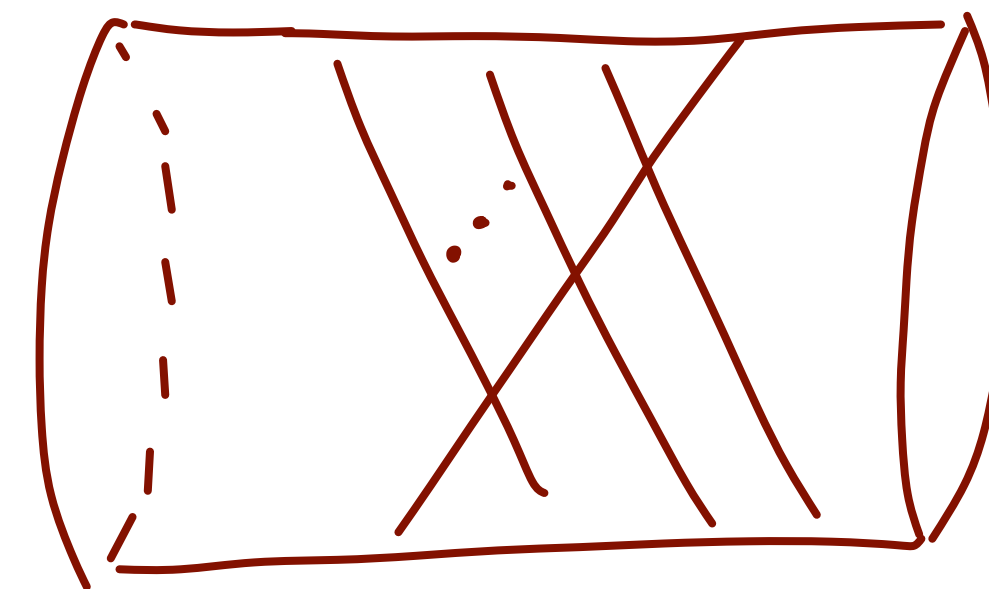
$$Z(\beta) = Z_0(\beta) - \frac{1}{2\pi i} \int dE \sum_n e^{-\beta E} \text{Tr} [T^{(n)}(E + i\epsilon)]$$

$$\left[\lim_{E \rightarrow 0}, T_R \right] \neq 0$$

$$\left[\int dE, \sum_n \right] \neq 0$$

$$\mathcal{F}(\beta) = \int dE e^{-\beta E} T_R \log e^{i \ell_s^2 \sum_{i < j} p_i \cdot p_j}$$

$$= \sqrt{1 - \frac{\ell_s^2 \pi}{\beta^2 3}}$$



IR finite
emergent thermal
mass.

Outlook

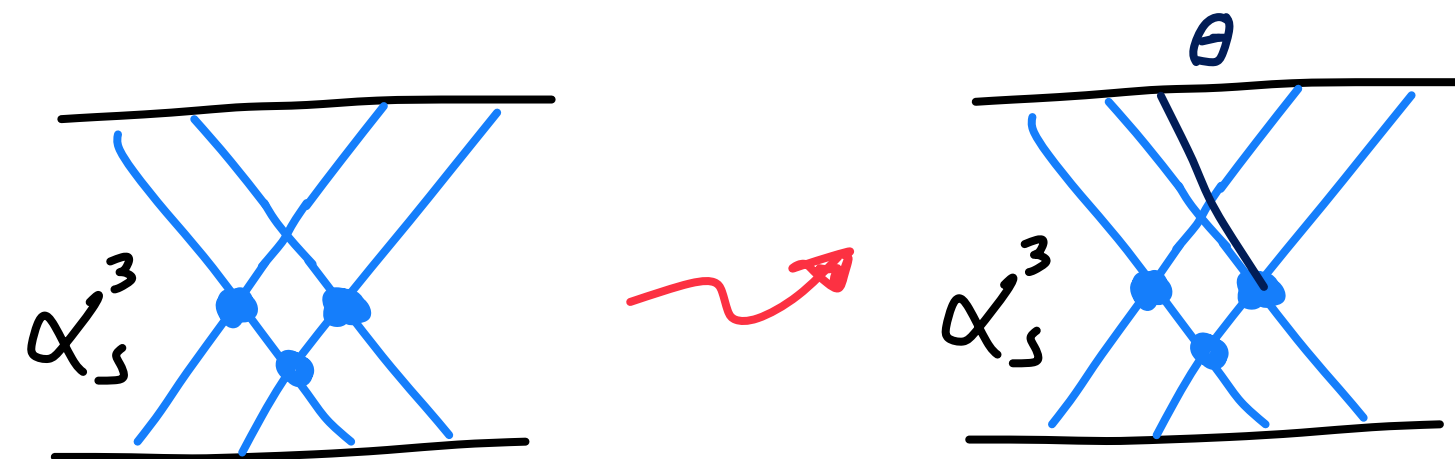
- 1 - It is possible to generalise to other observables. E.g. finite volume excited states from form factors.

$$S \rightarrow S + \int h(x) \Theta(x), \text{ then } \left. \frac{\delta F[h]}{\delta h(x)} \right|_{h=0} = -\frac{1}{2\pi i} \int dE \, e^{\beta E} \text{Tr}_c [S^{-1} \otimes F_0]$$

- 2 - Inverse problem: S-matrix in terms finite volume states. *Lüscher, Sharpe, Hansen, ...*

- It will be fun to continue pursuing a revision of the pQCD equation of state in view of this formalism.

➔ Speculation: $K \sim g^2 T \text{ div}_S$ or "Linde problem" may be treated in a collider-inspired approach.



Review of Asymptotic Bethe Ansatz

Two particle state in infinite volume

$$\psi(x_1, x_2) = \langle 0 | \phi(x_1) \phi(x_2) | p_1, p_2 \rangle$$

Consider $x_1 \ll x_2$ (mutatis mutandis $x_1 \gg x_2$)

$$\psi(x_1 \gg x_2) = e^{i p_1 x_1 + i p_2 x_2} + e^{i p_1 x_2 + i p_2 x_1} S_{12}(p_1, p_2)$$

Place the theory at finite volume w/ $x_1 \ll x_2 \ll x_1 + R$,

$$\psi(x_1, x_2) = \psi(x_1 + R, x_2) \Rightarrow e^{i p_1 R} S_{12}(p_1, p_2) = 1$$

Same logic for multiple rescatterings, all in all

$$p_i R + \sum_j 2 \delta_{ij}(p_i, p_j) = 2\pi N_i$$

Winding corrections exponentially suppressed for massive theories.

