NNNLO $t\bar{t}$ threshold cross section

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collaborations with ;
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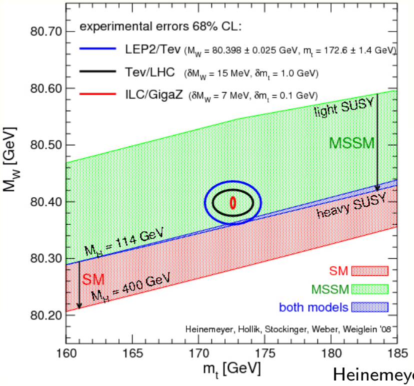
Motivation

- m_t measurement is a good test of physics understanding



Heinemeyer-Hollik-Stockinger-Weber-Weiglein (08)

- threshold scan of σ_{tt} at ILC provide an unique opportunity: $(\Delta m_t)_{\text{exp}} \leq 50\text{MeV} \rightarrow$ theory goal : $\delta\sigma/\sigma \leq 3\%$

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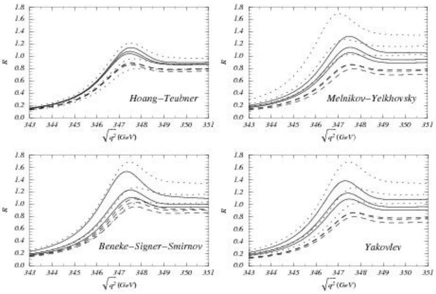

Figure 1: The total normalised photon-induced $t\bar{t}$ cross section

$\rightarrow$ NNLO result has large uncertainty $(\sim 20\%)$

(Beneke-Signer-Smirnov, Hoang-Teubner, Melnikov-Yelkovsky, Nagano-Ota-Sumino, Penin-Pivovarov, Yakovlev)

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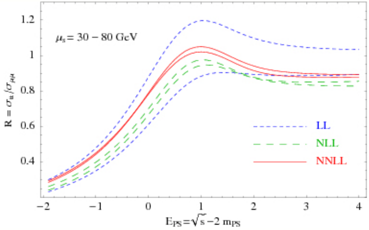

Pineda-Signer(2006)

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  $\rightarrow$ uncertainty reduced to $\pm 6\%$ (Hoang,et al.), and it was pointed out that the main effect is NNNLO logarithm (Pineda et al.).
- EW corrections also enter into the game for precision test
  - EW One-loop known (5% at most) (Guth-Kühn ('92), Hoang-Reißer(05))
  - Two-loop $\mathcal{O}(\alpha\alpha_s)$ due to H/Z and g (Eiras-Steinhauser (06))
  - Two-loop EW and QCD mixed corrections are not yet complete.

Recipe

Our goal is to complete NNNLO top cross section near threshold.

Threshold is characterized by $v \sim \alpha_s$, and size of EW correction is parametrized by $\alpha \sim \alpha_s^2$

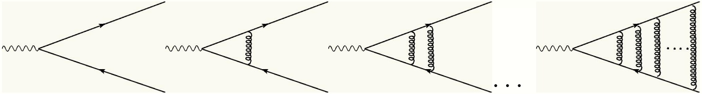
Recipe for NNNLO: $\mathcal{O}(v^3,\ v^2\alpha_s,\ \alpha_S^3,\ \alpha\alpha_s)$

- Bound-state correction to non-relativistic $t\bar{t}$ (long distance, but still perturbative) $\rightarrow$ this talk
- Three-loop Wilson coefficient: $\bar{t}\gamma^i t = C_V \bar{\psi}\sigma^i \chi$ (short distance $t\bar{t}$ production amplitude)
  $\mathcal{O}(\alpha_s^3 \times n_f)$ was done; Marquard-Piclum-Seidel-Steinhauser (06)
- $\mathcal{O}(\alpha\alpha_s)$ corrections to C_V $\rightarrow$ this talk
- An unstable particle effects (Γ_t) was calculated
  Hoang-Reißer (06) (formulations: Beneke-Chapovsky-Signer-Zanderighi (04), Beneke-Falgari-Schwinnn-Signer-Zanderighi(08))

Part II

EFT

Threshold cross section requires resummation of $\alpha_s/v \sim \mathcal{O}(1)$



- each gluon exchange yields Coulomb singularity, α_s/v

  $\mathrm{LO} \sim 1 + \frac{\alpha_s}{v} + (\frac{\alpha_s}{v})^2 + \cdots \sim \Sigma_n\,(\frac{\alpha_s}{v})^n$

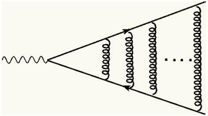
  $\mathrm{NLO} \sim \Sigma_n\,\{\alpha_s,\, v\} \times (\frac{\alpha_s}{v})^n$

  $\mathrm{NNLO} \sim \Sigma_n\,\{\alpha_s^2,\, \alpha_s v,\, v^2\} \times (\frac{\alpha_s}{v})^n$

  $\mathrm{NNNLO} \sim \Sigma_n\,\{\alpha_s^3,\, \alpha_s^2 v,\, \alpha_s v^2,\, v^3\} \times (\frac{\alpha_s}{v})^n$

Good-Old-Days:People knew the origin of α_s/v

That comes from Non-Relativistic (NR) régime $k_g \sim (0, \mathcal{O}(\vec{v}))$:



- Gluon propagator becomes QM potential:
  $$\frac{C_F\alpha_s}{k^2} \approx -\frac{C_F\alpha_s}{\vec{k}^{\,2}} \Rightarrow V(r) = -\frac{C_F\alpha_s}{r}$$
- $Q\bar{Q}$-propagator becomes $\widetilde{G}_0 = \frac{1}{\vec{p}^{\,2}/m - E}$
  $$\int \frac{dp^0}{2\pi i} \left(\frac{1}{p^0 + E/2 - \vec{p}^{\,2}/(2m) + i0}\right)\left(\frac{1}{p^0 - E/2 + \vec{p}^{\,2}/(2m) - i0}\right) = \widetilde{G}_0(\vec{p})$$

Emergence of Quantum Mechanics(QM) Fadin-Khoze ('87), Peskin-Strassler ('91)

$$G_0 - G_0\,V\,G_0 + G_0\,V\,G_0\,V\,G_0 + \cdots = \frac{1}{p^2/m + V(r) - E} \equiv G_C$$
$$\Rightarrow \sigma_{t\bar{t}} = \mathrm{Im}\,G_c(r=0, r=0) \sim v \times \left(1 + \frac{\alpha_s}{v} + (\frac{\alpha_s}{v})^2 + \cdots\right)$$

Why Effective Field Theory (EFT)?

Our EFT is Non-relativistic QCD (NRQCD) $\approx$ QM

- Systematic treatment of higher order is mandatory
- How do we maintain renormalizability?
  of non-renormalizable theory.
  $\Rightarrow$ understand QM ($\approx$ non-relativistic field theory)

Threshold-expansion: the connection between QCD and EFT.

Power Counting: systematic higher order corrections.

EFT: simplifies multi-scale problem, separation of physics by scales.

Threshold Expansion :(toy example 1)

$$I = \int_0^\infty dk \frac{k^\epsilon}{(k^2+M^2)(k^2+m^2)} = \left(\frac{\pi}{2mM}\right)\frac{1}{M+m} + O(\epsilon)$$

Consider a case $M \gg m$: $\left(\frac{\pi}{2mM^2}\right)\left[1 - \frac{m}{M} + \left(\frac{m}{M}\right)^2 - \left(\frac{m}{M}\right)^3 + \cdots\right]$

The result by Integration By Region

- Ultra-soft: $k < m$;
- Soft : $m \leq k < M$;
- Hard : $M < k$;

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The result by Integration By Region

- Ultra-soft: $k < m$; scale-less zero
  $$\int_{us} dk \frac{k^\epsilon}{(k^2+M^2)(k^2+m^2)} = \int dk k^\epsilon \left(1 + k + k^2 + \cdots\right) \equiv 0$$
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Summary of the method

- Taylor-expand your integrand by region
- Use Dim-Reg to regularize UV as well as IR
- Perform the integration from 0 to ∞

Threshold Expansion 2: $I = \int_0^\infty dk\, k^\epsilon \frac{\ln\left(\frac{\sqrt{k^2+M^2}}{k+M}\right)}{k^2+m^2}$

NIntegrate$= \{-4.04986, -3.37187, -2.98126, \cdots, -1.87487\}$ for $M = 1, \epsilon = 0, m = \{1, 2, 3, \cdots, 10\} \times 10^{-2}$

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$$= \frac{1}{\epsilon_{UV}}\left(\frac{1}{M} - \frac{m^2}{3M^2} + \frac{m^4}{5M^5}\right) + \frac{\ln(m)}{M} - \frac{m\pi}{2M^2} - \frac{m^2\ln(m)}{3M^3} + \frac{m^4\ln(m)}{5M^5} + \cdots$$

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  $= -\frac{1}{\epsilon_{IR}}\left(\frac{1}{M} - \frac{m^2}{3M^2} + \frac{m^4}{5M^5}\right) - \frac{2-\pi+2\ln(M)}{2M} + \frac{m^2\left(2+3\pi+6\ln(M)\right)}{18M^3} - \frac{m^4\left(2-5\pi+10\ln(M)\right)}{50M^5} + \cdots$

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- $I_{soft+hard} = \{-4.04986, -3.37187, -2.98126, \cdots, -1.87486\}$
  The agreement is 6(5) digits with $O(m^4)$-expansion

Integrating out the modes $m \gg mv \gg mv^2$

Now physics!

Our EFT contains modes of order $k^\mu \sim mv^2$ for gluon and $k \sim (mv^2, m\vec{v})$ for heavy quark. Let us integrating out the hard-loops: $k \sim m$

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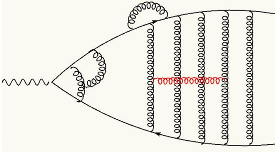

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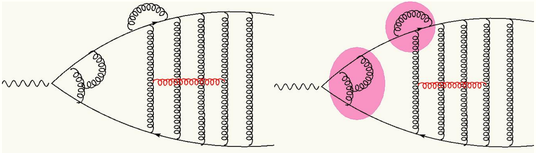

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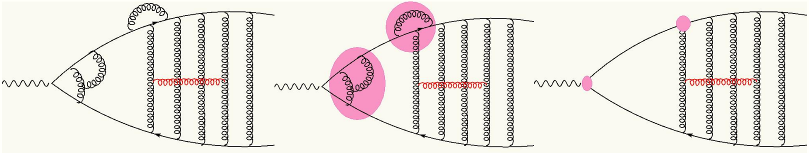

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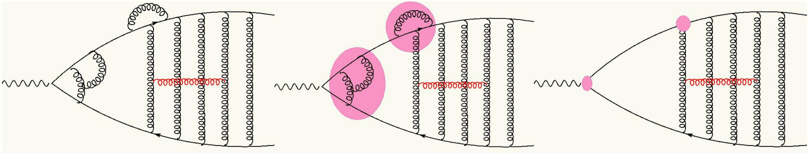

- Hard-loops near γ/Z-vertex $\Rightarrow \left[\bar{\Psi}\gamma^\mu\Psi\right] = C_v(\mu)\left[\psi^\dagger\sigma^\mu\chi\right]$
  $C_v(\mu) \sim \alpha_s^2\left[\frac{1}{\epsilon} + \ln\frac{\mu}{m}\right]$

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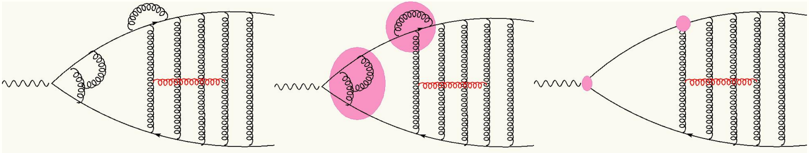

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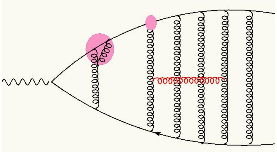

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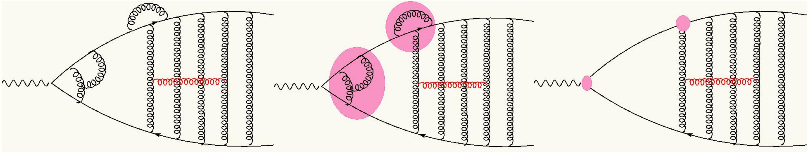

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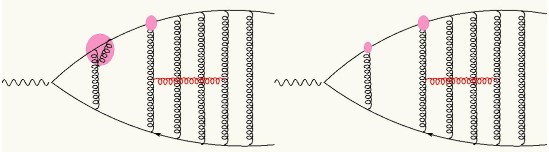
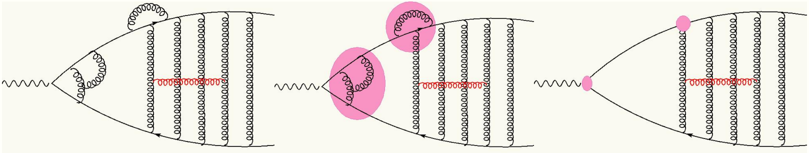
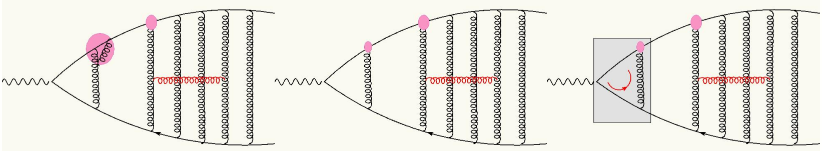


Integrating out the modes $m \gg mv \gg mv^2$

Hard-loops: $k \sim m$



- Hard-loops near γ/Z-vertex $\Rightarrow \left[\bar{\Psi}\gamma^\mu\Psi\right] = C_v(\mu)\left[\psi^\dagger\sigma^\mu\chi\right]$
  $C_v(\mu) \sim \alpha_s^2\left[\frac{1}{\epsilon} + \ln\frac{\mu}{m}\right]$



- Hard-loops distant from production vertex $\Rightarrow \delta\mathcal{L}_{\mathrm{NRQCD}}$
  $\langle\, \delta\mathcal{L}(x)\,[\psi^\dagger\sigma\chi](0)\,\rangle \sim -\alpha_s^2\left[\frac{1}{\epsilon} + \ln\frac{\mu}{p} + \cdots\right]$

Integrating out the modes $m \gg mv \gg mv^2$

- Hard-loops near γ/Z-vertex $\Rightarrow \left[\bar{\Psi}\gamma^\mu\Psi\right] = C_v(\mu)\left[\psi^\dagger\sigma^\mu\chi\right]$
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EFT Renormalization

Renormalization in EFT can be understood as cancelation of UV and/or IR $1/\epsilon$ among different modes.

Potential NRQCD: systematic threshold resummation

- Integrate out Hard (Caswell-Lepage('86))

$$\mathcal{L}_{\rm NRQCD} = \psi^\dagger \left(iD_0 + \frac{\vec{D}^2}{2m_t} \right) \psi + [\psi \to \chi] + \cdots$$

- Integrate out Soft/Potentia gluons

(Pineda-Soto('98); Luke-Manohar-Rothstein('99))

$$\mathcal{L}_{\rm pNRQCD} = \psi^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2m_t} \right) \psi + \int d\vec{r} \, [\psi^\dagger \chi] \, V_{pot}(r) \, [\chi^\dagger \psi]$$
$$+ig \, \psi^\dagger \left[A_{0,us} + \frac{\nabla \vec{A}_{us}}{m} \right] \psi - \frac{1}{4} F_{us}^2 + \cdots$$

- Remaining Mode is Ultra Soft gluon: $k \sim m(v^2, \vec{v}^2)$

Potentials are Wilson Coeff: $V_{pot}(r)\left[\psi^\dagger\chi\right](r)\left[\chi^\dagger\psi\right](0)$

$$V_{pot} = -\frac{C_F\alpha_s}{r} + \frac{C_2}{r^2} + C_3\delta(\mathbf{r}) + \cdots$$

- Higher order corr to the potential (case of Coulomb pot)

$$\widetilde{V}_C = -\frac{4\pi C_F\alpha_s(\mathbf{q})}{\mathbf{q^2}} \times \Big[1 + \frac{\alpha_s(\mathbf{q})}{4\pi}a_1 + \left(\frac{\alpha_s(\mathbf{q})}{4\pi}\right)^2 a_2 + \left(\frac{\alpha_s(\mathbf{q})}{4\pi}\right)^3 [a_3 + 8\pi^2 C_A^3 (\frac{1}{3\epsilon} + \ln\frac{\mu_{US}^2}{\mathbf{q}^2})]\Big]$$

ADM IR Divergence in QCD potential ('70)

Catastrophe of QCD QM, because it was supposed to be physical.

- a_2 Schröder('99); $a_{3,pade}$ Chishtie-Elias (01)
- ADM IR Div; Appelquist-Dine-Muzinich ('78)

  Brambilla-Pineda-Soto-Vairo('99), Kniehl-Penin-Smirnov-Steinhauser(02)

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$\rightarrow 1/\epsilon$ ADM Divergence is renormalized

ADM IR Divergence in QCD potential today

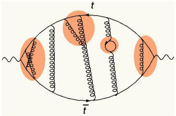
Its just one of scale dependent Wilson coefficients.

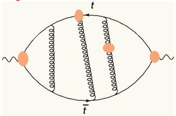
- a_2 Schröder('99); $a_{3,pade}$ Chishtie-Elias (01)
- ADM IR Div; Appelquist-Dine-Muzinich ('78)

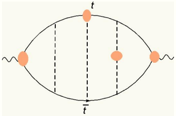
Brambilla-Pineda-Soto-Vairo('99), Kniehl-Penin-Smirnov-Steinhauser(02)

Threshold cross section $R_{t\bar{t}} \equiv \sigma_{t\bar{t}}/\sigma_{m=0} = \frac{4\pi e_t^2}{s}\,\mathrm{Im}\,\Pi(s)$

Principal quantity is $\Pi(q) = i\int d^4x e^{iqx}\langle 0|J^{\mu}(x)J_{\mu}(0)|0\rangle$







- Integrating hard mode $\longrightarrow$ NRQCD (Caswel-Lepage '86):
  $J^i(x) = \left[\bar{t}\gamma^i t\right] \quad \longrightarrow \quad c_v\left[\psi^{\dagger}\sigma^i\chi\right]$
- Integrating soft/potential modes $\longrightarrow$ PNRQCD
  (Pineda-Soto '97/Luke - Manohar-Rothstein'99):
  $\mathcal{L}_{\mathrm{PNRQCD}} = \psi^{\dagger}\left[i\partial_0 + \frac{\partial^2}{2m} + \frac{\partial^4}{8m^3} - g_s\mathbf{x}\mathbf{E}(t,\mathbf{0})\right]\psi + (\psi \leftrightarrow \chi)$
  $+\int d\mathbf{r}\left[\psi^{\dagger}\psi\right](x+\mathbf{r})\,V_{pot}(\mathbf{r})\left[\chi^{\dagger}\chi\right](x) + \cdots$

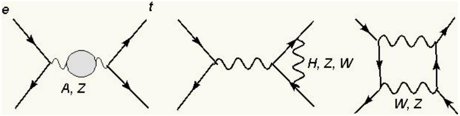
$$\Pi(q) = i\int d^4x e^{iEx}\, c_v^2\,\langle 0|\left[\psi^{\dagger}\sigma^i\chi\right](x)\ \left[\chi^{\dagger}\sigma_i\psi\right](0)|0\rangle$$

Part IV

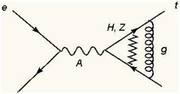
EW x QCD hard loop corrections

- One-loop EW is known since long

  Grzadkowski-Kühn-Krawczyk-Stuart('87), Guth-Kühn('92), Hoang-Reißer (05)

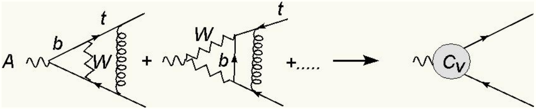


- 2-loop $\alpha\alpha_s$ corr (Z/H and g);Eiras-Steinhauser(06)



- This talk is on $\alpha\alpha_s$ (W and g) corrections to $At\bar{t}$-vertex

On threshold $t\bar{t}$ production, i.e. $s = 4m_t^2$

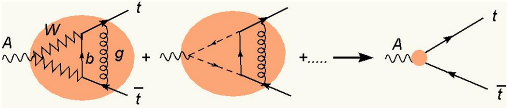


We match SM top pair production vertex to $c_v \psi^\dagger \sigma^i \chi$

- ψ, χ are NR 2-component spinor, e.g. $u(p) = \begin{pmatrix} \sqrt{\frac{E+m}{2m}}\psi \\ \frac{\sigma \mathbf{p}}{\sqrt{2m(E+m)}}\psi \end{pmatrix}$
- c_v is gauge dependent, but well-defined and one of building blocks for $R_{t\bar{t}}$
- Hard-loop is equivalent to $t\bar{t}$ on-threshold amplitude (+h.o.)
- Method:

  The result in expansion of small-(M_W^2/m_t^2) $(\sim 1/4)$
  Differential Eq.(Remid'97) + Mellin-Barnes Rep. are applied,
  $1/\epsilon$-poles canceled analytically, some finite parts numerically
  We used: QGRAF, q2e, exp, crusher ,MB, AMBRE, HypExp, Cuba

Result for 2-loop EW(W-boson) x QCD corrections



- Obtained 2-loop corrections give 0.1 % shift to R (negligible for LC study ;-)

2-loop EW(W) x QCD corrections (YK-Seidel-Steinhauser)

$$Q_t C^{(2)}_{EWQCD} = \frac{C_F \alpha_S}{4\pi} \frac{\alpha}{4\pi \sin^2\theta_W} \Bigg[\frac{-0.45 - 2.06i}{z} + (6.34 - 25.14i - 2.00 \ln z)$$

$$+ z(-6.27 - 6.10i + (2.16 + 4.10i)\ln z) + z^2(13.50 - 31.29i - (4.53 + 2.91i)\ln z) \Bigg]$$

$$= \Big[-0.64_{1/z} + 2.89_1 - 0.64_z - 0.30_{z^2} - 0.13_{z^3} + 0.014_{z^4}$$

$$+ i(-2.91_{1/z} - 7.73_1 - 0.83_z - 0.39_{z^2} - 0.081_{z^3} - 0.0013_{z^{13}}) \Big] \times 10^{-4}$$

Part V

Calculation of Π in PNRQCD

Loops, loops, $\cdots$ in PNRQCD: loops of ultrasoft $\bigoplus$ potential Modes

Ultra-soft correction is a part of NNNLO, which involved conceptually new for us. In the following we discuss renormalization of ultra-soft mode in detail.

Sub-leading potentials in PNRQDC are treated by means of Rayleigh-Schrödinger perturbation theory. They also need regularization and renormalization. $\rightarrow$ QM in DimReg and $\overline{\text{MS}}$ $(d = 4 - 2\epsilon)$.
(I skip this part in this talk because of lack of time)

Reminder

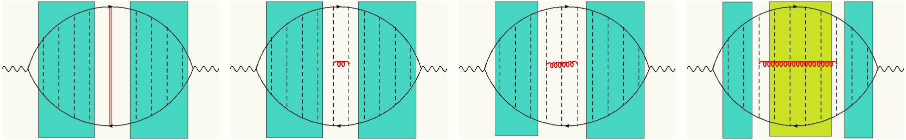
We use PNRQCD to compute $\Pi(q) \sim \int dx e^{iqx} \langle j^i(x) j_i(0) \rangle$.

PNRQCD Lagrangian (Pineda-Soto('98); Luke-Manohar-Rothstein('99))

$$\mathcal{L}_{\rm PNRQCD} = \psi^\dagger \left(i\partial_0 + \frac{\vec{\nabla}^2}{2m_t} \right) \psi + \int d\vec{r} \left[\psi^\dagger \psi \right] V_{pot}(r) \left[\chi^\dagger \chi \right]$$
$$+ig\, \psi^\dagger \left[A_{0,us} + \frac{\nabla \vec{A}_{us}}{m} \right] \psi - \frac{1}{4} F_{us}^2 + \cdots$$

- e.g. $\delta \tilde{V}_C = -\frac{4\pi\alpha_s}{\mathbf{q}^2} \left(\frac{\alpha_s(\mathbf{q})}{4\pi} \right)^3 \left[a_3 + 8\pi^2 C_A^3 \left(\frac{1}{3\epsilon} + \ln \frac{\mu_{US}^2}{\mathbf{q}^2} \right) \right]$
- Remaining Mode is Ultra Soft gluon: $k \sim m(v^2, \vec{v}^2)$

Ultrasoft renormalization I: Hamiltonian (static case)

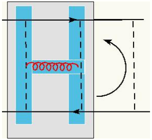


- quark-gluon vertex is $1/m$ suppressed; $\psi^\dagger(iD_0 + \frac{\vec{D}^2}{2m})\psi$
- n_g, number of potential exchange $\sim \Delta t$; $n_g > 1 \Leftrightarrow$ UV finite
- ADM $1/\epsilon$ of the Coulomb pot is a counter term for us corr

$$\delta V_C^{\mathrm{ADM}} = \frac{1}{\epsilon}\frac{C_A^3\alpha_s^4}{\mathbf{q}^2}$$

$$\mathrm{H}_0 = -\frac{\alpha_s^3 C_A^2}{\epsilon}\left(\frac{1}{mq} + \frac{2(p^2/m-E)}{\mathbf{q}^3} + \epsilon L_{\mathrm{Bethe}}\right) + O(\epsilon)$$

$$\mathrm{H}_1 = -\frac{1}{\epsilon}\frac{C_A^2(C_A-2C_F)\alpha_s^4}{\mathbf{q}^2} + L_{\mathrm{Bethe}} + O(\epsilon)$$



- UV cancelation happens tricky way:
  $\frac{(p^2/m-E)}{\mathbf{q}^3} \Rightarrow \frac{C_F\alpha_s}{\mathbf{q}^2}$ (Eq. of motion)

Ultrasoft renormalization II: vertex

$$\delta V_C^{\mathrm{ADM}} = \frac{1}{\epsilon}\frac{C_A^3\alpha_s^4}{\mathbf{q}^2}$$

$$\mathrm{H}_0 = -\frac{\alpha_s^3 C_A^2}{\epsilon}\left(\frac{1}{mq} + \frac{2(p^2/m-E)}{\mathbf{q}^3} + \epsilon L_{\mathrm{Bethe}}\right) + O(\epsilon)$$

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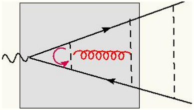

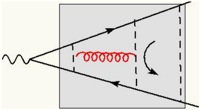

- Loop near photon vertices are more singular
  ⇔ Vertex Renormalization
- $1/\epsilon$ cancelation does not happen exactly anymore
  - due to vertex divergence, (3+1) dimensional Eq. of motion is invalid
  - diagrams with different loop order get mis-match of Eq. of motion

Ultrasoft renormalization II: vertex

$$\delta V_C^{\mathrm{ADM}} = \frac{1}{\epsilon}\frac{C_A^3\alpha_s^4}{\mathbf{q}^2}$$

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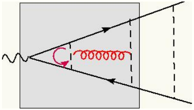
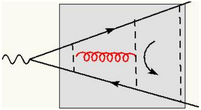
$$\mathrm{H}_1 = -\frac{1}{\epsilon}\frac{C_A^2(C_A - 2C_F)\alpha_s^4}{\mathbf{q}^2} + L_{\mathrm{Bethe}} + O(\epsilon)$$





- Loop near photon vertices are more singular
  $\Leftrightarrow$ Vertex Renormalization
- $1/\epsilon$ cancelation does not happen exactly anymore
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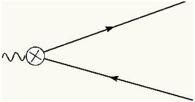
  - diagrams with different loop order get mis-match of Eq. of motion

Needs external current renormalization



Result: Ultrasoft corrections to Green Function

- All the logarithmic part were obtained analytically (Beneke-YK 08)
- Constant part numerically, a function of one dimensionless variable $\hat{E} \equiv (E + i\Gamma_t)/(m_t\alpha_s^2)$

$$
\begin{aligned}
\delta^{us} G(E) = \frac{2m^2\alpha_s^4}{9\pi^2} \Bigg\{ & \left[\frac{17\, i\hat{\Gamma}_t}{24} + \frac{527\, \hat{G}_C}{72} \right] \frac{1}{\epsilon^2} + \left[\frac{17\, i\hat{\Gamma}_t}{12} + \frac{221\, \hat{G}_C}{36} \right] \frac{L_\mu}{\epsilon} + \left[\left(\frac{19}{12} \ln 2 - \frac{91}{72} \right) i\hat{\Gamma}_t \right. \\
& \left. + \left(-\frac{119}{12} \ln 2 + \frac{2059}{108} \right) \hat{G}_C \right] \frac{1}{\epsilon} + \left[-\frac{34\, i\hat{\Gamma}_t}{3} - \frac{595\, \hat{G}_C}{9} \right] L_{\alpha_s}^2 + \left[-\frac{17\, i\hat{\Gamma}_t}{12} - \frac{833\, \hat{G}_C}{36} \right] L_\mu^2 \\
& + \left[\frac{34\, i\hat{\Gamma}_t}{3} + \frac{748\, \hat{G}_C}{9} \right] L_{\alpha_s} L_\mu + \left[\frac{2380\, \mathcal{P}^2}{27} + \left(\frac{272 \ln 2}{9} - \frac{23483}{162} + \frac{2380}{27\lambda} + \frac{272}{27\lambda^2} \right) \mathcal{P} \right. \\
& \left. + \left(\frac{27\lambda}{2} - \frac{16}{3\lambda} \right) \psi' + \frac{64}{27\lambda^3} + \frac{4\,(-1331 + 306 \ln 2)}{81\lambda} + \frac{4\,(-199 + 114 \ln 2)}{81\lambda^2} \right] L_{\alpha_s} \\
& + \left[-\frac{1496\, \mathcal{P}^2}{27} + \left(-\frac{34 \ln 2}{3} + \frac{5065}{72} - \frac{1496}{27\lambda} - \frac{136}{27\lambda^2} \right) \mathcal{P} + \left(\frac{8}{3\lambda} - \frac{81\lambda}{8} \right) \psi' \right. \\
& \left. - \frac{32}{27\lambda^3} + \frac{163 - 114 \ln 2}{27\lambda^2} + \frac{271 - 51 \ln 2}{9\lambda} \right] L_\mu + \delta^{us}(\hat{E}) \Bigg\},
\end{aligned}
$$

$$
L_\mu = \ln \frac{\mu}{m_t}, \quad L_{\alpha_s} = \ln \alpha_s, \quad \lambda = \frac{C_F}{2\sqrt{-\hat{E}}}, \quad \mathcal{P} = \ln\left(\frac{C_F}{\lambda} \right) + \gamma_E + \psi(1-\lambda),
$$

Result: Scale dependence of ultrasoft corrections to R

$E = \sqrt{s} - 2m_t$, $\Gamma_t = 1.4$ GeV, $m_t = 175$ GeV, $\alpha_s = 0.14$

Fig.1: Ultrasoft correction only.
Constant (solid line), log+cons (orange band) with $\mu = 32.6$ GeV (upper dashed), $\mu = 175$ GeV (lower dashed)

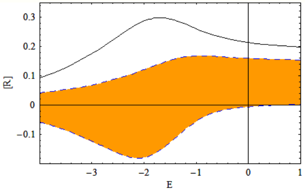
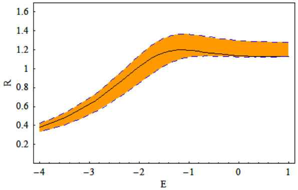
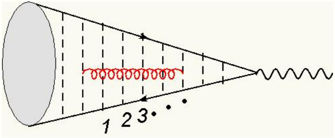


Fig.2: $R_{\rm LO} + [R]_{us}$; LO (solid line), LO+ ultrasoft (orange band) with $\mu = 32.6$ GeV (upper dashed) and $\mu = 175$ GeV (lower dashed)



- Ultrasoft contribution itself is not physical (scale dependent)
- Constant part is $+25\%$ in Fig.1 around peak position.

Result: Ultrasoft correction to Bound-state

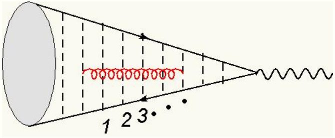


Quarkonium wave function at the origin (Beneke-YK-Penin 07)

$$\delta|\psi_1(0)|^2/|\psi_{C,1}|^2 = \frac{\alpha_s^3}{\pi}\Big[(-66.9_0 - 3.05_1)\,\mathrm{L_{us}} - 29.4\mathrm{L_p}$$
$$+\left(-55.2 + 21.8\ln\left(\frac{\mu}{m}\right)\right)\mathrm{L_p} - 2.73\ln^2\left(\frac{\mu}{m}\right) + 4.37\ln\left(\frac{\mu}{m}\right) + 357.7\Big],$$

non-log part is $\sim$ 7 % correction, $L_p = \ln\left(\frac{\mu}{m\alpha_s}\right)$, $L_{us} = \ln\left(\frac{\mu}{m\alpha_s^2}\right)$.

Result: Ultrasoft correction to Bound-state



Energy level(QCD Bethe-logarithm)

$$\delta E_1/E_C = \frac{\alpha_s^3}{\pi}\Big(\big(-42.81_0 - 1.784_1\big)\ln\Big(\frac{\mu}{m\alpha_s^2}\Big) + 88.86_0$$
$$+3.783_1 + 0.04426_{2-\infty} + \text{potential terms}\Big)$$

$L_E = -78.20_0 - 3.310_1 - 0.0280_{2-\infty} = -81.54$ agrees with Kniehl-Penin(2000).

Result: Potential insertion to the wave function

Quarkonium wave function at the origin: $\delta|\Psi(0)|^2 = |\Psi_C(0)|^2 \left(\frac{\alpha_s}{4\pi}\right)^3 f_3$

$$\frac{f_3^{nC}}{64\pi^2} = \left[\frac{7}{6}C_F^3 + \frac{37}{12}C_A C_F^2 + \frac{4}{3}C_A^2 C_F + \beta_0\left(\frac{4}{3}C_F^2 + 2C_A C_F\right)\right] L^2 + \left[C_F^3\left(-\frac{3}{2} + \frac{14}{3n} - \frac{7S_1}{3}\right)\right.$$
$$+ C_A C_F^2\left(\frac{226}{27} + \frac{8\ln 2}{3} + \frac{37}{3n} - \frac{5}{3n^2} - \frac{37S_1}{6} + C_A^2 C_F\left(\frac{145}{18} + \frac{4\ln 2}{3} + \frac{16}{3n} - \frac{8S_1}{3}\right)\right)$$
$$+ C_F^2\, T_F\left(\frac{2}{15} - \frac{59}{27}n_f\right) - \frac{109}{36}C_A C_F T_F\, n_f + \beta_0\left\{C_F^2\left(\frac{16}{3} + \frac{10}{3n} - \frac{75}{16n^2} - \frac{\pi^2 n}{9} - \frac{4S_1}{3} + \frac{2nS_2}{3}\right)\right.$$
$$\left.\left. + C_A C_F\left(\frac{15}{8} + \frac{5}{n} - \frac{\pi^2 n}{6} - 2S_1 + nS_2\right)\right\}\right] L + \left[\frac{1}{3}C_F^3 + \frac{1}{2}C_A C_F^2\right] L_m L + \left[\frac{1}{12}C_F^3 + \frac{1}{8}C_A C_F^2\right] L_m^2$$
$$+\left[C_F^3\left(\frac{1}{12} + \frac{2}{3n} - \frac{S_1}{3}\right) + C_A C_F^2\left(-\frac{5}{9} + \frac{1}{n} - \frac{S_1}{2}\right) + \frac{1}{15}C_F^2 T_F\right] L_m + \frac{c_{\psi,3}^{nC}}{64\pi^2},$$

(Beneke-YK-Schuller07)

Result: Potential insertion to the wave function

Quarkonium wave function at the origin: $\delta|\Psi(0)|^2 = |\Psi_C(0)|^2 \left(\frac{\alpha_s}{4\pi}\right)^3 f_3$

$$
\begin{aligned}
\frac{c^{nC}_{\psi,3}}{64\pi^2} = &\Big[-\frac{137}{36} - \frac{49\pi^2}{432} - \frac{25}{6n} + \frac{35}{12n^2} + S_1\left(\frac{3}{2} - \frac{14}{3n} + \frac{7S_1}{6}\right) - \frac{7S_2}{6}\Big] C_F^3 + \Big[\frac{7061}{486} - \frac{50\pi^2}{81} + \frac{1475}{108n} + \frac{\pi^2}{9n} \\
&- \frac{321}{32n^2} + \ln 2\left(\frac{353}{54} + \frac{16}{3n} - \frac{16\ln 2}{9}\right) - S_1\left(\frac{226}{27} + \frac{8\ln 2}{3} + \frac{37}{3n} + \frac{1}{n^2} - \frac{37S_1}{12}\right) - S_2\left(\frac{37}{12} + \frac{2}{3n}\right)\Big] C_A C_F^2 \\
&+ \Big[\frac{3407}{432} - \frac{5\pi^2}{18} + \frac{133}{9n} + \ln 2\left(\frac{187}{108} + \frac{8}{3n} - \frac{8\ln 2}{9}\right) - \frac{4S_2}{3} - S_1\left(\frac{145}{18} + \frac{4\ln 2}{3} + \frac{16}{3n} - \frac{4S_1}{3}\right)\Big] C_A^2 C_F \\
&+ \Big[\frac{1}{15} + \frac{4}{15n} - \frac{2S_1}{15}\Big] C_F^2 T_F + \Big[-\frac{361}{108} + \frac{49\ln 2}{108} - \frac{109}{18n} + \frac{109S_1}{36}\Big] C_A C_F T_F n_f + \Big[-\frac{3391}{486} + \frac{5\pi^2}{648} \\
&- \frac{2\ln 2}{27} - \frac{118}{27n} + \frac{125}{24n^2} + \frac{59S_1}{27}\Big] C_F^2 T_F n_f + \beta_0 \Big[\Big\{\frac{1027}{648} + \frac{19}{6n} + \frac{25}{24n^2} - \frac{35\pi^2}{108} - \frac{11\pi^2 n}{27} + \frac{5\pi^2}{16n} \\
&+ \frac{4nS_3}{3} - \frac{2nS_{2,1}}{3} - S_1\left(\frac{10}{9} + \frac{1}{3n} + \frac{45}{16n^2} - \frac{\pi^2 n}{9} + \frac{2nS_2}{3}\right) + S_2\left(1 + \frac{22n}{9} - \frac{15}{8n}\right)\Big\} C_F^2 \\
&+ \Big\{\frac{7}{24} - \frac{91\pi^2}{144} - \frac{1}{4n} - \frac{5\pi^2 n}{24} - S_1\left(\frac{3}{8} + \frac{1}{2n} - \frac{\pi^2 n}{6} + nS_2\right) + S_2\left(\frac{3}{2} + \frac{5n}{4}\right) - nS_{2,1} + 2nS_3\Big\} C_A C_F\Big] \\
&+ \left(\frac{v_m^{(1,\epsilon)}}{8} + \frac{v_q^{(1,\epsilon)}}{12} + \frac{v_p^{(1,\epsilon)}}{12}\right) C_F^2 - \frac{C_F}{6} b_2^{(\epsilon)}.
\end{aligned}
$$

(Beneke-YK-Schuller07)

Result: Potential insertion to the wave function

Quarkonium wave function at the origin: $\delta|\Psi(0)|^2 = |\Psi_C(0)|^2 \left(\frac{\alpha_s}{4\pi}\right)^3 f_3$

- "Toponium" wave function:

$$\frac{\delta_3|\psi_1(0)|^2_{nC}}{|\psi_1^{(0)}(0)|^2} = \frac{\alpha_s^3(\mu_B)}{\pi}\Big(-165.1 + 0.8\,\ln(\alpha_s C_F) + 0.9\,\ln^2(\alpha_s C_F)\Big) = -0.14$$

Result: Potential insertion to the wave function

Quarkonium wave function at the origin: $\delta|\Psi(0)|^2 = |\Psi_C(0)|^2 \left(\frac{\alpha_s}{4\pi}\right)^3 f_3$

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- Bottomonium wave function:

$$\frac{\delta_3|\psi_1(0)|^2_{nC}}{|\psi_1^{(0)}(0)|^2} = \frac{\alpha_s^3(\mu_B)}{\pi}\Big(-162.0 + 0.8\,\ln(\alpha_s C_F) + 0.9\,\ln^2(\alpha_s C_F)\Big) = -1.4$$

Result: Potential insertion to the wave function

Quarkonium wave function at the origin: $\delta|\Psi(0)|^2 = |\Psi_C(0)|^2 \left(\frac{\alpha_s}{4\pi}\right)^3 f_3$

- "Toponium" wave function:

$$\frac{\delta_3|\psi_1(0)|^2_{nC}}{|\psi_1^{(0)}(0)|^2} = \frac{\alpha_s^3(\mu_B)}{\pi}\Big(-165.1 + 0.8\,\ln(\alpha_s C_F) + 0.9\,\ln^2(\alpha_s C_F)\Big) = -0.14$$

- Bottomonium wave function:

$$\frac{\delta_3|\psi_1(0)|^2_{nC}}{|\psi_1^{(0)}(0)|^2} = \frac{\alpha_s^3(\mu_B)}{\pi}\Big(-162.0 + 0.8\,\ln(\alpha_s C_F) + 0.9\,\ln^2(\alpha_s C_F)\Big) = -1.4$$

- However the wave function is not physical, namely μ-dependence.

Mathematica Code: TTbarXSection.m

All the corrections were assembled, and TTbarXSection.m will calculate the threshold cross section for $e^+e^- \to t\bar{t}$

- From K.Schuller's Thesis

```
TTbarXSection[En,Mu,Constants→{m,w,as},
              Order→{ord,Potentials},MassDef→MD,
              PoleResum→PR,Production→Prod]
```

- *Mandatory Input*:
  - En: The energy at which the cross section is calculated.
  - Mu: The renormalization scale (typical value: 30 GeV).
  - m: The mass of the top quark in the chosen scheme.
  - w: The top quark width.
  - ord: The order of the calculation. It has to be a number from 0 to 3, where 0 stands for LO and 3 for NNNLO.
- *Optional Input*:
  - as: The value of the coupling constant at the scale M_Z (mass of the Z-boson); default value: value defined in the file "TTbarConstants.m"

Part VI

NNNLO QCD Phenomenology

Now we assemble all the corrections:

- Hard mode (Wilson coefficients)
- Soft/Potential modes (QM corrections)
- Ultra-soft mode (dynamical gluon propagation)

We are combining all the QCD effect to build up scale invariant quantity for phenomenology. Following parameters in (P)NRQCD will be used;

$$J^i = c_v \psi^\dagger \sigma^i \chi + d_v \frac{1}{6m_t^2} \psi^\dagger \sigma^i \mathbf{D}^2 \chi \qquad \mathcal{L}_{\text{QCD}} \Leftrightarrow \mathcal{L}_{\text{PNRQCD}}$$

$c^{(2)}$: Beneke-Signer-Smirnov('97), Czanecki-Melnikov('97)

$c^{(3)}_{n_f}$: Marquard-Piclum- Seidel-Steinhauser(06)

$d_v^{(1)}$: Luke-Savage('97)

a_2: Schröder('98)

$a_{3,\text{pade}}$: Chishtie-Elias (01) (New: $a_{3,\,n_f}$ Smirnov-Smirnov-Steinhauser (Sep.08))

$\delta\mathcal{L}^{(1)}$: Manohar('97), Beneke-Signer-Smirnov('99), Wüster-Schuller('03)

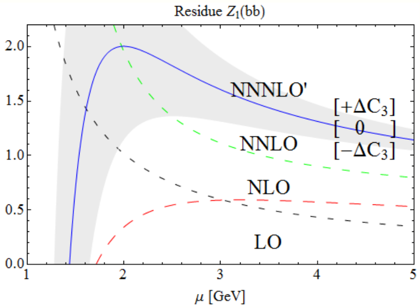
$\delta\mathcal{L}^{(2)}$: Kniehl-Penin- Smirnov-Steinhauser(02) ($\delta\mathcal{L}^{(2)} = \mathcal{O}(\epsilon)$ not known)

$\delta\mathcal{L}^{(us)}$: Brambilla-Pineda-Soto-Vairo('99), Kniehl-Penin- Smirnov-Steinhauser(02)

- We use $a_{3,pade}$ and set unknown $\mathcal{O}(\epsilon)$-potential terms zero (numerical difference is expected to be small)
- It will turn out that effect of $c^{(3)}$ is very important.
  We use $\pm c^{(3)}_{n_f}$ as an order estimate of unknown terms.
  Comment: At two loop non-n_f term of $c^{(2)}$ is larger than n_f-term in magnitude and its sign is opposite to n_f-term.

$\Upsilon(1S)$ residue: $\Pi(q) \overset{E\to E_n}{=} \frac{N_c}{2m_b^2}\frac{Z_n}{E_n - E - i\epsilon}$

Residue of $\Pi(q)$ is physical quantity, which can be extracted from leptonic decay width of Υ. Scale dependence of $Z_n(\mu)/Z_n(\mu_B)$ is plotted $(\mu_B = 2\text{GeV})$ (Beneke-YK-Penin-Schuller(07)).



$$Z_n = \left[c_v - \frac{E_n}{2m_b}\left(1 + \frac{d_v}{3}\right)\right]^2 |\Psi_n(0)|^2$$

$$\Gamma\left(\Upsilon(nS) \to l^+l^-\right) = \frac{4\pi N_c Q_t^2 \alpha^2}{3m_b^2} Z_n$$

$m_b = 5\text{GeV}$ used

$\Gamma(\Upsilon(1S))|_{\text{exp}} = 1.4 \pm 0.1\text{KeV}$

$\rightarrow$ aqua band

$\Upsilon(1S)$ residue: $\Pi(q) \overset{E\to E_n}{=} \frac{N_c}{2m_b^2}\frac{Z_n}{E_n-E-i\epsilon}$

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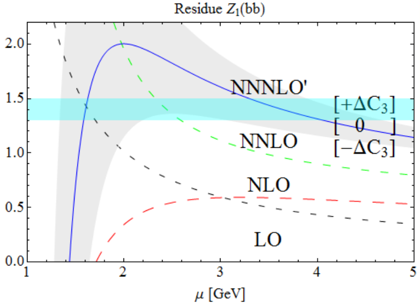


$$Z_n = \left[c_v - \frac{E_n}{2m_b}\left(1+\frac{d_v}{3}\right)\right]^2 |\Psi_n(0)|^2$$

$$\Gamma\left(\Upsilon(nS)\to l^+l^-\right) = \frac{4\pi N_c Q_t^2\alpha^2}{3m_b^2} Z_n$$

$m_b = 5\text{GeV}$ used

$\Gamma(\Upsilon(1S))|_{\text{exp}} = 1.4 \pm 0.1\text{KeV}$

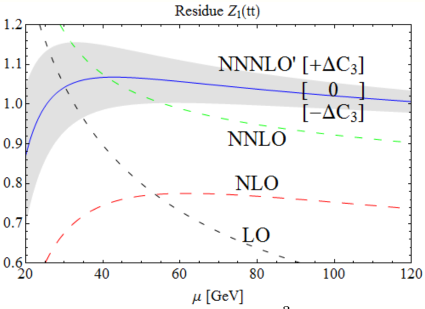
$\rightarrow$ aqua band

- Uncertainty (gray band) due to unknown $c_v^{(3)}$ is large.
- Scale-dependence can be reduced if $c_v^{(3)}$ is small (if non-n_f term has opposite sign to cancel n_f-term).

$t\bar{t}(1S)$ residue: $\Pi(q) \overset{E\to E_n}{=} \frac{N_c}{2m_t^2} \frac{Z_n}{E_n - E - i\Gamma_t}$

The first residue Z_1 of $\Pi(q)$ governs magnitude of R.

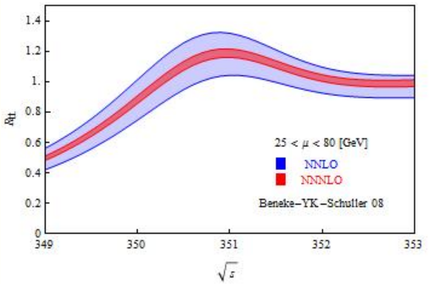
Below $Z_1(\mu)/Z_1(\mu_B)$ is plotted ($\mu_B = 32.62\text{GeV}$) (Beneke-YK-Penin-Schuller(07)).



- Third order correction is 10-15% shift to NNLO (depends on value of $c^{(3)}$)
- Scale-dependence is mild (much better if non-n_f term is negative).

$$Z_{t\bar{t}(1S)}|_{\mu_B} = \frac{(C_F m_t \alpha_s)^3}{8\pi}\left[1 - 2.13\alpha_s + 22.7\alpha_s^2 + \left(-38.8 + 5.8_{a3} + 37.6_{c3,n_f}\right)\alpha_s^3\right]$$

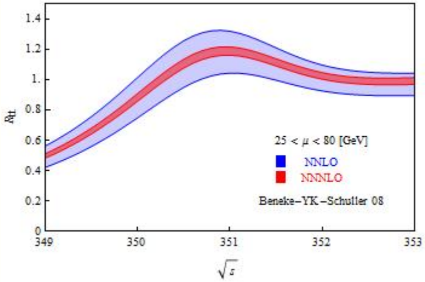
Top cross section



- $m_{t,\text{PS}}(20\text{GeV}) = 175\text{GeV},\ \Gamma_t = 1.4\text{GeV}, \alpha_s(M_z) = 0.1189.$

  (We included all the known corrections, e.g. $c^{(3)}_{n_f}$. All the logarithm term of c_v is known)
- The scale-dependence is significantly reduced to 10% at NNNLO'.
- Constant part of NNNLO is also important:

Top cross section



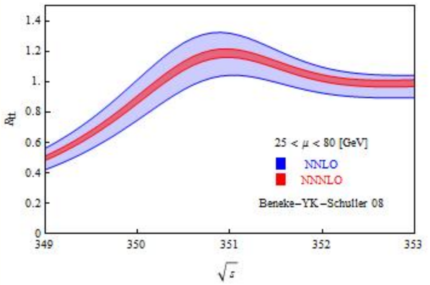
- $m_{t,\mathrm{PS}}(20\mathrm{GeV}) = 175\mathrm{GeV},\ \Gamma_t = 1.4\mathrm{GeV}, \alpha_s(M_z) = 0.1189.$
  (We included all the known corrections, e.g. $c_{n_f}^{(3)}$. All the logarithm term of c_v is known)
- The scale-dependence is significantly reduced to 10% at NNNLO'.
-

$$Z_1 = \frac{(m_t \alpha_s C_F)^3}{8\pi} \Big[1 + \alpha_s \Big(-2.13 + 3.66L \Big) + \alpha_s^2 \Big(8.38 - 7.26 \ln \alpha_s - 13.40L + 8.93L^2 \Big)$$

$$+\alpha_s^3 \Big(11.01 + [37.58]_{c_{3,n_f}} - 9.79 \ln \alpha_s - 16.35 \ln^2 \alpha_s$$

$$+(53.17 - 44.27 \ln \alpha_s) L - 48.18L^2 + 18.17L^3 \Big) \Big], \qquad (L = \ln(\mu/(m_t \alpha_s C_F)))$$

Top cross section



- $m_{t,\mathrm{PS}}(20\mathrm{GeV}) = 175\mathrm{GeV}$, $\Gamma_t = 1.4\mathrm{GeV}, \alpha_s(M_z) = 0.1189.$
  (We included all the known corrections, e.g. $c^{(3)}_{nf}$. All the logarithm term of c_v is known)
- The scale-dependence is significantly reduced to 10% at NNNLO'.
- Constant part of NNNLO is also important:

Part VII

Summary

QCD part:

- We have completed NNNLO QCD corrections to NR Green function.
- There are some missing coefficients for the threshold cross section. The most important piece is (probably) $c_v^{(3)}$.
- Remaining scale uncertainty is about 10% (if $c^{(3)}$ is NOT too large).

EW part:

- We have started 2-loop EW (W boson) and QCD mixed computations.
- (Hard-loop) Corrections to the $\gamma t\bar{t}$ vertex due to W and g shifts the threshold cross section about 0.1%. (Remaining most challenging part is $\mathcal{O}(\alpha\alpha_s)$ box diagrams)
- There are some to be done concerning to EW corrections $\Leftrightarrow$ Unstable top quark effect, etc.

Part VII

Summary

There are still large uncertainty. So...

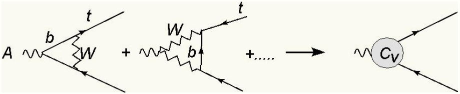
- Hope: Once C_3 is obtained, the uncertainty will reduce ?
- Some work: Combined analysis with renormalization group improvement?
- Tricky?: Find out good scale choice? (EFT scale setting is not so easy)
- What?.......?

Thanks for your attention.

Part VIII

Backup

1-loop lesson: Convergence of z-expansion ($z = \frac{M_W^2}{m_t^2}$)



$$Q_t\, c_v^{(1)}|_{\rm Wboson} = \frac{\alpha}{4\pi s_w^2}\Big[\frac{0.20}{z} + (0.48 + 0.79i + 0.25\ln z)$$
$$+z(-0.0024 - 1.37 - 0.44\ln z) + z^2(-0.07 + 1.39i + 0.44\ln z) + \mathcal{O}(z^3)\Big]$$

- Inclusion of first five terms ≈ exact result (red line) (Guth-Kuühn)

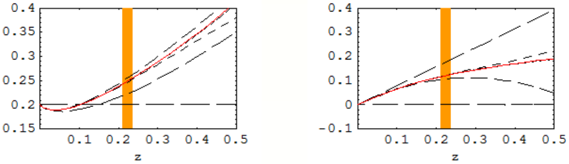
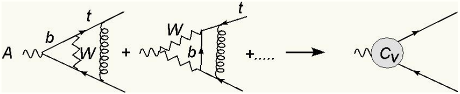


Fig.1: Real (left) and imaginary (right) parts of $C_v/[\frac{\alpha}{4\pi s_w^2}\ \frac{1}{z}]$. The shortest dashed-line is $\mathcal{O}(z^3)$ shitting on exact line.

- Leading (1/z)-term is due to ($\phi_W t\bar{t}$-) Yukawa coupling
- Imaginary part contains un-wanted $b\bar{b}, W^+W^-$ cuts (Hoang-Reisser)

2-loop EW x QCD in z-expansion ($z = \frac{M_W^2}{m_t^2}$)



$$Q_t\, C_V^{(2)}|_{\mathrm{W\times g}} = \frac{\alpha}{4\pi s_w^2}\frac{\alpha_s C_F}{4\pi}\Bigg[\frac{-0.45-2.06i}{z} + (6.34 - 25.14i - 2.00\ln z)$$
$$+z(-6.27-6.10i+(2.16+4.10i)\ln z)+z^2(13.50-31.29i-(4.53+2.91i)\ln z)+\mathcal{O}(z^3)\Bigg]$$

- Inclusion of successive terms shows a sign of convergence

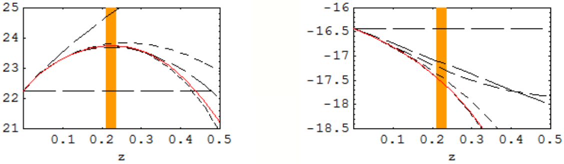


Fig.1: Real (left) and imaginary (right) parts of $C_V/[\frac{\alpha}{4\pi s_w^2}\frac{\alpha_s C_F}{4\pi}\frac{1}{z}]$.
Orange band is physical mass range for top quark $m_t = 165 - 175$ GeV.
$\mathcal{O}(z^4) \approx 2\%$ (red line)

- Leading 1/z due to Yukawa coupling

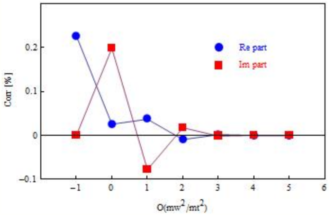
EW 1-loop

- W-loop corr to hard matching coefficient of $t\bar{t}$ X-Section:

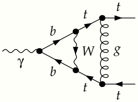
$$(C_V \times C_V) \sim 0.460 + \alpha(0.56 + 0.29\,i)_{\triangle} + \alpha(-2.61 - 3.46\,i)_{\square} + \mathrm{GBCont}$$
$$(C_A \times C_A) \sim 0.022 + \alpha(0.19 + 0.08\,i)_{\triangle} + \alpha(-0.56 - 0.75\,i)_{\square} + \mathrm{GBCont}$$

The imaginary part from (t → W-b)-cut should be extracted (Hoang-Reißer)

- In fig. corr to tt-vertex by W-loop is shown in %



EW x QCD Feynman Diagrams



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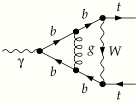

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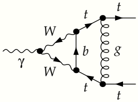

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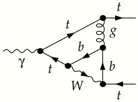

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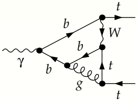

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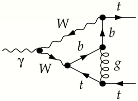

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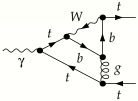

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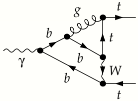

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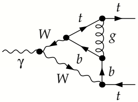

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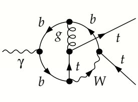

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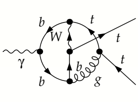

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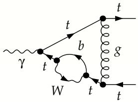

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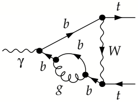

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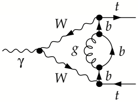

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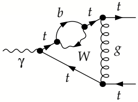

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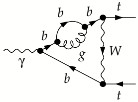

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