Effective Theories for Nonperturbative Physics @ Durham, 2026

Decays of excited states in metastable vacua from large-order perturbation theory

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Based on the work with

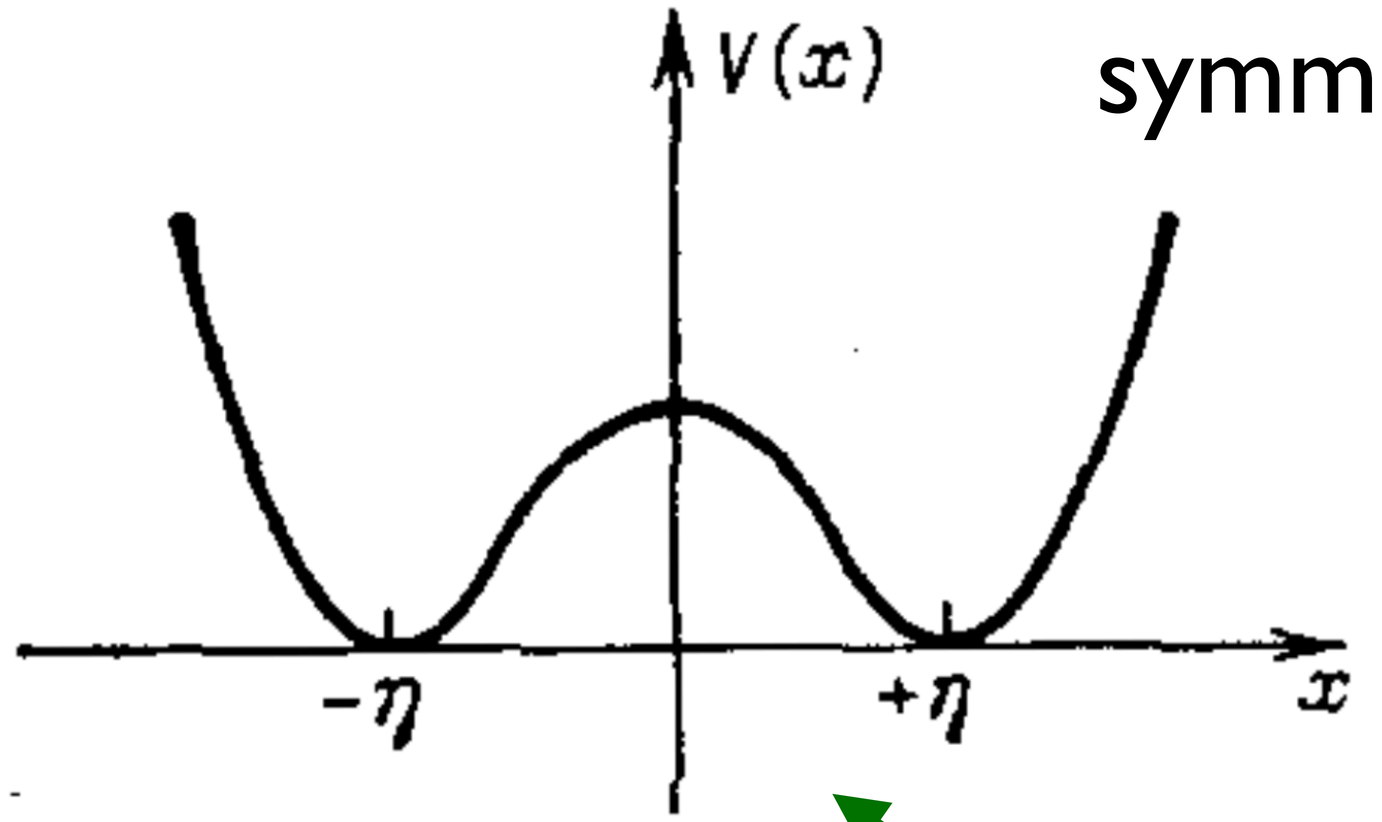
Sebastian Schenk, and Felix Yu

2609.XXXXX

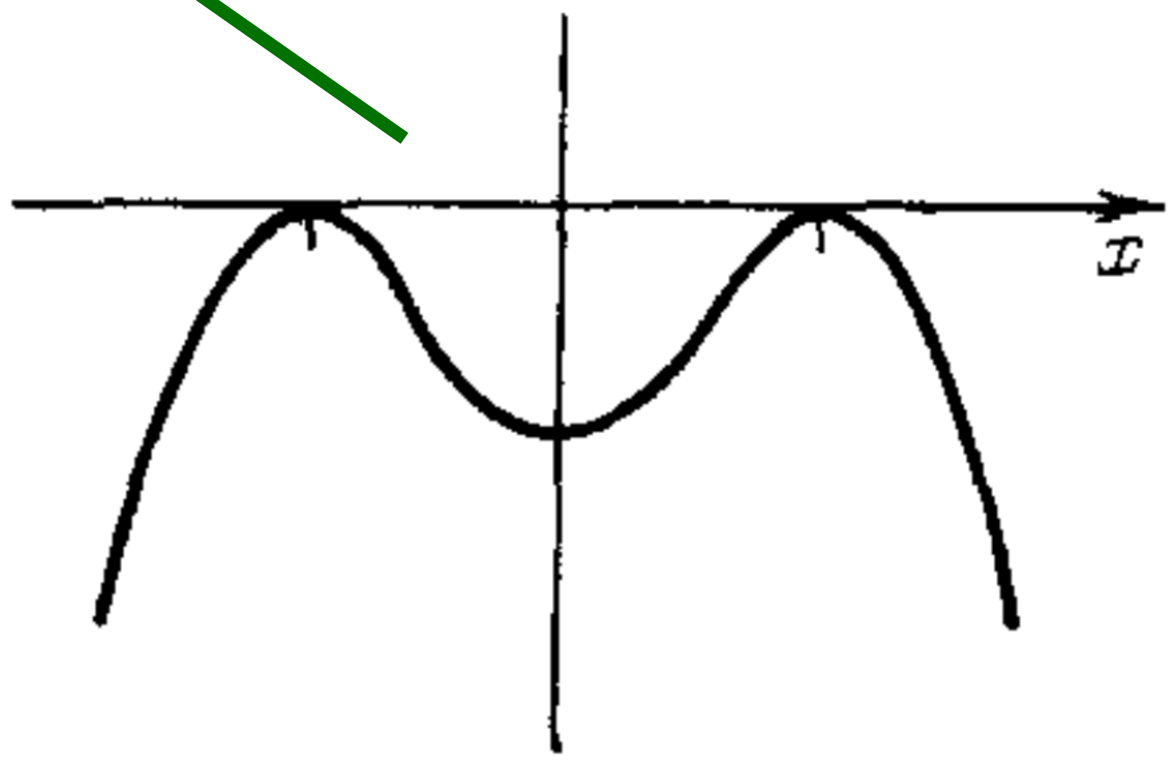
Introduction

- Weaknesses of the perturbation series
  - Nonperturbative corrections

symmetric double well



$$\psi(x) \approx \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar} \int p(x)\, dx}.$$

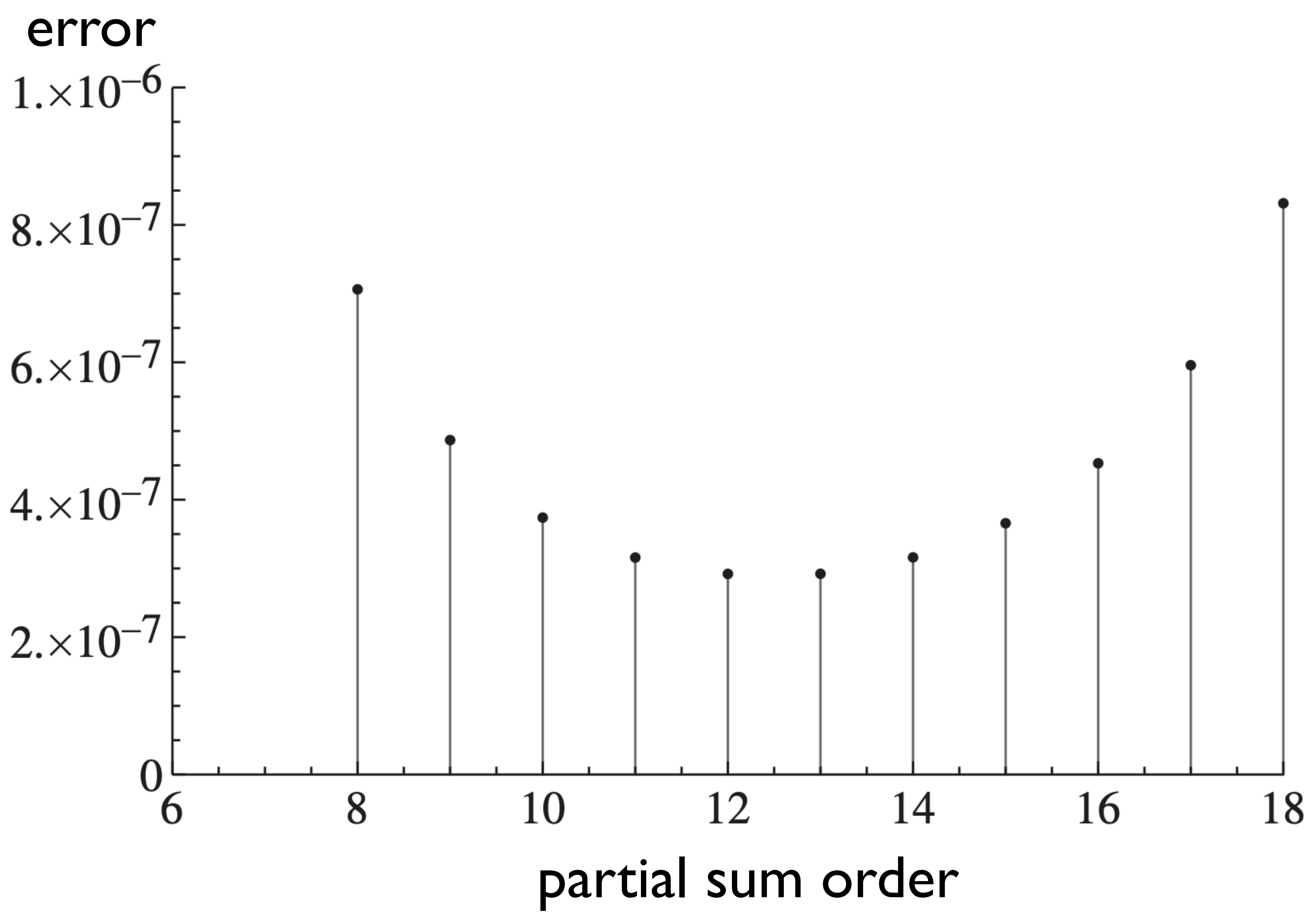


A. Vaĭnshteĭn, et al. ABC of instantons

- Asymptotic

Dyson's argument

$$\phi(z) = \sum_{k=0}^{\infty} a_k z^k \qquad a_k \sim A^{-k} k!$$



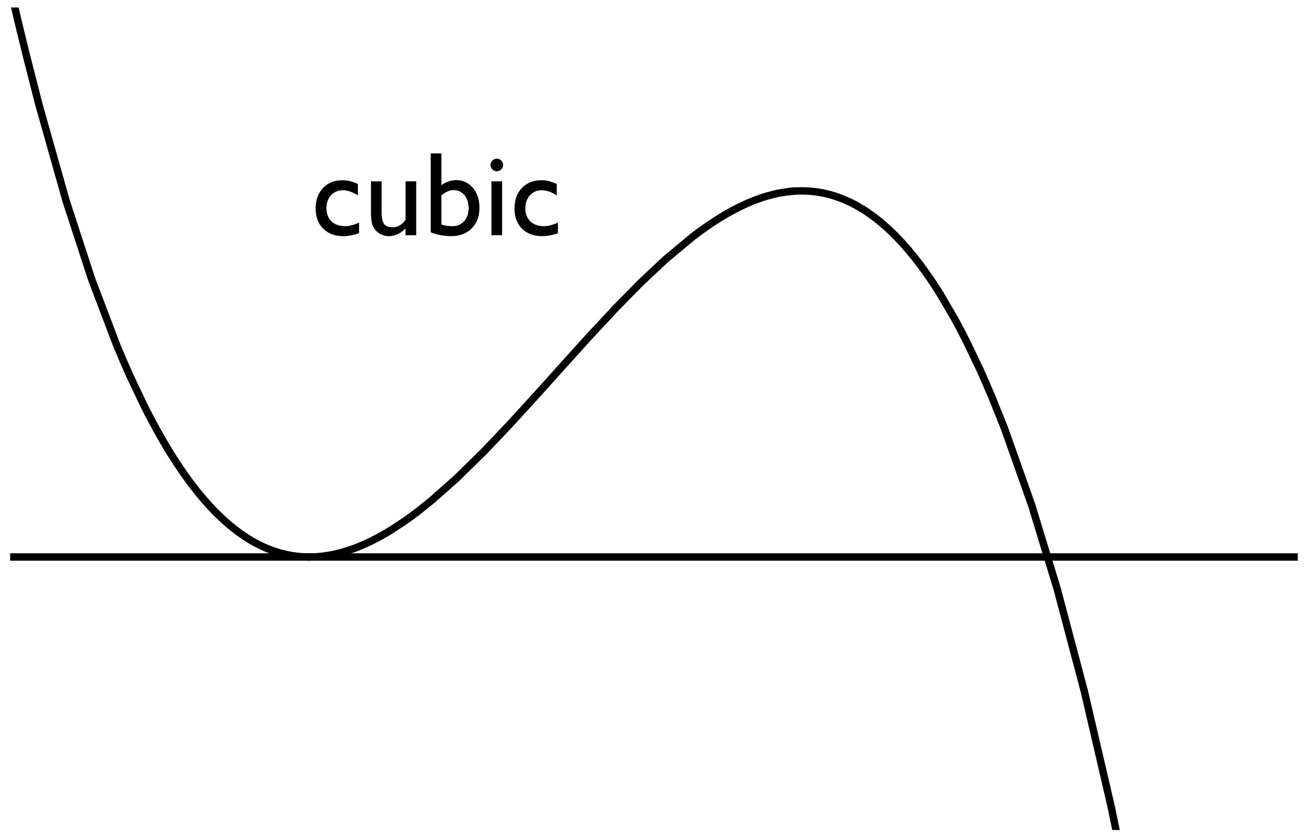
M. Mariño. Advanced Topics in Quantum Mechanics

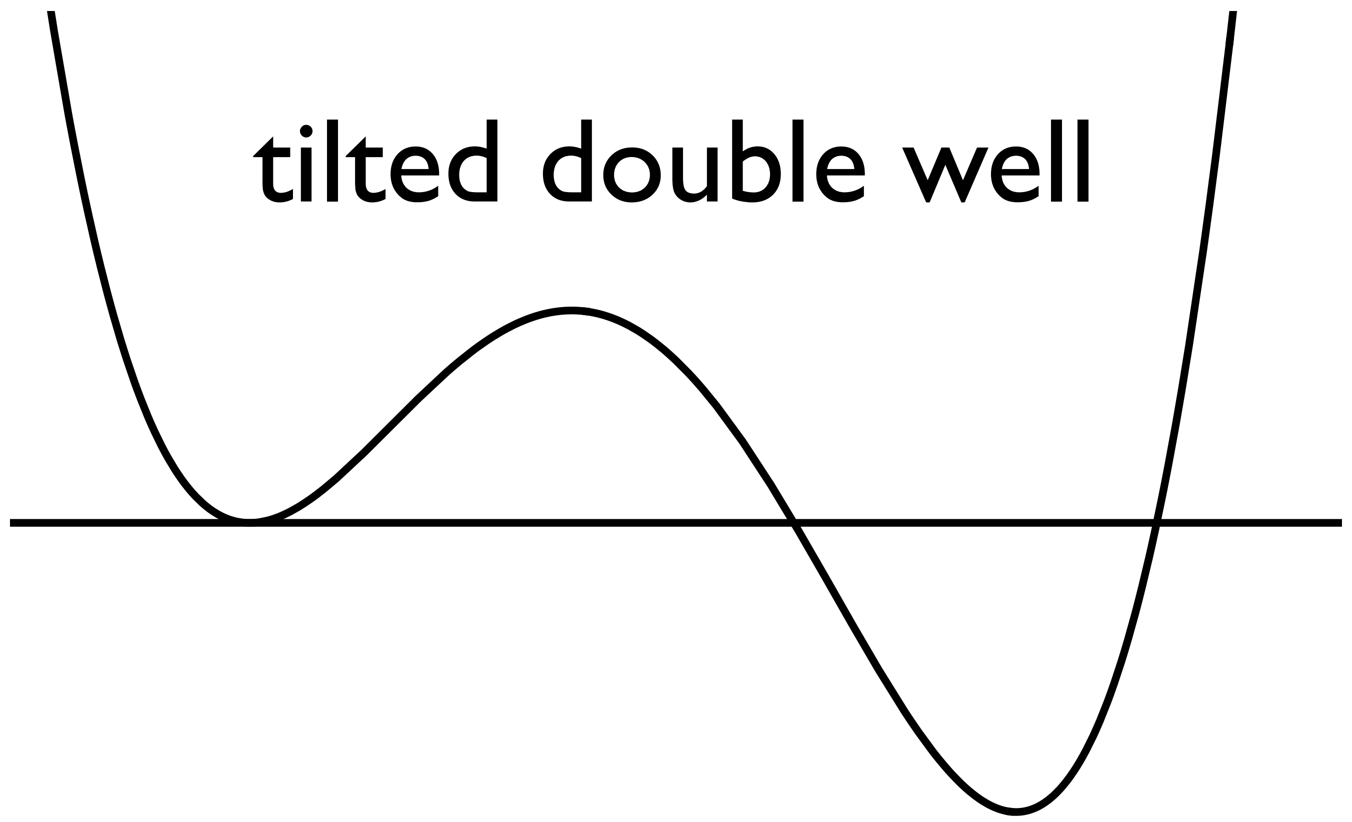
Introduction

- Perturbation theory can do more!

It encodes the **nonperturbative** information after we tame the **asymptotics**:

Resurgence by Jean Écalle, 80s





Standard nonperturbative approaches:

- bounce solution
- WKB

We will compute the decay rate using perturbation theory alone.

Introduction

- Flow

Resurgence framework:

Borel resummation, and uniform WKB

Determine the energy transseries:

for two specific potentials, using uniform WKB

Resum the energy transseries:

demonstrate the cancellation of ambiguities

Numerical fit:

extract the decay rate from perturbation series

Resurgence framework

- Borel resummation

$$\phi(z) = \sum_{k=0}^{\infty} a_k z^k \xrightarrow{\text{Borel transformation}} \hat{\phi}(u) = \sum_{k=0}^{\infty} \frac{a_k}{k!} u^k$$

Laplace transformation

$$s(\phi)(z) = \int_0^{\infty} e^{-w} \hat{\phi}(zw) dw$$

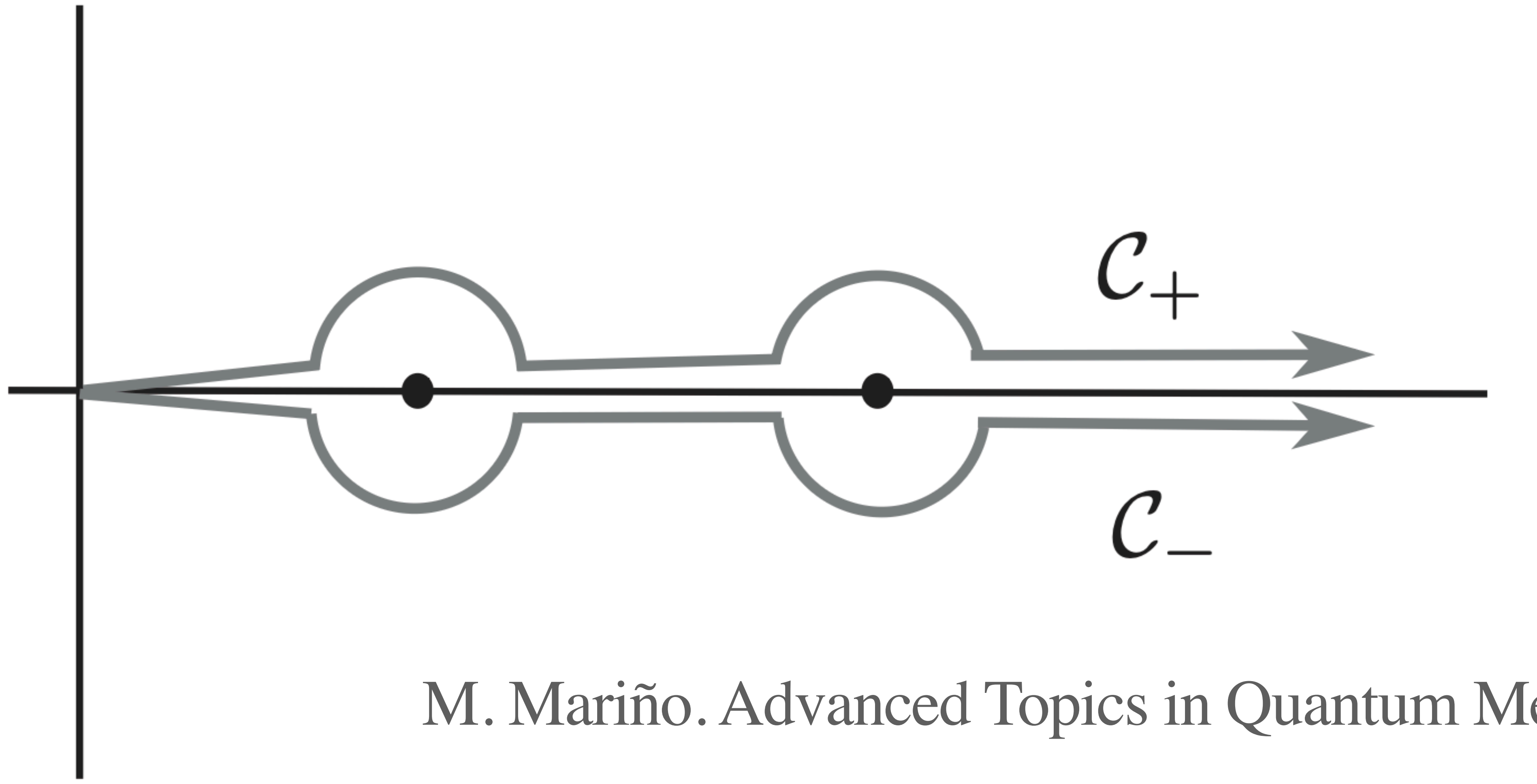
expand

Caveat: if there are poles/branch cuts on the positive real axis of Borel plane (u)?

not Borel resummable

- lateral Borel resummation

$$s_{\pm}(\phi)(z) = z^{-1} \int_{C_{\pm}^{\theta}} e^{-w/z} \hat{\phi}(w) dw$$



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Resurgence framework

- Large order and resummation ambiguity

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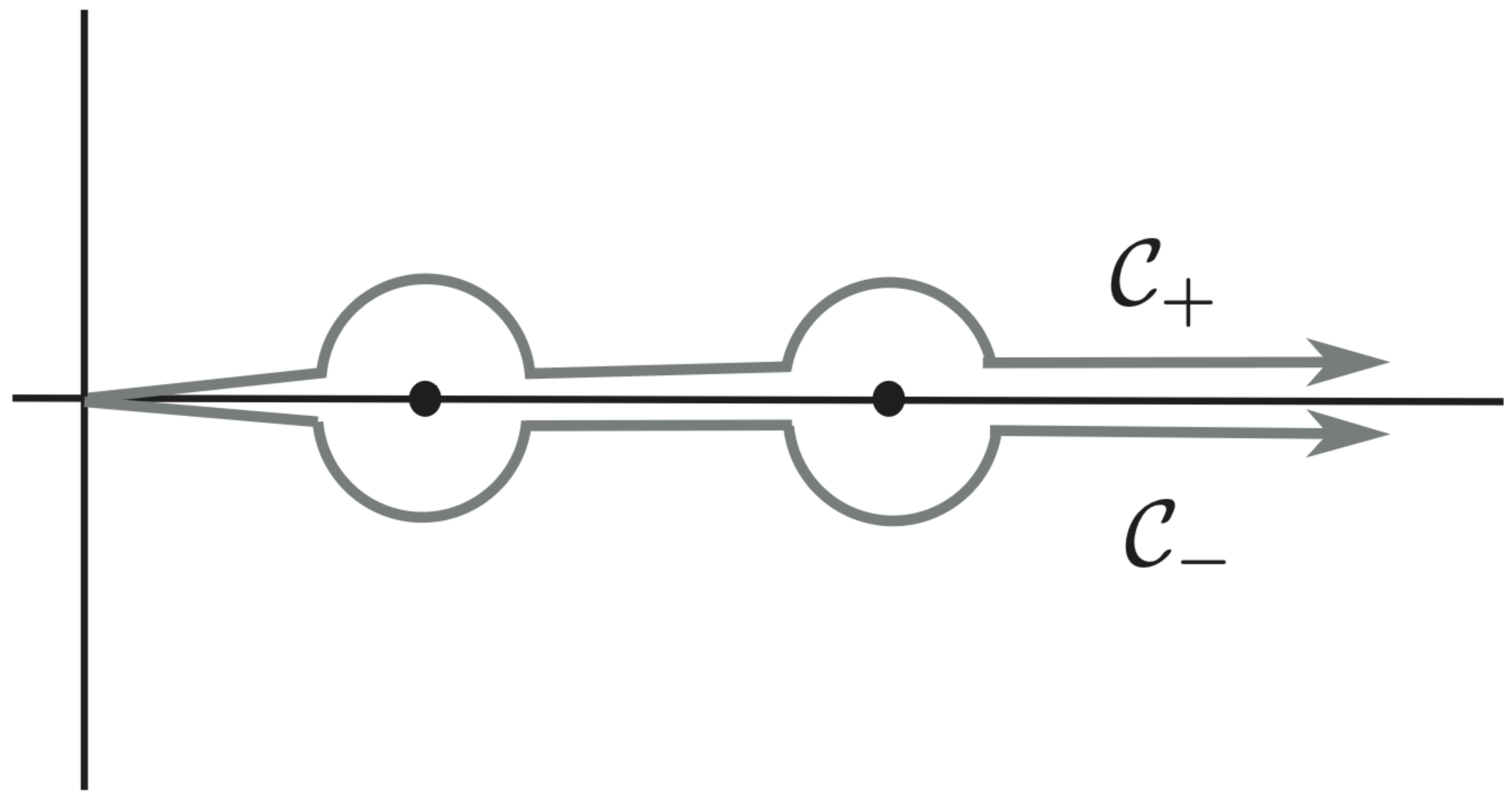
$$E_n = \sum_{k=0}^{\infty} g^{2k} \epsilon_{n,k} \equiv \sum_{k=0}^{\infty} \alpha^k \epsilon_{n,k}$$

large order behavior

$$\boxed{\epsilon_k \sim C a^k \Gamma(k+b)}$$

Borel transformation

$$\hat{E}_n(\alpha) = C \sum_{k=0}^{\infty} \frac{\Gamma(k+b)}{k!} (a\alpha)^k = C\Gamma(b) a^{-b} \left(\frac{1}{a} - \alpha \right)^{-b}$$



after lateral Borel resummation

$$\begin{aligned} \mathrm{disc}(E_n)(\alpha) &= s_+(E_n)(\alpha) - s_-(E_n)(\alpha) \\ &= 2i\,\alpha^{-1} \int_{C_+ - C_-} e^{-z/\alpha} \hat{B}(E_n)(z) dz \\ &= \boxed{2\pi i C a^{-b} g^{-2b} e^{-\frac{1}{ag^2}},} \end{aligned}$$

ambiguity:

$$E(v) \to E_R \pm i E_I$$

but the energy must be unique, why?

Resurgence framework

- Energy transseries

in this case perturbation series itself is ill-defined, because it is not Borel summable

global B.C. will correct the energy level parameter $\hat{\nu} = \nu + \delta\nu$

$$E(\nu) \to E(\hat{\nu})$$

nonperturbative multi-instanton expansion

$$E^{(N)}(g^2) = E^{(N)}_{\text{pert}}(g^2) + \sum_{k=1}^{\infty}\sum_{l=0}^{k-1}\sum_{p=0}^{\infty}\left(\frac{1}{g^{2N+1}}\exp\left[-\frac{c}{g^2}\right]\right)^k\left(\ln\left[\frac{1}{g^2}\right]\right)^l c^{\pm}_{k,l,p}\, g^{2p}$$

G. Dunne. 1401.5202

Resurgence framework

- Energy transseries

G. Dunne. 1401.5202

nonperturbative multi-instanton expansion

$$E^{(N)}(g^2) = \underbrace{E^{(N)}_{\text{pert}}(g^2)}_{\text{seperate 0-instanton}} + \sum_{k=1}^{\infty}\sum_{l=0}^{k-1}\sum_{p=0}^{\infty} \underbrace{\left(\frac{1}{g^{2N+1}}\exp\left[-\frac{c}{g^2}\right]\right)^k}_{\text{k-instanton}} \underbrace{\left(\ln\left[\frac{1}{g^2}\right]\right)^l}_{\text{quasi-zeromode}} \underbrace{c^{\pm}_{k,l,p}\, g^{2p}}_{\text{perturbative fluctuations}}$$

this is also ambiguous: has two branches

$$E_+(g^2) = E_{\text{pert}}(g^2) + E_{\text{np},+}$$

$$E_-(g^2) = E_{\text{pert}}(g^2) + E_{\text{np},-}$$

The ambiguity in perturbation series actually **cancels** the ambiguity in the explicit nonperturbative corrections

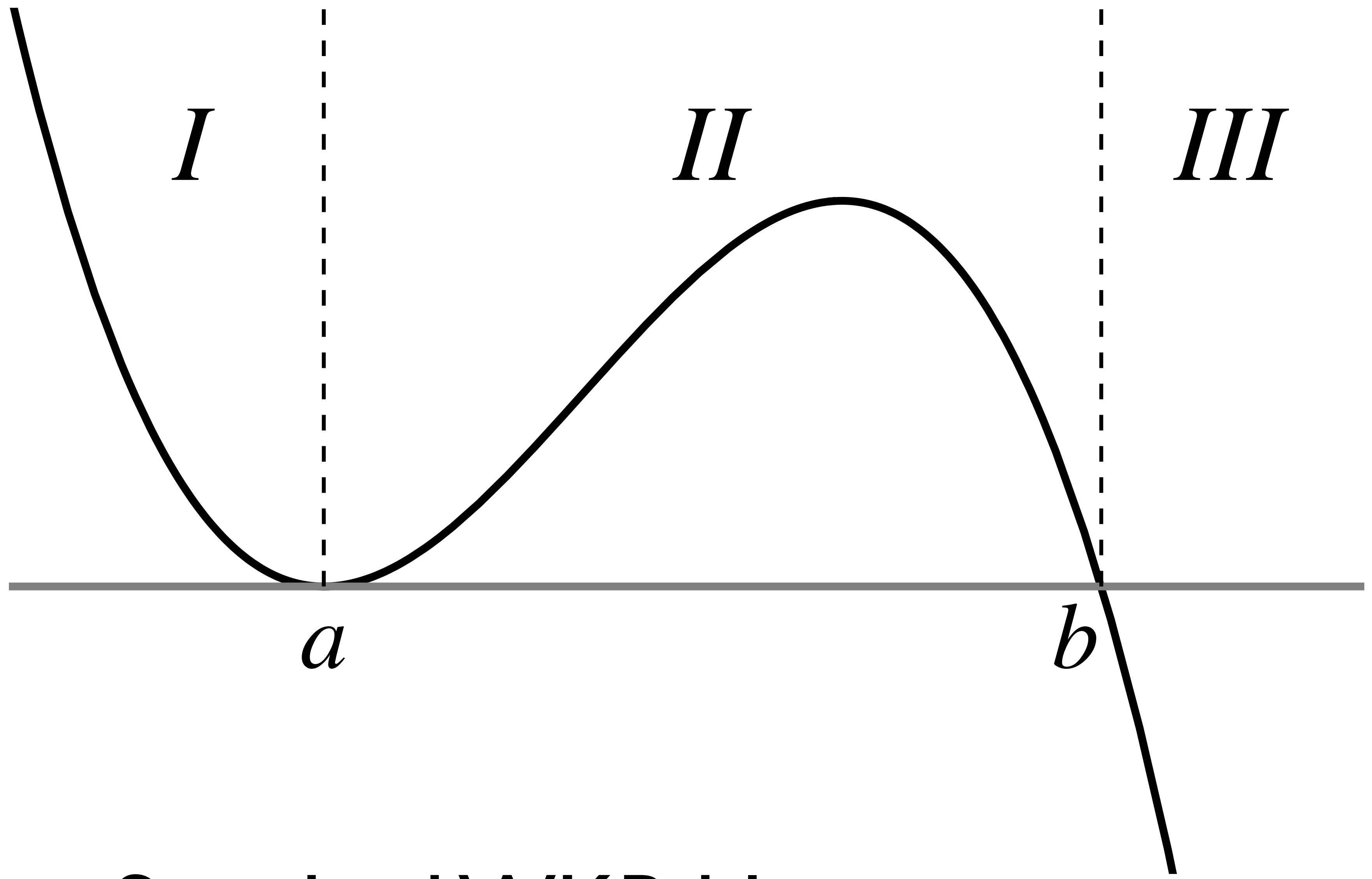
Resurgence

We need to resum the full transseries, not only perturbation series. → unique energy

How to get the energy transseries: uniform WKB

Resurgence framework

- Uniform WKB



Standard WKB blow up

at turning points $p = 0$

$$\psi(x) = \frac{C}{\sqrt{p(x)}} e^{\pm \frac{i}{\hbar} \int p(x) dx}$$

- uniform WKB:

choose a comparison equation with the same turning point structure

ansatz

$$\psi(z) = \frac{1}{\sqrt{\phi'(z)}} f(\phi(z))$$

reference equation

$$f''(\phi) + \frac{1}{\hbar^2} \Pi^2(\phi) f(\phi) = 0$$

- master equation

Schwarzian derivative

$$\Pi^2(\phi)(\phi'(z))^2 + \frac{\xi^2}{2}\{\phi(z), z\} + \left(V_{\text{eff}}(z) - 2\xi\mathscr{E}\right) = 0$$

at leading order $$(\phi'(z))^2 = -\frac{V_{\text{eff}}(z)}{\Pi^2(\phi)}$$

so e.g., if $V_{\text{eff}}(z) \sim z - z_0$ choose $\Pi^2(\phi) \sim \phi - \phi_0$

map order of the zero $\phi_0 \equiv \phi(z_0)$

then the solution is regular across the turning point

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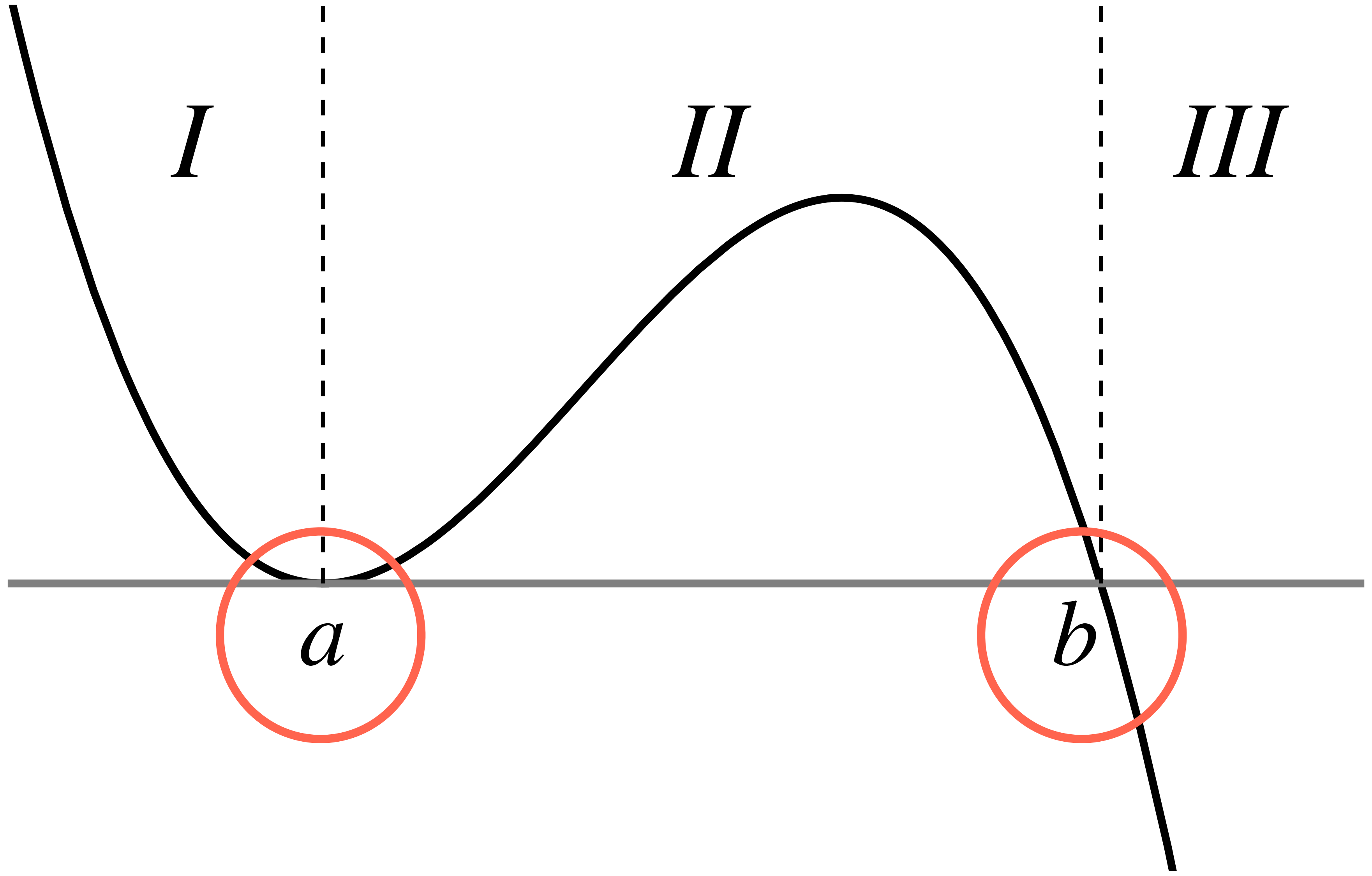
Resurgence framework

- Uniform WKB



- master equation

$$\Pi^2(\phi)(\phi'(z))^2 + \frac{\xi^2}{2}\{\phi(z), z\} + \left(V_{\text{eff}}(z) - 2\xi\mathscr{E}\right) = 0$$

procedure

- turning point structure → comparison equation
  → wave function ansatz
  double → parabolic cylinder D_ν
  single → Airy functions
- solve the master equation perturbatively in each turning-point region

$$\phi(z) = \sum_{n\geq 0} \phi_n(z)\xi^n, \quad \mathscr{E} = \sum_{n\geq 0} \mathscr{E}_n(\nu)\left(\frac{1}{2}\xi\right)^n$$

- match the uniform solutions in the overlap region
  asymptotic behavior of special functions

→ Exact Quantization Condition (EQC)

Transseries of metastable potential

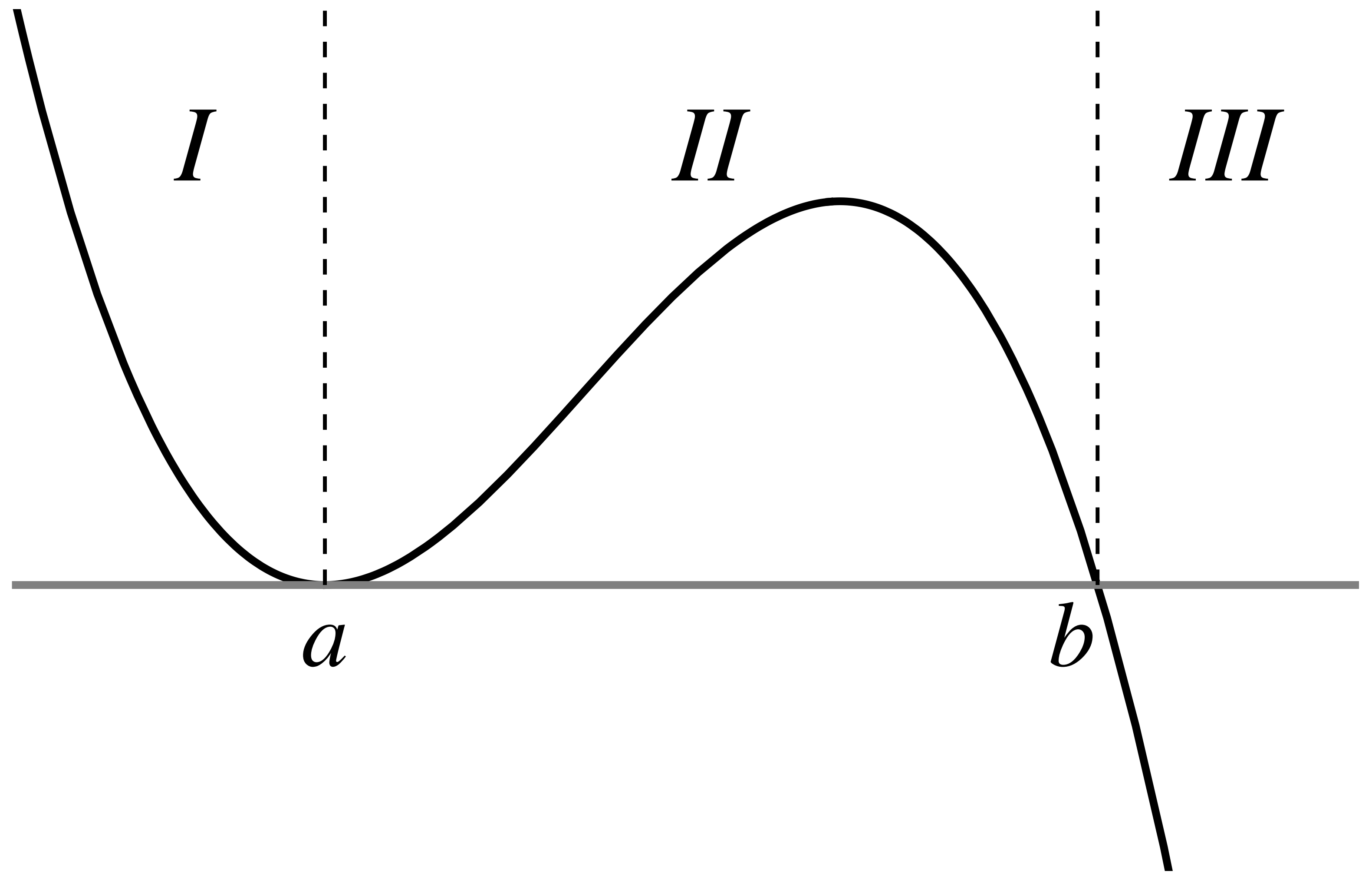
join uniform WKB solutions in different regions

$\longrightarrow$ exact quantization condition (EQC) $\longrightarrow$ energy transseries

- Potential setup

$$V_1(x) = \frac{1}{2}x^2 - gx^3$$

$$E(\nu) = \hbar\nu - \left(\frac{15}{4}\nu^2 + \frac{7}{16}\right)\hbar^2 g^2 - \left(\frac{705}{16}\nu^3 + \frac{1155}{64}\nu\right)\hbar^3 g^4 + \mathcal{O}(g^6)$$



Transseries of metastable potential

join uniform WKB solutions in different regions

⟶ exact quantization condition (EQC) ⟶ energy transseries

- Potential setup

$$V_1(x) = \frac{1}{2}x^2 - gx^3$$

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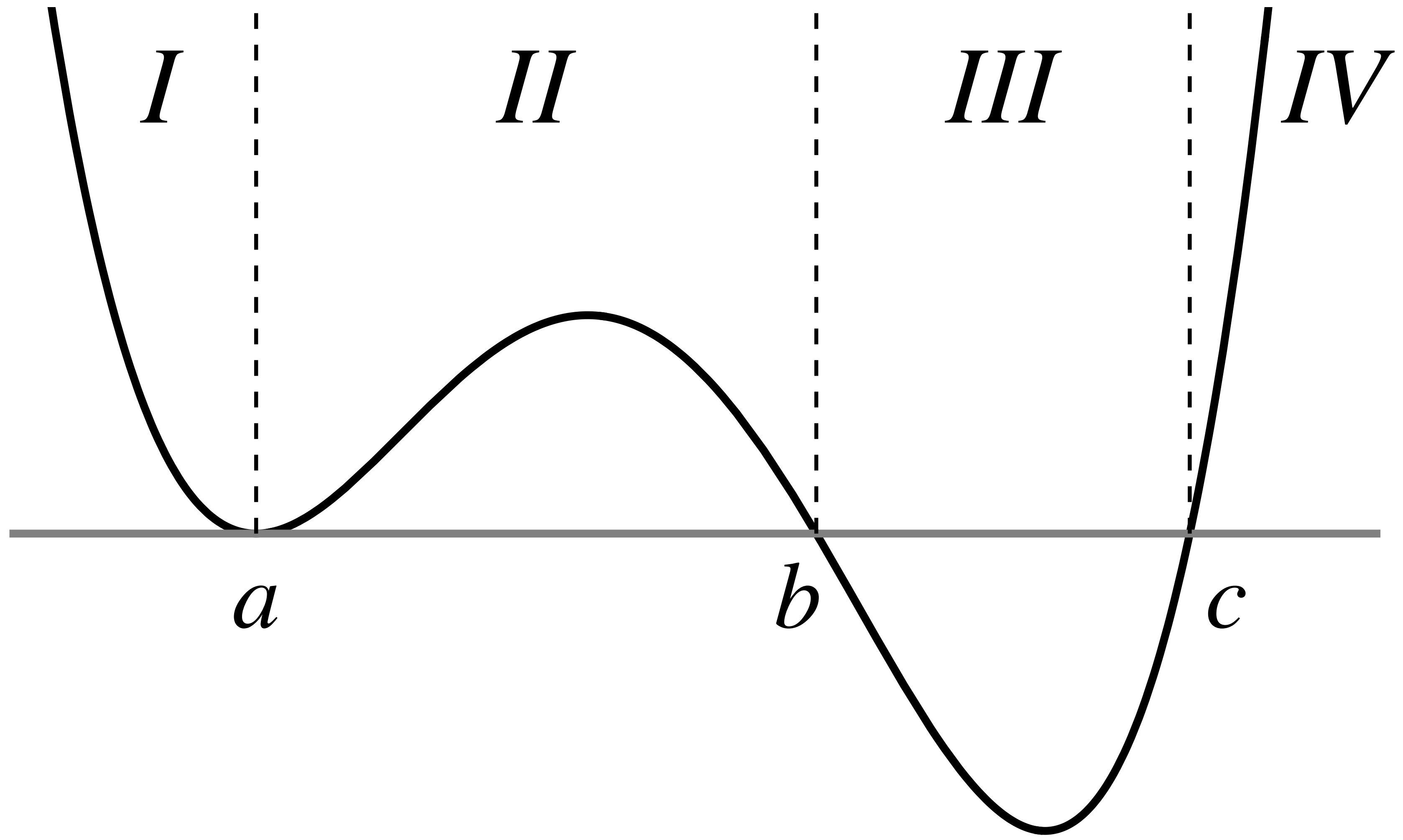
$$V_2(x) = \frac{1}{2}x^2 - gx^3 + \frac{1}{4}g^2x^4$$

$$E(\nu) = \hbar\nu - \left(\frac{27}{8}\nu^2 + \frac{11}{32}\right)\hbar^2 g^2 - \left(\frac{1937}{64}\nu^3 + \frac{2851}{256}\nu\right)\hbar^3 g^4 + \mathcal{O}(g^6)$$



Transseries of metastable potential

- Uniform WKB for the cubic

$$\Pi^2(\phi)(\phi'(z))^2 + \frac{\xi^2}{2}\{\phi(z), z\} + \left(z^2 - z^3 - 2\xi\mathscr{E}\right) = 0$$

$$z = 2gx \qquad \xi = 4g^2$$

double turning point $z = 0$:

parabolic cylinder function $\qquad \phi = u(z)/\sqrt{\xi}$

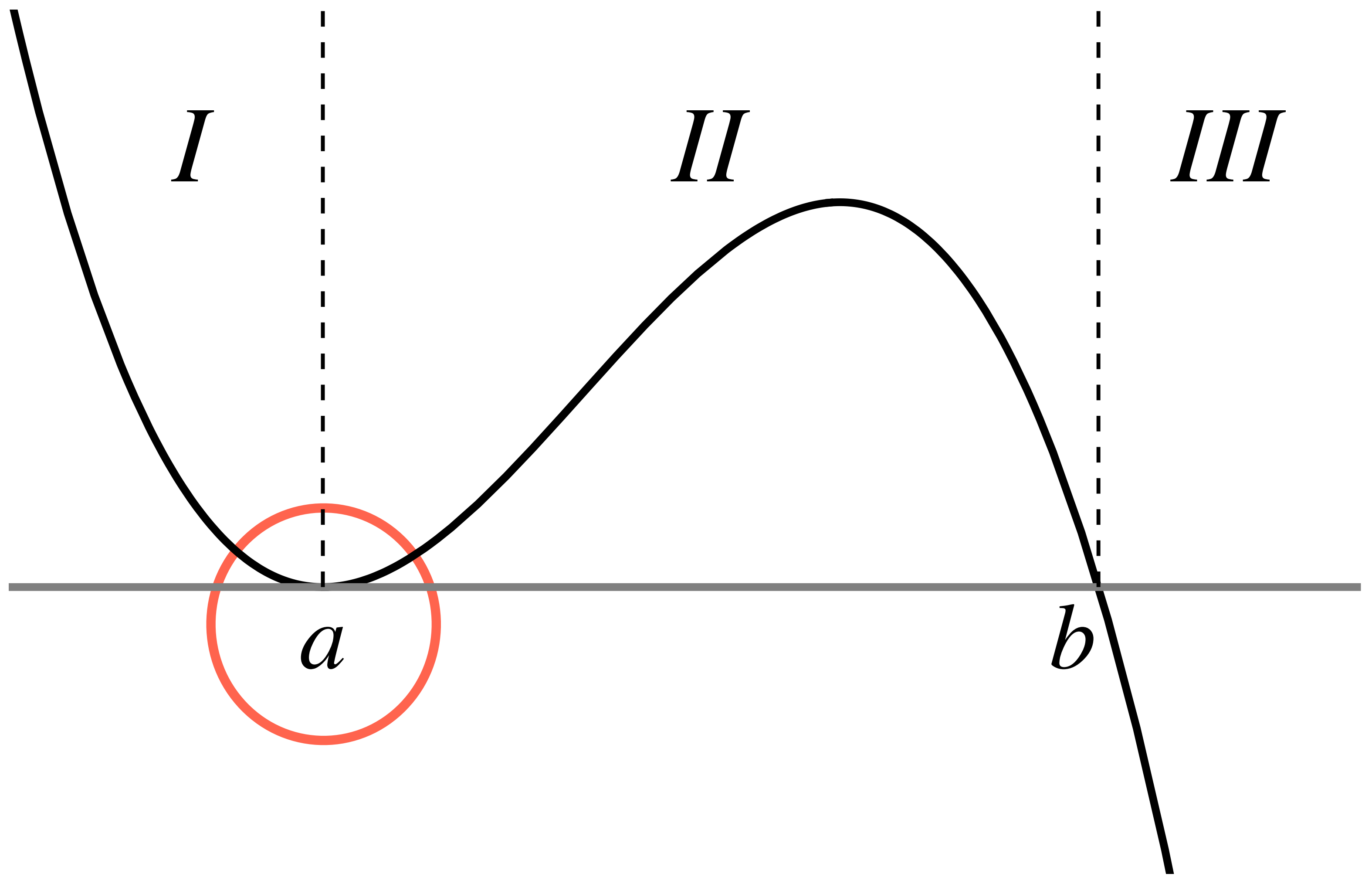
$$\Pi^2(\phi) = \xi^2(2\nu - \phi^2) \qquad \psi_1(z) = \frac{A}{\sqrt{u'(z)}} D_{\nu-\frac{1}{2}}\left(-\sqrt{\frac{2}{\xi}}u(z)\right)$$

fulfill decreasing boundary condition at $z \to -\infty$

$$D_\nu(\tilde{x} \to \infty) \to e^{-\tilde{x}^2/4}$$

master equation becomes

$$u'^2 u^2 = z^2 - z^3 - 2\xi(\mathscr{E} - \nu u'^2) + \frac{\xi^2}{2}\{u, z\}$$



solve it by expansion $\qquad U(z) = \frac{1}{2}u^2(z)$

$$U_0 = \frac{4}{15} - \frac{2}{15}(2+3z)(1-z)^{\frac{3}{2}}$$

$$U_1(z) = \nu \ln u_0(z) - \nu \ln \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}} - 2\nu \ln 2$$

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Transseries of metastable potential

- Uniform WKB for the cubic

$$\Pi^2(\phi)(\phi'(z))^2 + \frac{\xi^2}{2}\{\phi(z), z\} + \left(z^2 - z^3 - 2\xi\mathscr{E}\right) = 0$$

single turning point $z = 1$: Airy function

$$\Pi^2(\phi) = -\phi \qquad \psi_2(z) = \frac{B}{\sqrt{v'(z)}}\mathrm{Ai}^{(+)}\left(\xi^{-2/3}v(z)\right)$$

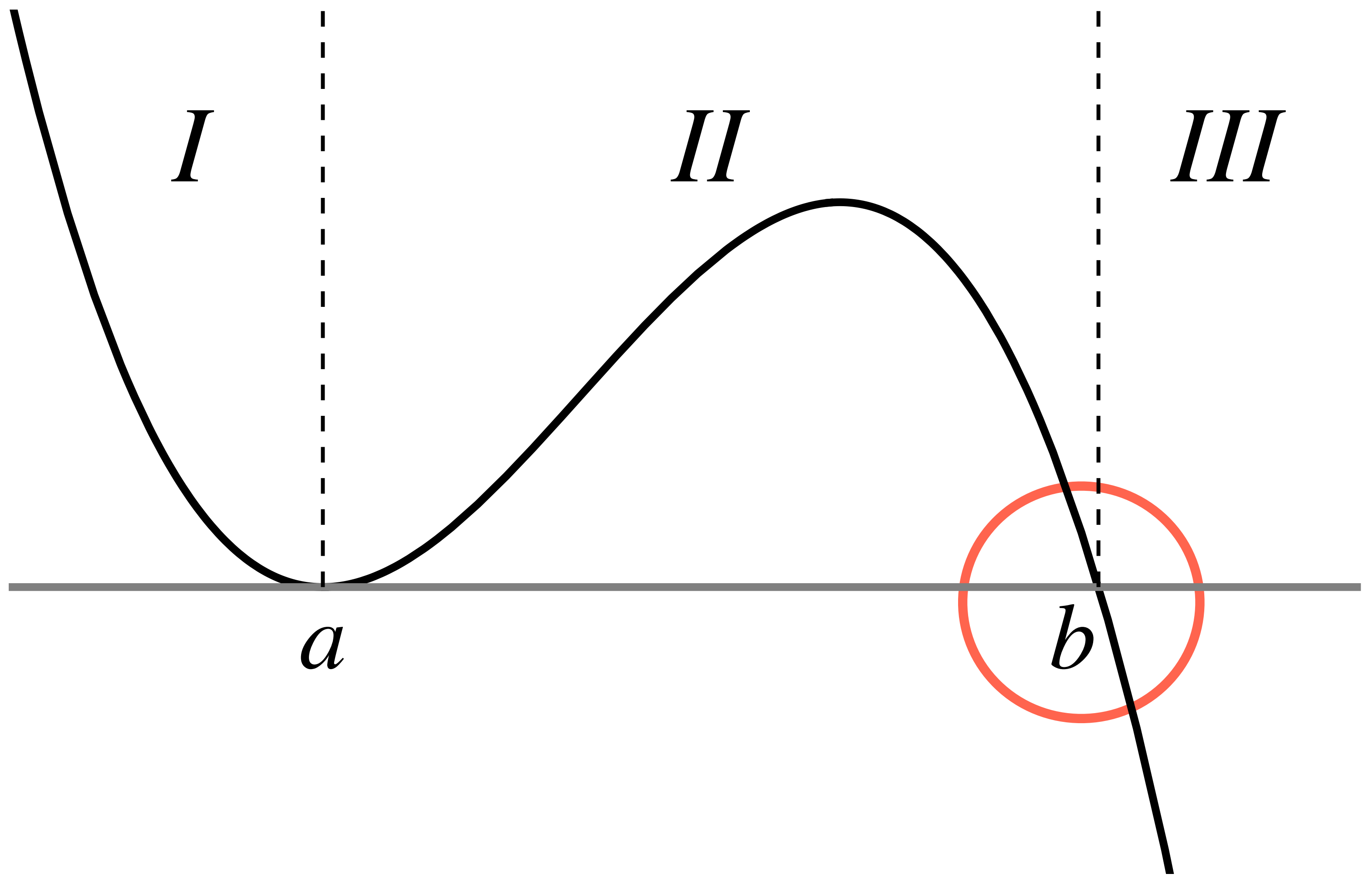
Gamow-Siegert B.C. at $z \to \infty$ $\qquad \mathrm{Ai}^{(+)}(x) \equiv \mathrm{Ai}(x) + \mathrm{Bi}(x)$

$$\mathrm{Ai}^{(+)}(-\tilde{x}) \sim \frac{1}{\sqrt{\pi}\tilde{x}^{1/4}}\exp\left(i\left(\frac{2}{3}\tilde{x}^{\frac{3}{2}} + \frac{\pi}{4}\right)\right)$$

only outgoing wave

master equation becomes

$$vv'^2 = z^2 - z^3 - 2\xi\mathscr{E} + \frac{\xi^2}{2}\{v, z\}$$



solve it by expansion

$$V_0(z) = \frac{2}{15}(2 + 3z)(1 - z)^{\frac{3}{2}}$$

$$V_1 = -\nu\ln\frac{2 - z + 2\sqrt{1 - z}}{z}$$

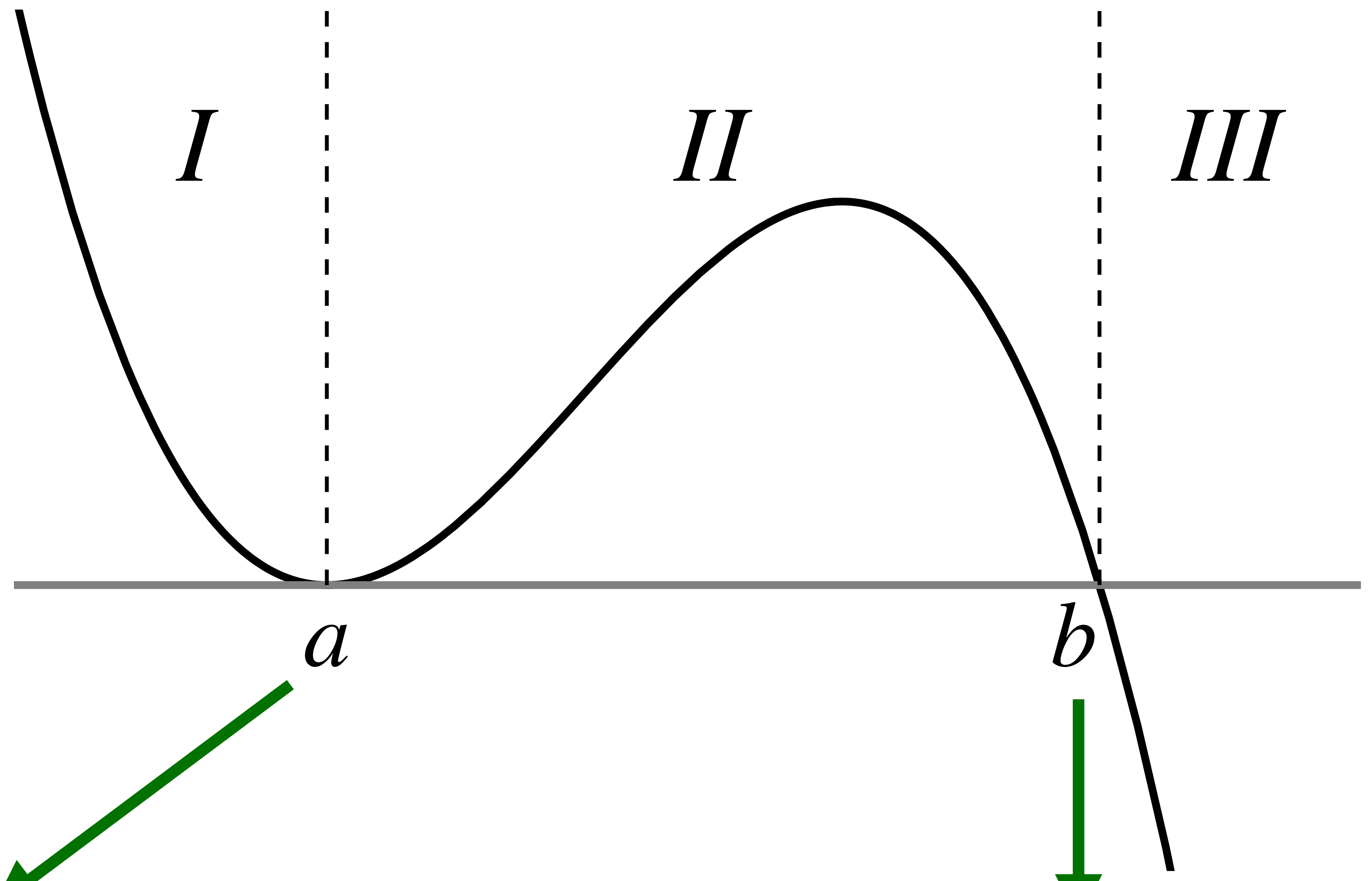
Transseries of metastable potential

- Uniform WKB for the cubic
  - joining the solutions in different regions

    joining the solutions around a and b

    in the barrier region II

    using the asymptotics of the special functions



$$\psi_1(z) = \frac{A}{\sqrt{u'(z)}} D_{\nu-\frac{1}{2}}\left(-\sqrt{\frac{2}{\xi}}u(z)\right) \qquad \psi_2(z) = \frac{B}{\sqrt{v'(z)}} \mathrm{Ai}^{(+)}\left(\xi^{-2/3}v(z)\right)$$

$$\mathrm{Ai}^{(+)}(\tilde{x}) \sim \frac{1}{\sqrt{\pi}\tilde{x}^{1/4}} e^{\frac{2}{3}\tilde{x}^{3/2}} {}_2F_0\left(\frac{1}{6},\frac{5}{6};;\frac{3}{4}\tilde{x}^{-\frac{3}{2}}\right), \qquad -\frac{2}{3}\pi < \arg(\tilde{x}) < 0$$

vanishing

$$\mathrm{Ai}^{(+)}(\tilde{x}) \sim \frac{1}{\sqrt{\pi}\tilde{x}^{1/4}} e^{\frac{2}{3}\tilde{x}^{3/2}} {}_2F_0\left(\frac{1}{6},\frac{5}{6};;\frac{3}{4}\tilde{x}^{-\frac{3}{2}}\right)$$

exponential decaying

$$+\frac{i}{\sqrt{\pi}\tilde{x}^{1/4}} e^{-\frac{2}{3}\tilde{x}^{3/2}} {}_2F_0\left(\frac{1}{6},\frac{5}{6};;-\frac{3}{4}\tilde{x}^{-\frac{3}{2}}\right), \quad 0 < \arg(\tilde{x}) < \frac{2}{3}\pi,$$

For $g = g_R + i0^+$

$$D_{\nu-\frac{1}{2}}(-\tilde{x}) \sim \frac{\sqrt{2\pi}}{\Gamma\left(\frac{1}{2}-\nu\right)} e^{\tilde{x}^2/4}\tilde{x}^{-\frac{1}{2}-\nu} {}_2F_0\left(\frac{1}{4}+\frac{\nu}{2},\frac{3}{4}+\frac{\nu}{2};;2\tilde{x}^{-2}\right)$$

Stokes jump

$$\pm i e^{\mp i\pi\nu} e^{-\tilde{x}^2/4}\tilde{x}^{\nu-\frac{1}{2}} {}_2F_0\left(\frac{1}{4}-\frac{\nu}{2},\frac{3}{4}-\frac{\nu}{2};;-2\tilde{x}^{-2}\right), \quad 0 < \pm\arg(\tilde{x}) < \frac{\pi}{2}.$$

$$\frac{1}{\Gamma\left(\frac{1}{2}-\nu\right)} = 0 \qquad \nu = n + \frac{1}{2}$$

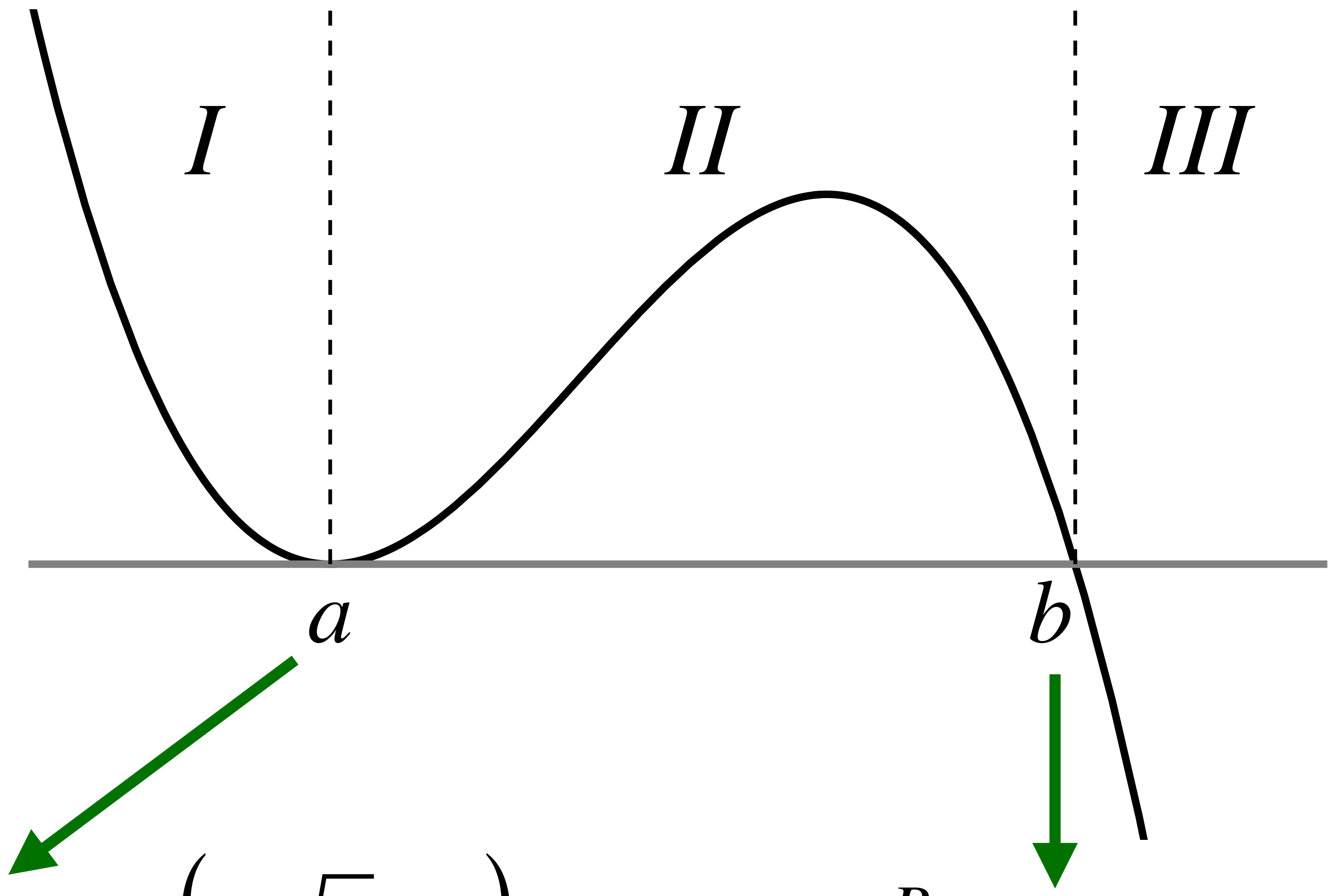
Transseries of metastable potential

- Uniform WKB for the cubic
  - joining the solutions in different regions

    joining the solutions in region II and region IV

    in the barrier region III

    using the asymptotics of the special functions



Stokes jump

$$\psi_1(z) = \frac{A}{\sqrt{u'(z)}} D_{\nu-\frac{1}{2}}\left(-\sqrt{\frac{2}{\xi}}u(z)\right) \qquad \psi_2(z) = \frac{B}{\sqrt{v'(z)}} \mathrm{Ai}^{(+)}\left(\xi^{-2/3}v(z)\right)$$

For $g = g_R - i0^+$

$$-\frac{\Gamma\left(\frac{1}{2}-\nu\right)}{\sqrt{2\pi}e^{i\pi\nu}} e^{-\frac{u^2}{\xi}} \left(\frac{2u^2}{\xi}\right)^{\nu} \frac{{}_2F_0\left(\frac{1}{4}-\frac{\nu}{2},\frac{3}{4}+\frac{\nu}{2};;-\xi u^{-2}\right)}{{}_2F_0\left(\frac{1}{4}+\frac{\nu}{2},\frac{3}{4}-\frac{\nu}{2};;\xi u^{-2}\right)} = e^{\frac{4v^{3/2}}{2\xi}} \frac{{}_2F_0\left(\frac{1}{6},\frac{5}{6};;\frac{3\xi}{4}v^{-\frac{3}{2}}\right)}{{}_2F_0\left(\frac{1}{6},\frac{5}{6};;-\frac{3\xi}{4}v^{-\frac{3}{2}}\right)}$$

$$\boxed{-f(\nu) = e^{2\pi i\nu} + 1}$$

EQC for the cubic

For $g = g_R + i0^+$

$$\boxed{\frac{1}{\Gamma\left(\frac{1}{2}-\nu\right)} = 0} \qquad \nu = n + \frac{1}{2}$$

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Transseries of metastable potential

- Uniform WKB for the cubic
  - energy transseries

for a typical EQC $\quad 1 + e^{2\pi i\hat{\nu}} = f(\hat{\nu})$

solving it as the correction of ν $\quad \hat{\nu} = \nu + \Delta\nu \quad \nu = n + \frac{1}{2}$

for the cubic $E\_$

$$\Delta\nu = \frac{i}{2\pi}\ln(1 + i\epsilon\lambda f(\nu + \Delta\nu))$$

λ counts the multi-instanton sector

$$= -\frac{i}{2\pi}f(\nu) + \left(\frac{i}{4\pi}f^2(\nu) - \frac{1}{4\pi^2}f(\nu)f'(\nu)\right)$$

up to 2-instanton sector

$$\boxed{E_+(\hat{\nu}) = E(\nu)}$$

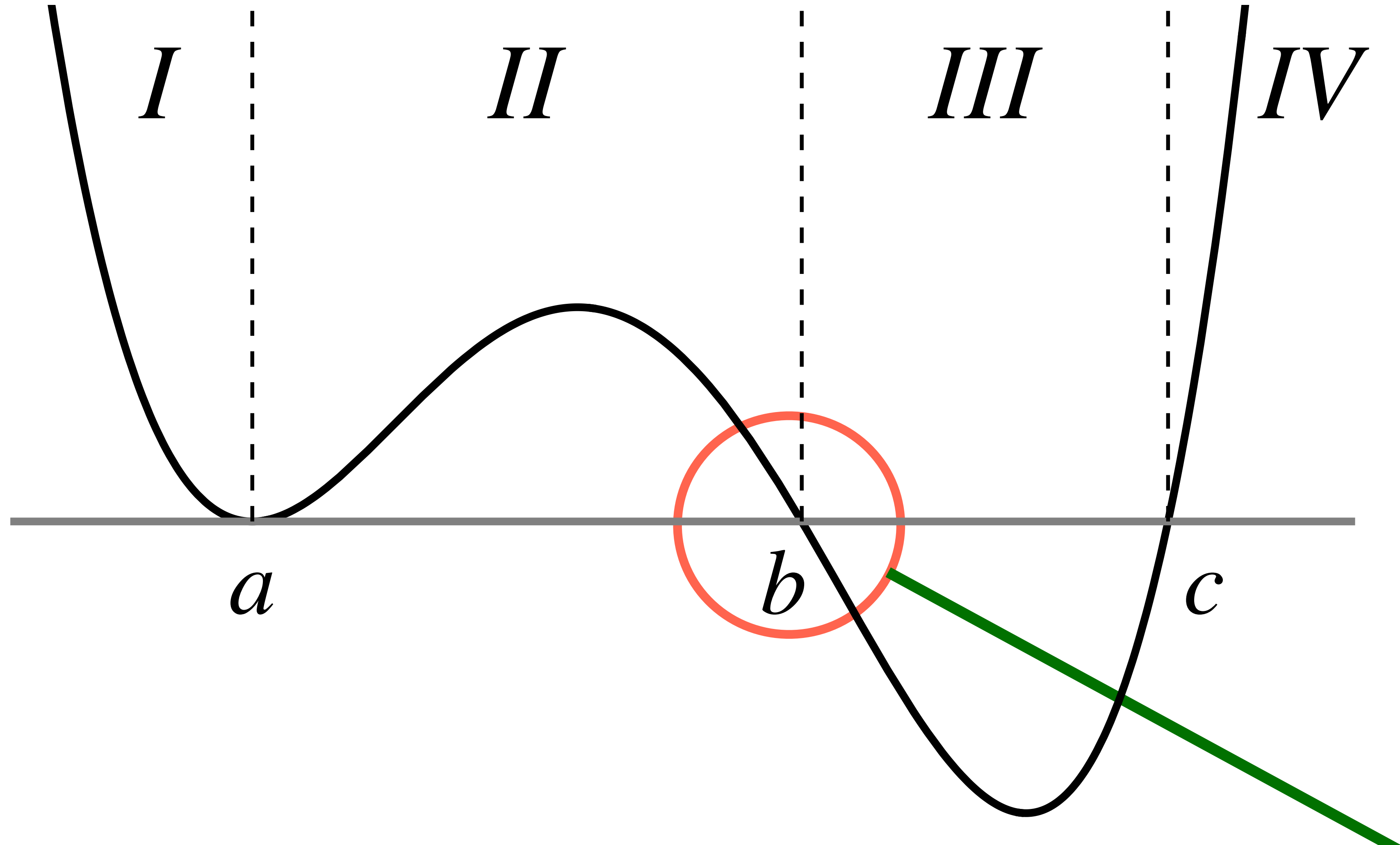
$$E\_(\hat{\nu}) = E(\nu) - \frac{i}{2\pi}f(\nu)\frac{\partial E}{\partial\nu} + \left[\frac{i}{4\pi}f^2(\nu)\frac{\partial E}{\partial\nu} - \frac{1}{8\pi^2}\frac{\partial}{\partial\nu}\left(f^2(\nu)\frac{\partial E}{\partial\nu}\right)\right],$$

- Gamow-Siegert B.C. gives this structure

determined by the asymptotics of Airy function

Transseries of metastable potential

- Uniform WKB for the tilted double well



the same parabolic cylinder

$$\psi_1(z) = \frac{A}{\sqrt{u'(z)}} D_{\nu-\frac{1}{2}}\left(-\sqrt{\frac{2}{\xi}}u(z)\right)$$

but now different B.C. for $z \to \infty$

exponential decaying

$$\psi_3(z) = \frac{B}{\sqrt{v'(z)}} \mathrm{Ai}\left(\xi^{-\frac{2}{3}}v(z)\right) \qquad \mathrm{Ai}(\tilde{x}) \sim \exp\left(-\frac{2}{3}\tilde{x}^{\frac{3}{2}}\right)$$

a new turning point, no B.C. selection, so in general

$$\psi_2(z) = \frac{1}{\sqrt{w'(z)}}[C\mathrm{Ai}(\xi^{-2/3}w(z)) + D\mathrm{Bi}(\xi^{-2/3}w(z))]$$

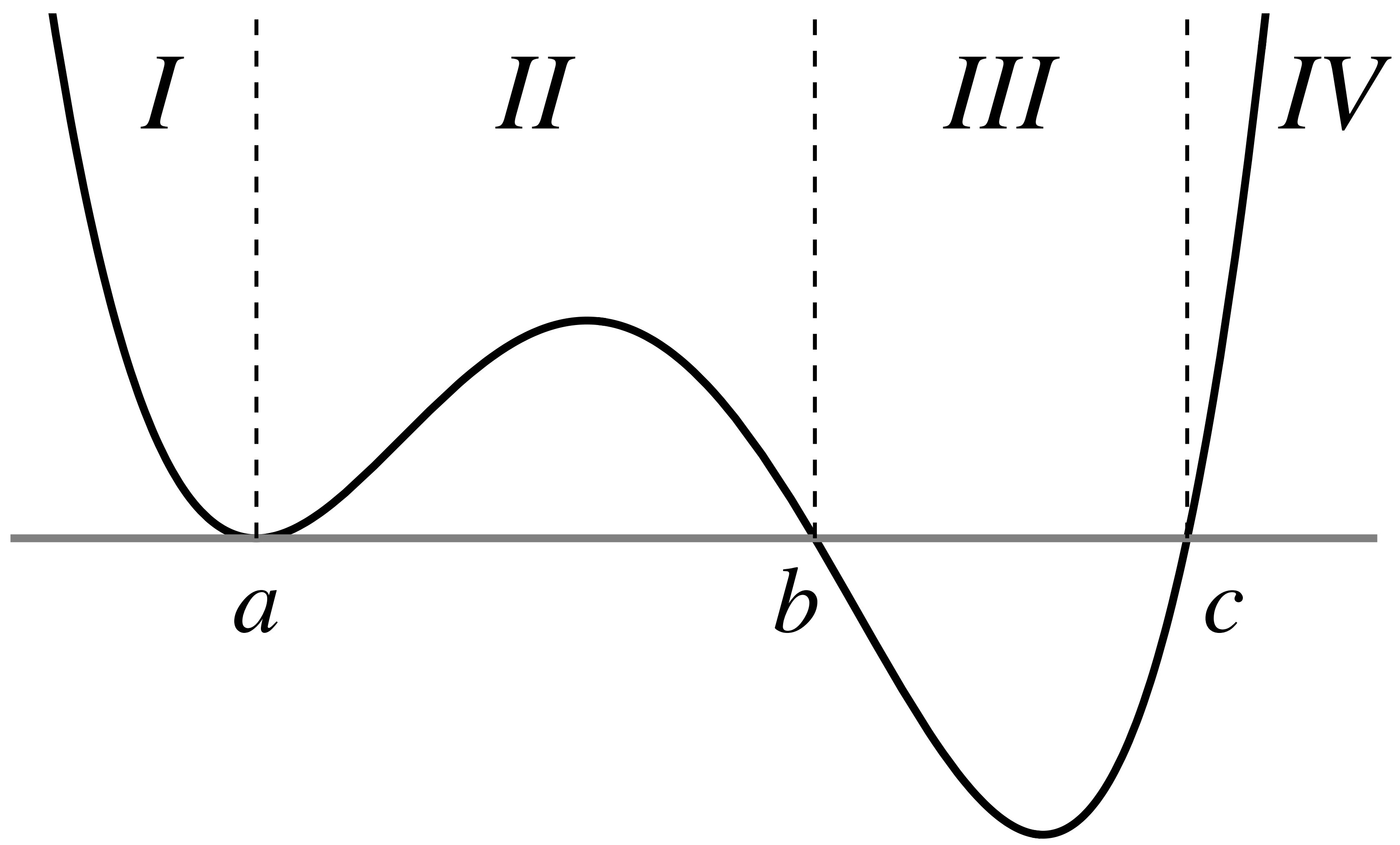
Two regions joining:

joining at region III:

$$\frac{e^{-i\pi/4} e^{\frac{2i}{3\xi}x^{\frac{3}{2}}} {}_2F_0\left(\frac{1}{6},\frac{5}{6};;\left(i\frac{4}{3\xi}x^{\frac{3}{2}}\right)^{-1}\right)}{e^{i\pi/4} e^{-\frac{2i}{3\xi}x^{\frac{3}{2}}} {}_2F_0\left(\frac{1}{6},\frac{5}{6};;\left(-i\frac{4}{3\xi}x^{\frac{3}{2}}\right)^{-1}\right)} = \frac{\left(Ce^{i\frac{\pi}{4}} + De^{-i\frac{\pi}{4}}\right) e^{-\frac{2i}{3\xi}y^{\frac{3}{2}}} {}_2F_0\left(\frac{1}{6},\frac{5}{6};;\left(-i\frac{4}{3\xi}y^{\frac{3}{2}}\right)^{-1}\right)}{\left(Ce^{-i\frac{\pi}{4}} + De^{i\frac{\pi}{4}}\right) e^{\frac{2i}{3\xi}y^{\frac{3}{2}}} {}_2F_0\left(\frac{1}{6},\frac{5}{6};;\left(i\frac{4}{3\xi}y^{\frac{3}{2}}\right)^{-1}\right)} \longrightarrow \frac{C}{D}$$

Transseries of metastable potential

- Uniform WKB for the tilted double well



joining at region II:

$$\boxed{\begin{aligned} e^{2\pi i\hat{\nu}} + 1 &= \frac{-f_2}{1+f_1} \\ e^{-2\pi i\hat{\nu}} + 1 &= \frac{-f_1 f_2}{1+f_1} \end{aligned}}$$

$$f_1 \equiv \exp\left(\frac{2(V+W)}{\xi}\right)\frac{F_1(2V/\xi)F_1(2W/\xi)}{F_1(-2V/\xi)F_1(-2W/\xi)}$$

$$f_2 \equiv \frac{\sqrt{2\pi}}{\Gamma\left(\nu+\frac{1}{2}\right)}e^{-\frac{2(U+W)}{\xi}}\left(\frac{2u^2}{\xi}\right)^{\nu}\frac{F_1(-2W/\xi)F_3(4U/\xi)}{F_1(2W/\xi)F_2(4U/\xi)}$$

EQC for the tilted double well

- energy transseries

$$E_+ = E(\nu) + \frac{1}{2\pi i}\frac{\partial E}{\partial \nu}\frac{f_2}{1+f_1} - \frac{\partial E}{\partial \nu}\left[\frac{1}{4\pi i}\frac{f_2^2}{(1+f_1)^2} + \frac{1}{4\pi^2}\frac{f_2 f_2'(1+f_1) - f_2^2 f_1'}{(1+f_1)^3}\right] - \frac{1}{8\pi^2}\frac{\partial^2 E}{\partial \nu^2}\frac{f_2^2}{(1+f_1)^2}$$

$$E_- = E(\nu) - \frac{1}{2\pi i}\frac{\partial E}{\partial \nu}\frac{f_1 f_2}{1+f_1} + \frac{\partial E}{\partial \nu}\left[\frac{1}{4\pi i}\frac{f_1^2 f_2^2}{(1+f_1)^2} - \frac{1}{4\pi^2}\frac{f_1 f_2((f_1' f_2 + f_1 f_2')(1+f_1) - f_1 f_2 f_1')}{(1+f_1)^3}\right] - \frac{1}{8\pi^2}\frac{\partial^2 E}{\partial \nu^2}\frac{f_1^2 f_2^2}{(1+f_1)^2}$$

Resummation of the transseries

consistency: $\tilde{E}_+ = \tilde{E}_-$,

gives the relation between P and NP

- Consistency tower for the cubic

lateral Borel resum

$$E_+(\hat{\nu}) = E(\nu)$$

$$E_-(\hat{\nu}) = E(\nu) - \frac{i}{2\pi} f(\nu) \frac{\partial E}{\partial \nu} + \left[\frac{i}{4\pi} f^2(\nu) \frac{\partial E}{\partial \nu} - \frac{1}{8\pi^2} \frac{\partial}{\partial \nu} \left(f^2(\nu) \frac{\partial E}{\partial \nu} \right) \right],$$

Resummation of the transseries

- Consistency tower for the cubic

lateral Borel resum $E_\pm \mapsto \tilde{E}_\pm \equiv S_\pm E_\pm$

$$\tilde{E}_+ = S_+(E(\nu))$$

$$\tilde{E}_- = S_-\left(E(\nu) - \frac{i}{2\pi} f(\nu)\frac{\partial E}{\partial \nu} + \left[\frac{i}{4\pi} f^2 \frac{\partial E}{\partial \nu} - \frac{1}{8\pi^2}\frac{\partial}{\partial \nu}\left(f^2 \frac{\partial E}{\partial \nu}\right)\right]\right)$$

consistency: $\tilde{E}_+ = \tilde{E}_-$,

gives the relation between P and NP

$$S_\pm y = \mathrm{PV}(y) + S_\pm\left(D^{(1)}_\pm y + D^{(2)}_\pm y\right)$$

$$= \mathrm{PV}(y) + \mathrm{PV}\left(D^{(1)}_\pm y\right) + S_\pm\left(D^{(1)}_\pm D^{(1)}_\pm y + D^{(2)}_\pm y\right)$$

generate new asymptotic series

Resummation of the transseries

- Consistency tower for the cubic

lateral Borel resum $E_\pm \mapsto \tilde{E}_\pm \equiv S_\pm E_\pm$

$$\tilde{E}_+ = \boxed{S_+(E(\nu))}$$

$$\tilde{E}_- = \boxed{S_-\left(E(\nu) - \frac{i}{2\pi} f(\nu) \frac{\partial E}{\partial \nu}} + \left[\frac{i}{4\pi} f^2 \frac{\partial E}{\partial \nu} - \frac{1}{8\pi^2} \frac{\partial}{\partial \nu}\left(f^2 \frac{\partial E}{\partial \nu}\right)\right]\right)$$

resum the perturbation series → one-instanton sector

$$\tilde{E}_+(\hat{\nu}) = \tilde{E}(\nu,0) + \boxed{S_+\left(E(\nu,1)} + D_-^{(2)} E\right)$$

$$\tilde{E}_-(\hat{\nu}) = \tilde{E}(\nu,0) + S_-\left(E(\nu,1) + \left[\frac{i}{4\pi} f^2 \frac{\partial E}{\partial \nu} - \frac{1}{8\pi^2} \frac{\partial}{\partial \nu}\left(f^2 \frac{\partial E}{\partial \nu}\right) + D_+^{(2)} E\right]\right)$$

consistency: $\tilde{E}_+ = \tilde{E}_-$,

gives the relation between P and NP

$$S_\pm y = \mathrm{PV}(y) + S_\pm\left(D_\pm^{(1)} y + D_\pm^{(2)} y\right)$$

$$= \mathrm{PV}(y) + \mathrm{PV}\left(D_\pm^{(1)} y\right) + S_\pm\left(D_\pm^{(1)} D_\pm^{(1)} y + D_\pm^{(2)} y\right)$$

generate new asymptotic series

$$\mathrm{disc}_{(1)}(E(\nu)) = S_+(E(\nu)) - S_-(E(\nu))$$

$$= -\frac{i}{2\pi} f \frac{\partial E}{\partial \nu}$$

discontinuity (P) = one-instanton NP

Resummation of the transseries

- Consistency tower for the cubic

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resum the one-instanton sector

$$\tilde{E}_+(\hat{\nu}) = \tilde{E}(\nu,0) + \tilde{E}(\nu,1) + \boxed{S_+\left(E(\nu,2)\right.} + D_-^{(2)}E(\nu,1) + \ldots\left.\right)$$

$$\tilde{E}_-(\hat{\nu}) = \tilde{E}(\nu,0) + \tilde{E}(\nu,1) + S_-\left(E(\nu,2) + D_+^{(2)}E(\nu,1) + \ldots\right).$$

consistency: $\tilde{E}_+ = \tilde{E}_-$,

gives the relation between P and NP

$$S_\pm y = \mathrm{PV}(y) + S_\pm\left(D_\pm^{(1)}y + D_\pm^{(2)}y\right)$$

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generate new asymptotic series

$$\mathrm{disc}_{(1)}(E(\nu)) = S_+(E(\nu)) - S_-(E(\nu))$$

$$= -\frac{i}{2\pi}f\frac{\partial E}{\partial \nu}$$

discontinuity (P) = one-instanton NP

Re part $\quad \mathrm{disc}_{(1)}(E(\nu,1)) = -\frac{1}{8\pi^2}\frac{\partial}{\partial \nu}\left(f^2\frac{\partial E}{\partial \nu}\right)$

Im part $\quad \mathrm{disc}_{(2)}(E(\nu)) = \frac{i}{4\pi}f^2\frac{\partial E}{\partial \nu}$

$$E(\nu,1) = = -\frac{i}{4\pi}f\frac{\partial E}{\partial \nu} \quad E(\nu,2) = \frac{i}{8\pi}f^2\frac{\partial E}{\partial \nu} - \frac{1}{16\pi^2}\frac{\partial}{\partial \nu}\left(f^2\frac{\partial E}{\partial \nu}\right)$$

Yan Luo

Resummation of the transseries

- The decay rate of the cubic

to the leading order, the resummed energy transseries is

$$\tilde{E}(\hat{\nu}) = E_R \boxed{- \frac{i}{4\pi} f \frac{\partial E}{\partial \nu} + \frac{i}{8\pi} f^2 \frac{\partial E}{\partial \nu} - \frac{1}{16\pi^2} \frac{\partial}{\partial \nu}\left(f^2 \frac{\partial E}{\partial \nu}\right)}$$

$$f \sim e^{-S_E/g^2}$$

leading pole discontinuity + subleading pole discontinuity $\longrightarrow$ give the decay rate

$$E_+(\hat{\nu}) = E(\nu)$$

$$E\_(\hat{\nu}) = E(\nu) - \frac{i}{2\pi} f(\nu) \frac{\partial E}{\partial \nu} + \left[\frac{i}{4\pi} f^2(\nu) \frac{\partial E}{\partial \nu} - \frac{1}{8\pi^2} \frac{\partial}{\partial \nu}\left(f^2(\nu) \frac{\partial E}{\partial \nu}\right)\right],$$

$\boxed{\text{discontinuity} = 2 \times \mathrm{Im} E = \text{decay rate } \Gamma}$

Gamow-Siegert B.C. and Airy asymptotics determine this

Resummation of the transseries

- Resurgence tower for the tilted double well

$$E_+ = E(\nu) + \frac{1}{2\pi i}\frac{\partial E}{\partial \nu}\frac{f_2}{1+f_1}$$

$$E_- = E(\nu) - \frac{1}{2\pi i}\frac{\partial E}{\partial \nu}\frac{f_1 f_2}{1+f_1}$$

discontinuity $\quad (E_+ - E_-)_{1I} = \frac{1}{2\pi i}\frac{\partial E}{\partial \nu} f_2(\nu)$

resurgence only gives

discontinuity = $E_- - E_+$

Moreover, we can check

$$\mathrm{Im}S_\pm E(\nu) = \frac{\mp 1}{2\pi i}\frac{\partial E}{\partial \nu}\frac{f_2}{1+f_1}$$

Imaginary parts vanish after resummation

Where is the decay rate?

The spectrum of the system is real, as we impose the **global** B.C.

We can only get the metastable decay rate once we impose the **artificial** Gamow-Siegert B.C.

$$E_+(\hat{\nu}) = E(\nu)$$
$$E_-(\hat{\nu}) = E(\nu) + E_{\mathrm{NP}},$$

The same structure.

The same perturbation series as that if we impose the global B.C.

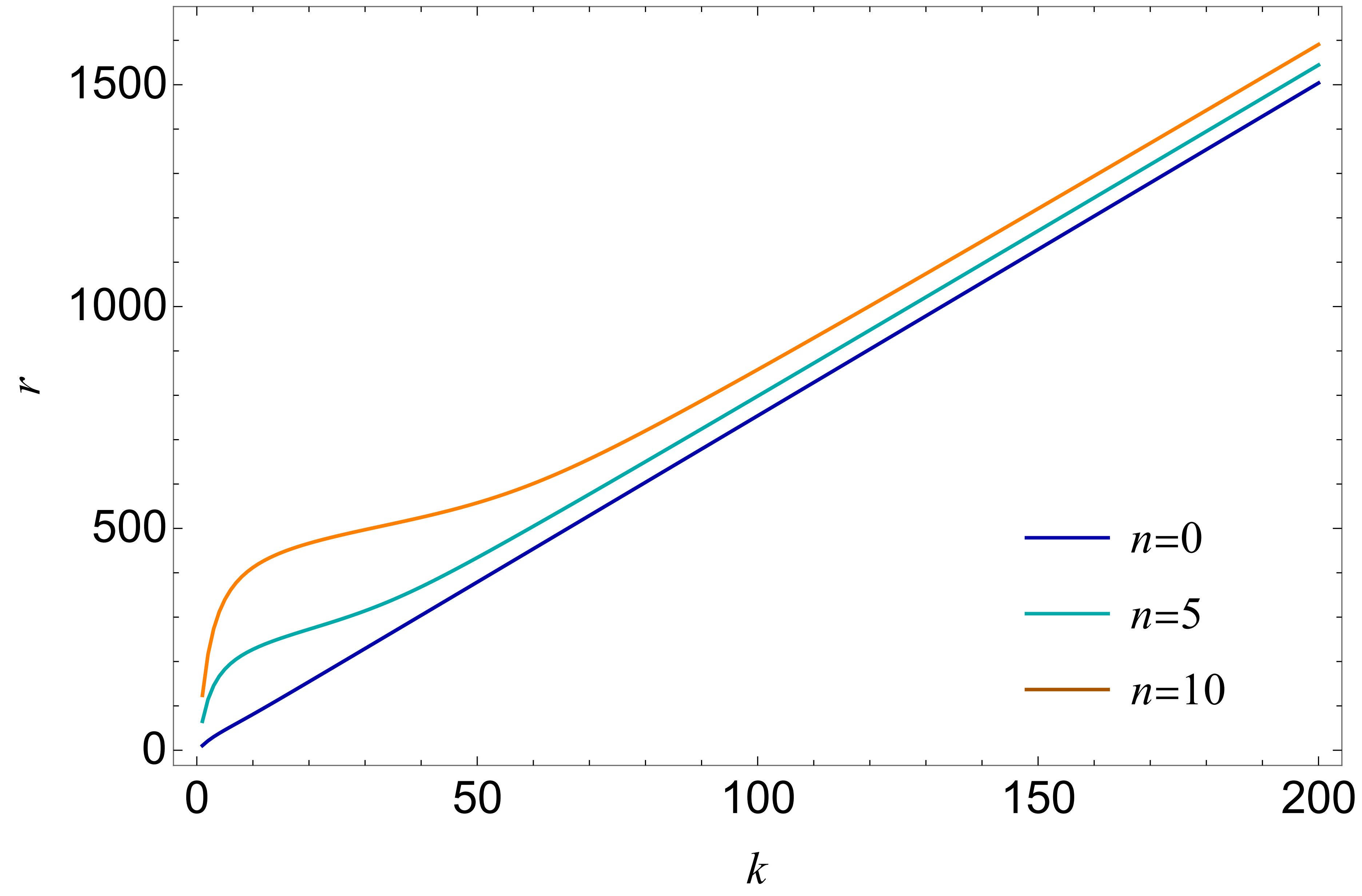
so discontinuity still gives the decay rate

Numerics: Leading Instanton

instanton calculation B. Garbrecht and N. Wagner. 2412.20431

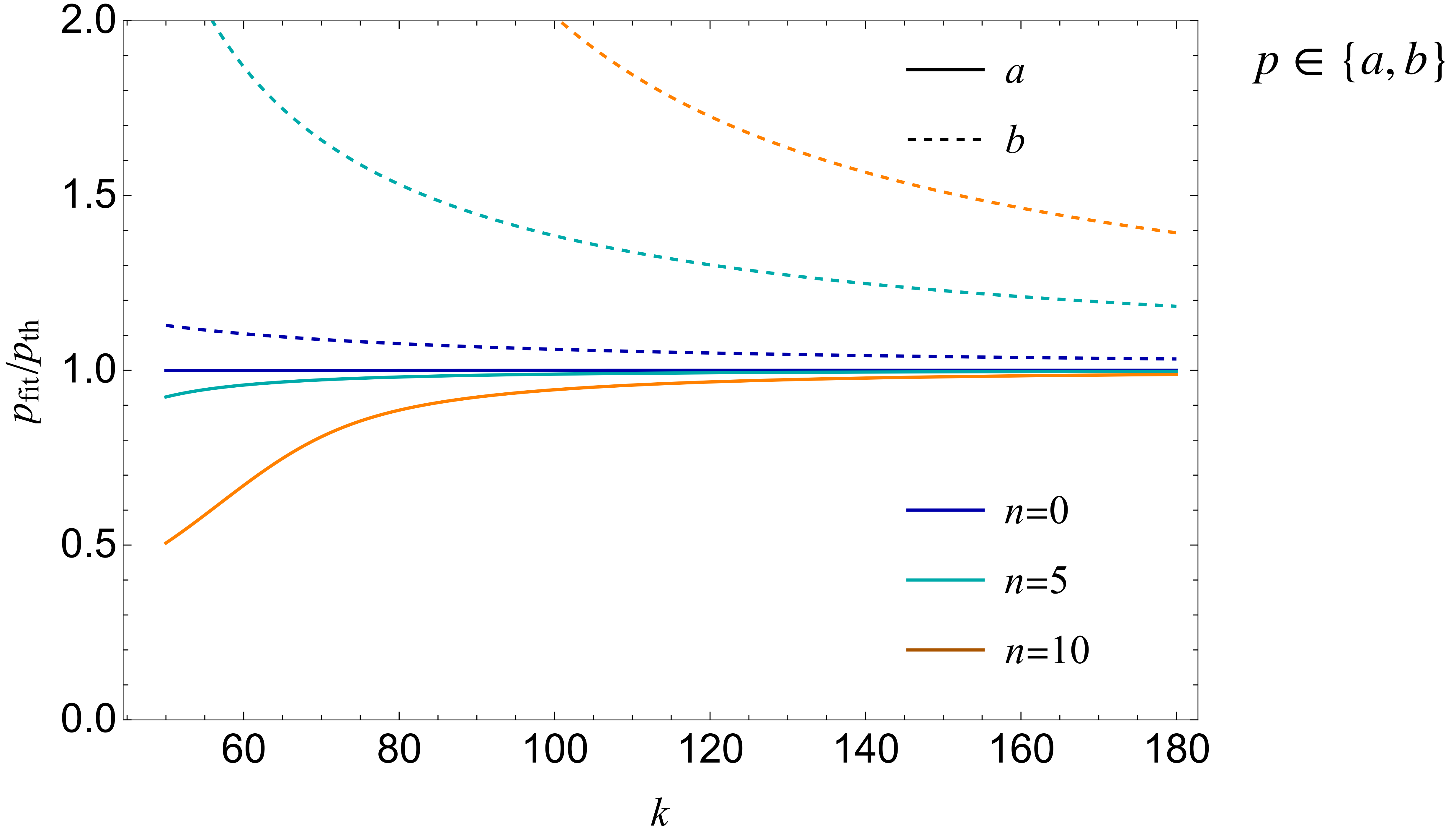
$$\Gamma_n = -\frac{1}{n!}\left(\frac{2m\omega A^2}{\hbar}\right)^n \sqrt{\frac{m\omega^2 A^2}{\pi\hbar}} e^{-\frac{B}{\hbar}}$$

$$E_k \sim -\frac{2^{3n}}{\pi^{3/2} n!}\left(\frac{15}{2}\right)^{k+n+\frac{1}{2}} \Gamma\left(k+n+\frac{1}{2}\right)$$



$$r_k^{(1)} = \frac{\epsilon_{n,k+1}}{\epsilon_{n,k}} \sim a_1(k+b_1)$$

Large order → discontinuity → decay rate



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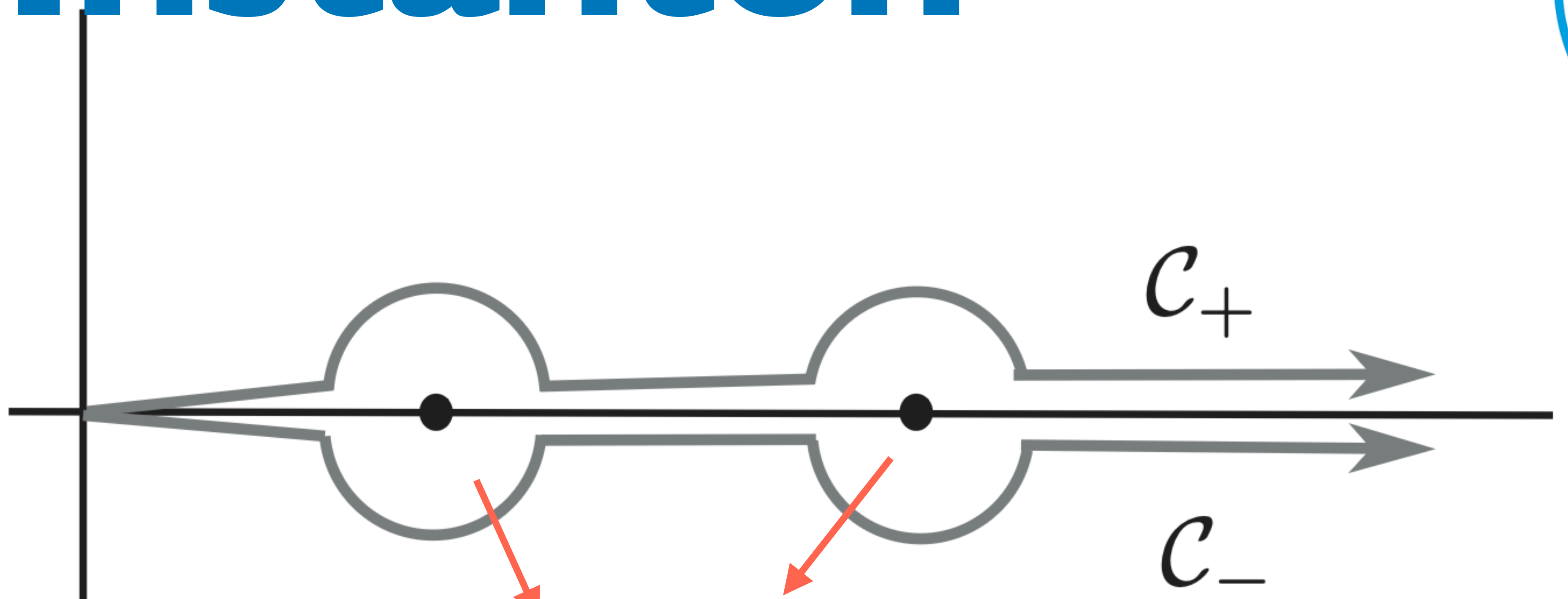
Numerics: Subleading Instanton

- ## Leading Fit: Tail effect

one-instanton correction itself is also an asymptotic series

what we just fit

$$\frac{i}{4\pi}f(\nu)\frac{\partial E}{\partial \nu} = \boxed{\frac{i}{\sqrt{\pi}}\frac{1}{g}e^{-\frac{2}{15g^2}}\left(1\right.}\left. - \frac{169}{16}g^2 - \frac{44507}{512}g^4 - \frac{86071851}{40960}g^6 + \dots\right)$$



$E_k \sim L_1 + L_2 + L_3 + \dots$ large order

M. Mariño. Advanced Topics in Quantum Mechanics

corresponds to the asymptotics in the large-order

$$L_1(k) = \boxed{-\frac{1}{\pi}\sqrt{\frac{15}{2\pi}}\left(\frac{15}{2}\right)^k\left[\Gamma\left(k+\frac{1}{2}\right)\right.}\left. - \frac{169}{120}\Gamma\left(k-\frac{1}{2}\right) - \frac{44507}{28800}\Gamma\left(k-\frac{3}{2}\right) - \frac{9563539}{1920000}\Gamma\left(k-\frac{5}{2}\right)\right]$$

behave as $1/k$ tail corrections in the ratio fit

$$\frac{L_1(k+1)}{L_1(k)} = \boxed{a_1 k + a_1 b_1} - \frac{a_1 d_1}{k} + \frac{a_1(d_1^2 + b_1 d_1 - d_1 - 2d_2)}{k^2} + \dots \equiv a_1(k+b_1) + \sum_{i=1} \frac{c_i}{k^i}$$

To reveal L_2,

we must subtract L_1 first

numerical fit gives

$$d_1 = -1.4083, \quad d_2 = -1.5454, \quad d_3 = -4.9810$$

match

WKB gives

$$d_1 = -\frac{169}{120}, d_2 = -\frac{44507}{28800}, d_3 = -\frac{9563539}{1920000}$$

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Numerics: Subleading Instanton

- Resum the leading

the Borel-Padé resummation $\mathscr{P}\mathscr{B}_N[E_n](z) \equiv \frac{P_l(z)}{Q_l(z)}$

$$\mathcal{S}_\pm\Phi(\alpha) = \frac{1}{\alpha}\int_0^{\infty e^{\pm i0}} e^{-t/\alpha}\mathscr{P}\mathscr{B}[\Phi](t)dt$$

$$D_{1,\pm}(k) = \frac{\Gamma\left(k+b_1\right)}{\pi^{3/2}}\left(a_1\right)^{k+b_1}\left[\left(k+b_1\right)\int_0^{\infty e^{\pm i0}}\frac{\mathscr{P}\mathscr{B}[\Phi](a_1 u)}{(1+u)^{k+b_1+1}}du\right]$$

a_1, b_1 from leading fit

$$D_1^{\text{med}}(k) = \frac{1}{2}[D_{1,+}(k) + D_{1,-}(k)]$$

the residue

$$R_1(k) = E_k - D_1^{\text{med}}(k)$$

We expect from WKB:

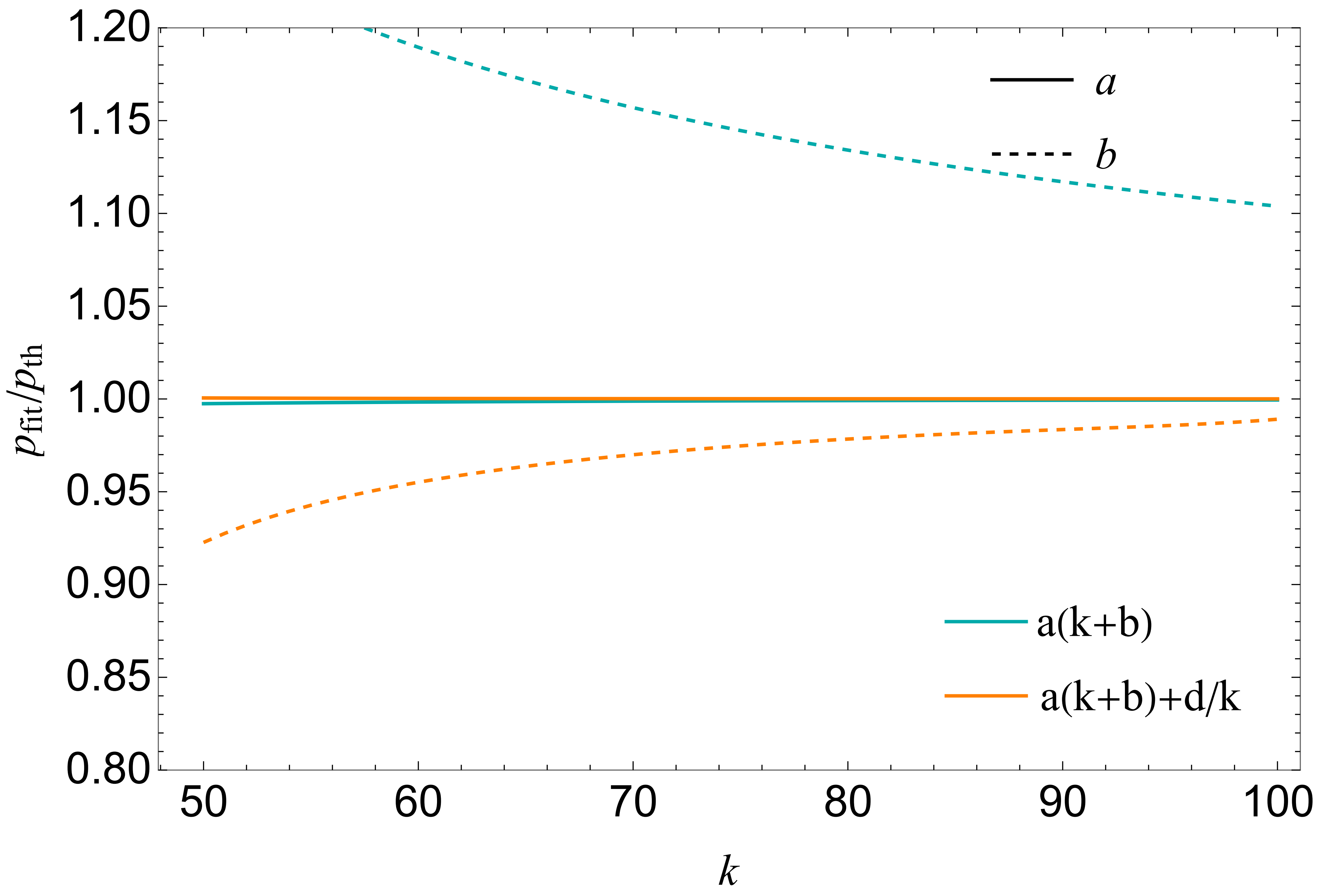
$$R_1(k) \sim \frac{15}{2\pi}\left(\frac{15}{4}\right)^k\Gamma(k+1)$$

- Fit the residue

the ratio $a_2 = 2/S_E$

2-instanton effect

$$r_k^{(2)} = \frac{R_1(k+1)}{R_1(k)} \sim a_2(k+b_2)$$



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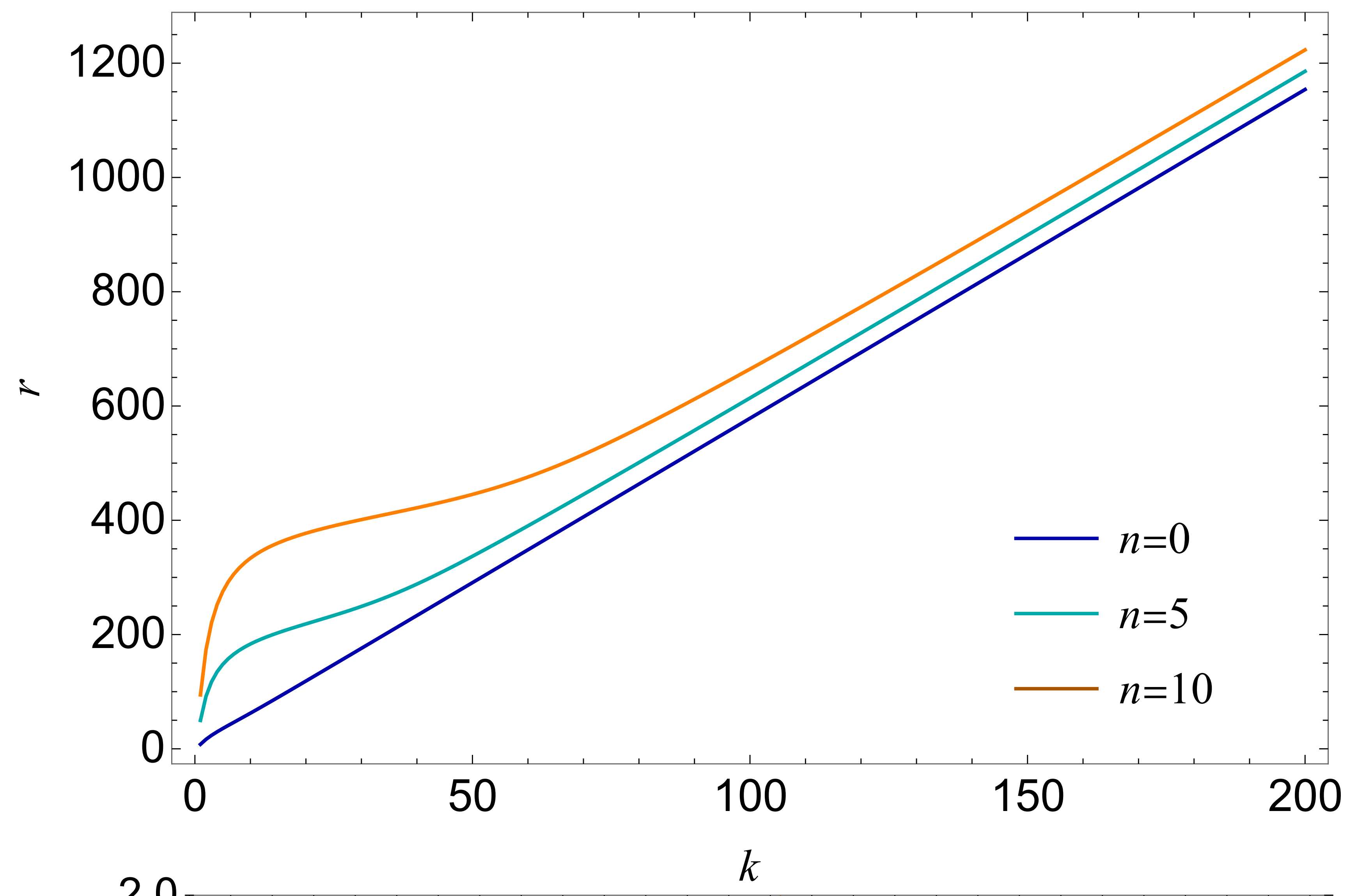
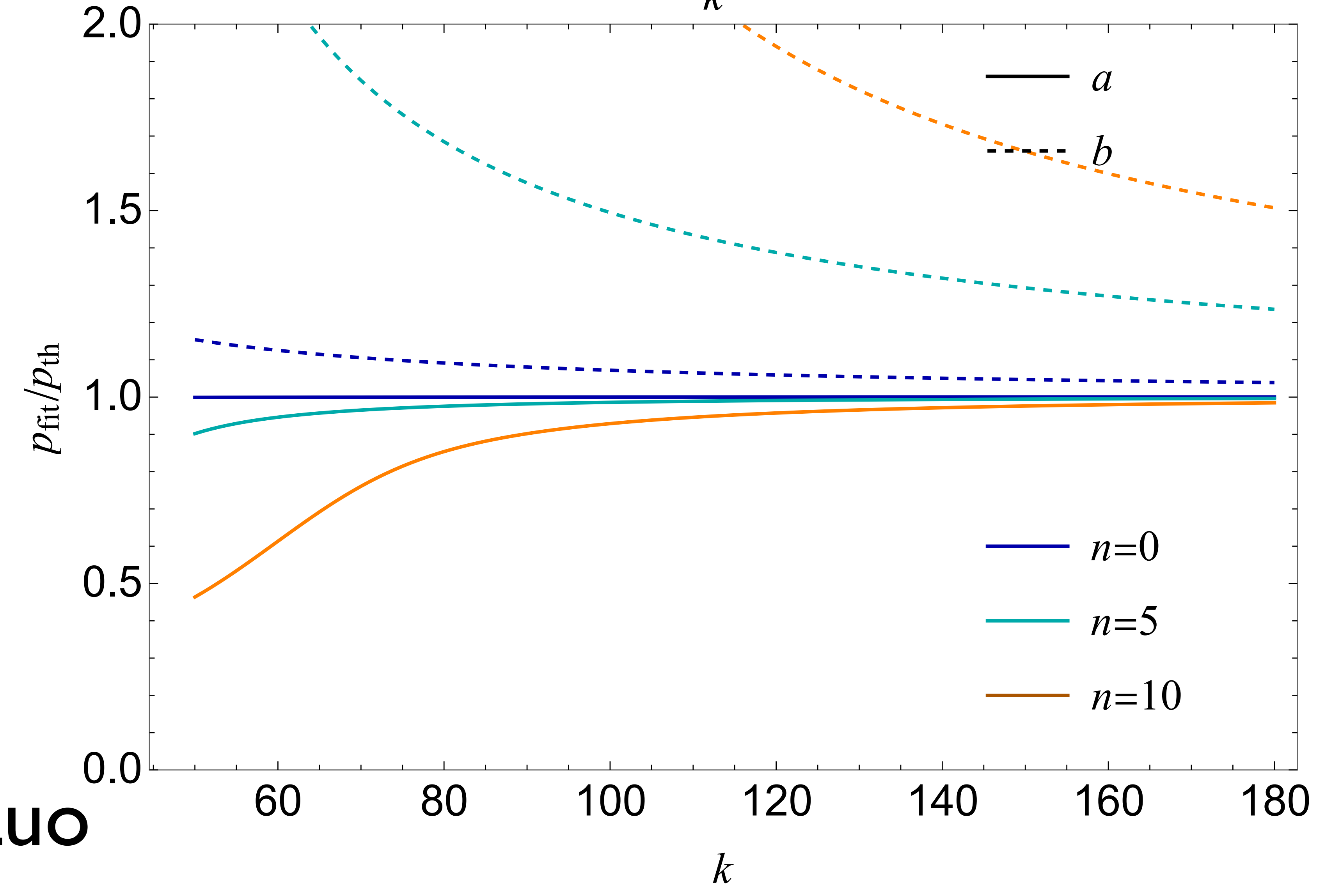
Numerics

- The tilted double well

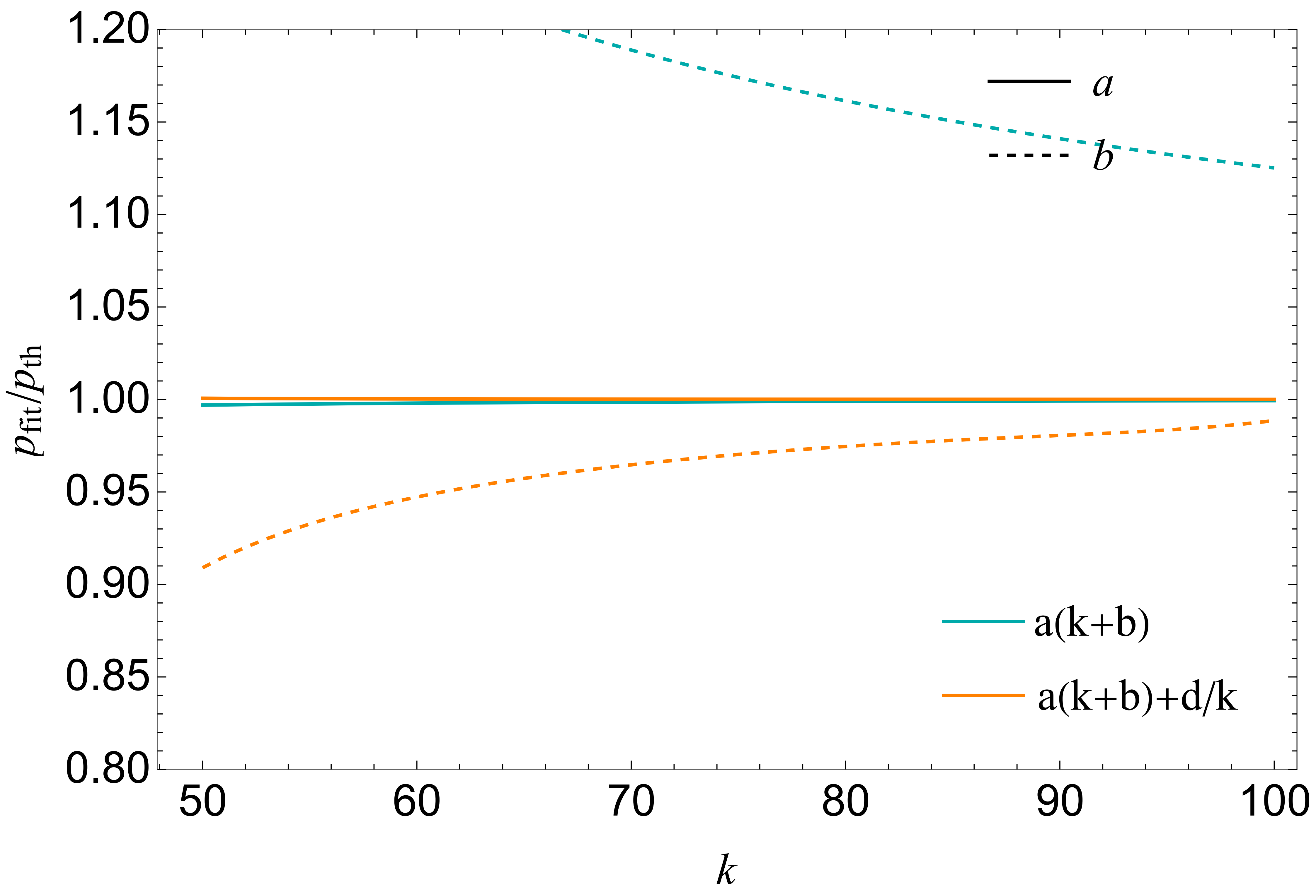




$$S_E = \frac{8}{3} + \sqrt{2}\ln 2 - 2\sqrt{2}\ln(2+\sqrt{2})$$

$$E_k \sim (\sqrt{2}-2)\pi^{-3/2}\sqrt{\frac{9+6\sqrt{2}}{8+\sqrt{2}\ln 8 - 6\sqrt{2}\ln(2+\sqrt{2})}}\Gamma\left(k+\frac{1}{2}\right)S_E^{-k}$$

$$R_1(k) \sim \frac{-2}{\pi S_E}(2S_E)^{-k}\Gamma\left(k+1\right)$$



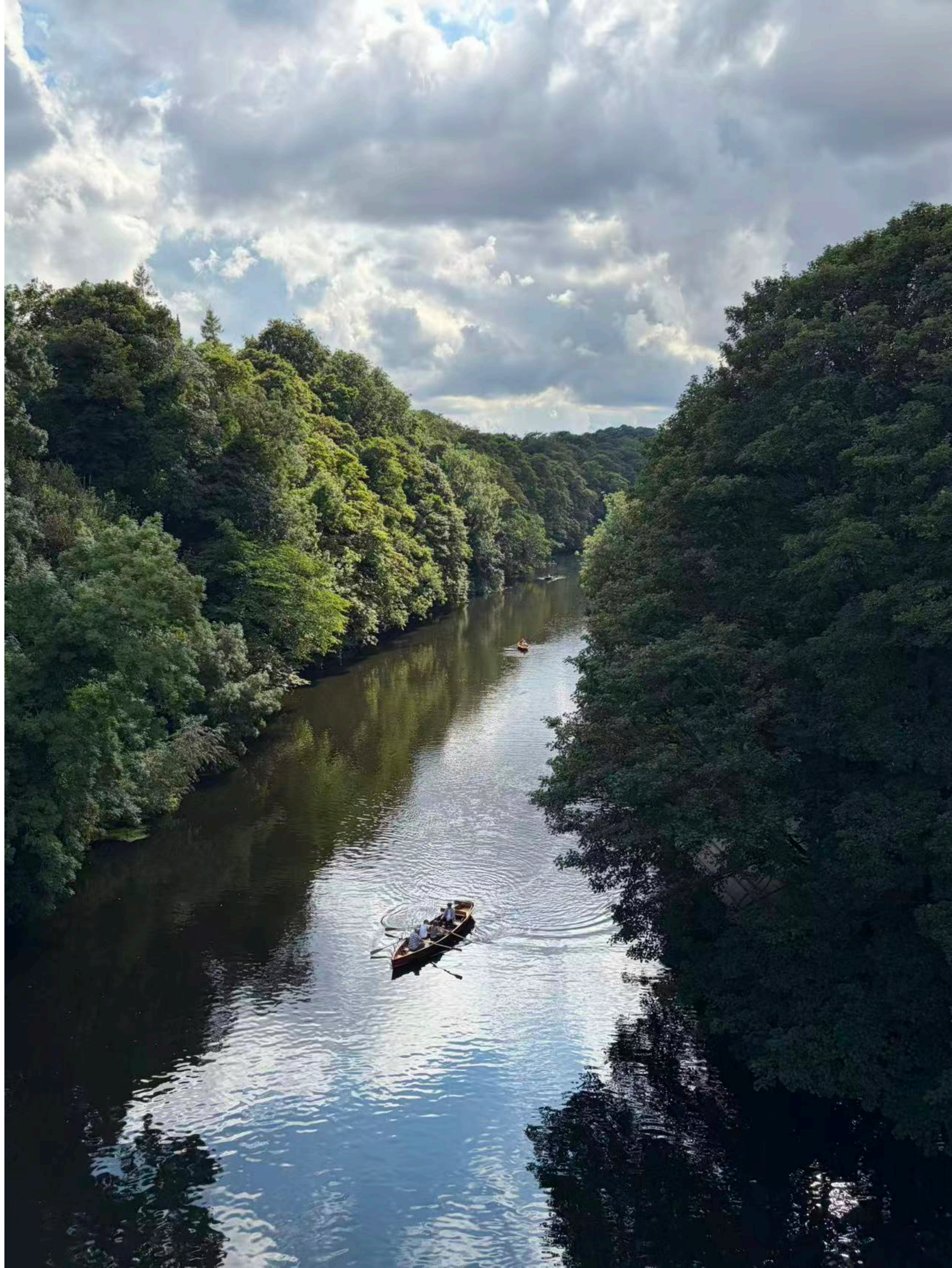
Summary

- The perturbative series is asymptotic, and its lateral Borel resummation can be ambiguous.

- The ambiguity of the perturbative sector is canceled by ambiguities in the nonperturbative sectors of the energy transseries.
  This consistency is the essence of resurgence and allows us to extract nonperturbative information from perturbative data.

- We extract the decay rate, up to the two-instanton sector, from the resummed perturbative series.

Thank you!



Yan Luo