A duality relation between loops and trees

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multiparticle final states at next-to-leading order (NLO)

- LO @ LHC: 100% uncertainty typically
- NLO @ LHC necessary for 2→3 (many recent results) and 2→4 (not yet a cross section)
- **Radiative Return @ NLO:** at least 2→3

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$$\sigma^{NLO} = \int_{m+1} d\sigma^{R} + \int_{m} d\sigma^{V}$$

new feature wrt LO: combine m with m+1

real radiation

virtual contribution

kinematics: momentum conservation + observable dependent function

real radiation

$$\int_{m+1} d\sigma^R = \int d\Phi^{(m+1)}(\{p_i\}) \quad \underbrace{M^{(m+1)}(\{p_i\}) \quad F^{(m+1)}(\{p_i\})}$$

several well known/tested working methods (subtraction, dipole, slicing, mixed, ...)

split phase-space integrand in two parts:
$(...)_{\mathrm{fin}} + (...)_{\mathrm{div}}$

IR finite: computable numerically as LO

IR singular: analytically computable up to O(ε)

virtual contribution

$$\int_m d\sigma^V = \int d\Phi^{(m)}(\{p_i\}) \quad \underbrace{\int d^d q \quad M^{(m)}(\{p_i\})} \quad F^{(m)}(\{p_i\})$$

loop integral: in multiparton processes (m≥5) regarded as main practical bottleneck
many new developments in recent years (OPP, generalized Unitarity, ...)

general goal

I transform loop integral into customary phase space integral for real radiation (loop ⇔ phase-space duality)

$$\int_{loop} d^d q \;\; M^{(m)}(\{p_i\},q) = \int d\Phi(q) \;\; M^{(m+q)}(\{p_i\},q)$$

$$d\Phi(q) \leftarrow d^d q \;\; \delta_+(q^2)$$

II then treat $\int_{m+q}(\ldots)$ similarly to the real emission contribution $\int_{m+1}(\ldots)$

III Monte Carlo integration

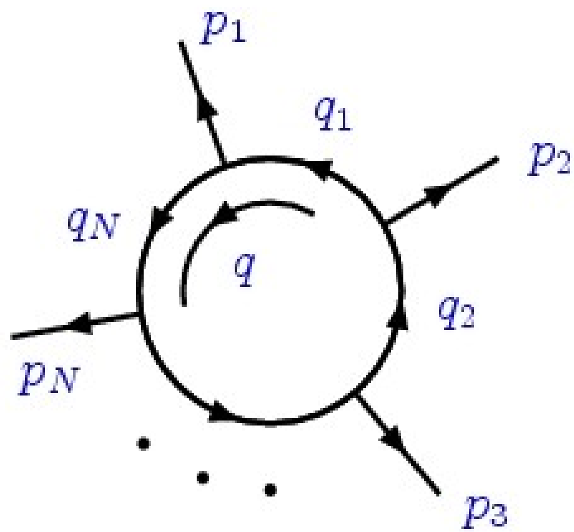
[see also Soper, Nagy, Kramer, Kleinschmidt, Moretti, Piccinini, Polosa]

Outline

- The Feynman Tree Theorem
- A duality theorem between one-loop integrals and single-cut phase-space integrals
- Relating the FTT and the duality relation
- Massive integrals, unstable particles, and gauge poles
- Duality at the amplitude level
- Final remarks

Notation

To simplify the presentation: massless internal lines only (more on massive particles later)



Scalar one-loop integral

$$L^{(N)}(p_1,\dots,p_N) = -i \int \frac{d^d q}{(2\pi)^d} \prod_{i=1}^{N} \frac{1}{q_i^2+i0}$$

q^μ is the loop momentum (anti-clockwise)

internal lines

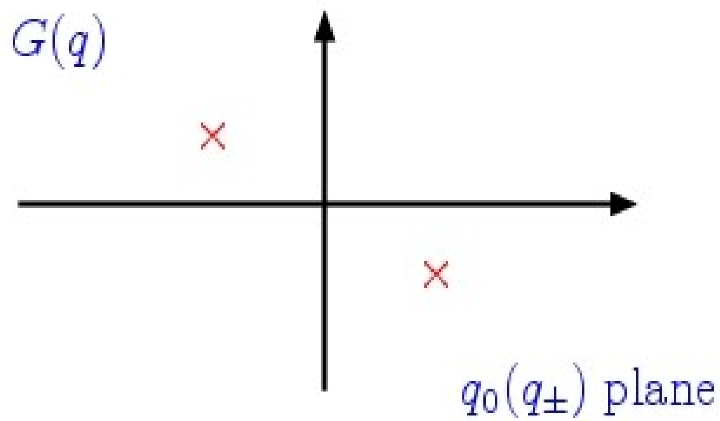
$$q_i = q + \sum_{k=1}^{i} p_k \ , \qquad \sum_{i=1}^{N} p_i = 0 \ , \qquad p_{N+i} = p_i \ .$$

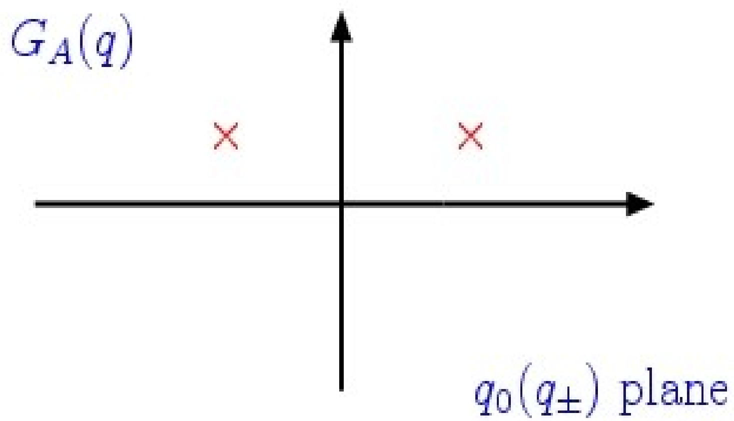
shorthand notation:

$$-i \int \frac{d^d q}{(2\pi)^d} \ \cdots \ \equiv \int_q \cdots \ , \qquad -i \int_{-\infty}^{+\infty} dq_0 \int \frac{d^{d-1}\boldsymbol{q}}{(2\pi)^{d-1}} \ \cdots \ \equiv \int dq_0 \int_{\boldsymbol{q}} \cdots$$

$$\boxed{\tilde{\delta}(q) \equiv 2\pi \ i \ \delta_+(q^2)}$$

Feynman and Advanced propagators





Feynman propagator

$$G(q) \equiv \frac{1}{q^2+i0}$$

+i0: positive frequencies are propagated forward in time, and negative frequencies backward

Advanced propagator

$$G_A(q) \equiv \frac{1}{q^2-i0\,q_0}$$

both poles displaced above the real axis (independently of the sign of the energy)

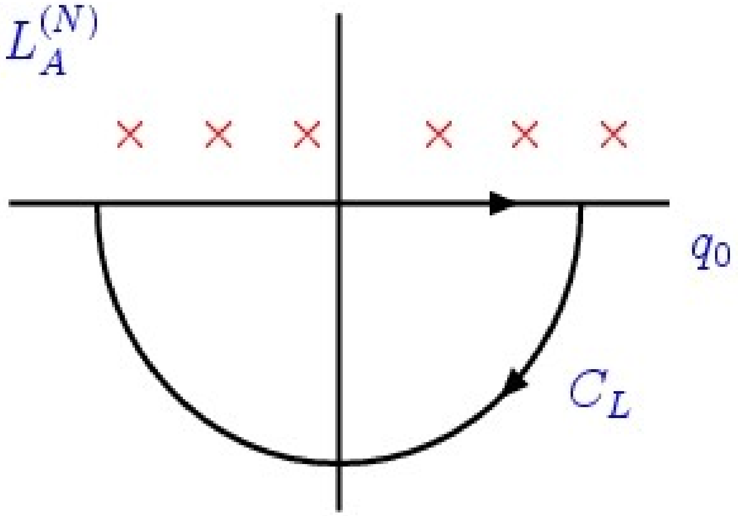
and are related by

$$\frac{1}{x\pm i0} = PV\left(\frac{1}{x}\right) \mp i\pi\delta(x)$$

$$G_A(q) = G(q) + \tilde{\delta}(q)$$

Feynman Tree Theorem

RP Feynman, Acta Phys. Polon. 24 (1963) 697



Advanced one-loop integral:
Feynman propagators replaced by advanced propagators

$$L_A^{(N)}(p_1, \ldots, p_N) = \int_q \prod_{i=1}^{N} G_A(q_i)$$

Cauchy residue theorem

$$L_A^{(N)}(p_1, \ldots, p_N) = 0$$

$$= \int_q \prod_{i=1}^{N} [G(q_i) + \tilde{\delta}(q_i)] = L^{(N)} + L^{(N)}_{1-cut} + L^{(N)}_{2-cut} + \ldots + L^{(N)}_{N-cut}$$

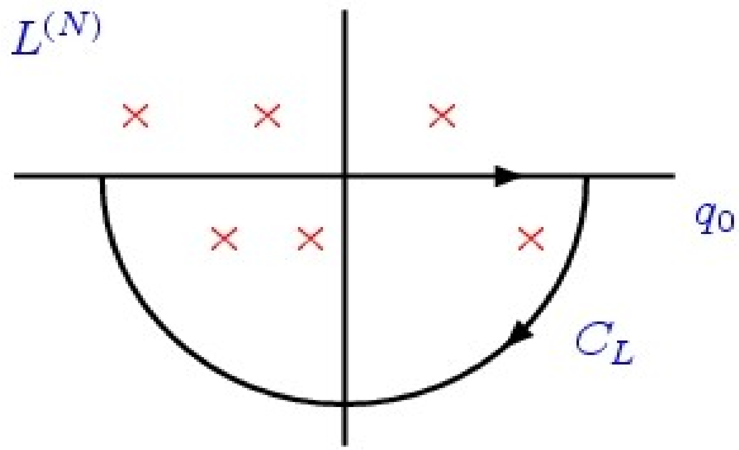
then

$$L^{(N)}(p_1, \ldots, p_N) = -\left[L^{(N)}_{1-cut}(p_1, \ldots, p_N) + \ldots + L^{(N)}_{N-cut}(p_1, \ldots, p_N) \right]$$

in four-dimensions, 4-cut at most (4 delta functions)

Duality Theorem

S.Catani et al. JHEP09(2008)065



Cauchy residue theorem

close the contour at ∞ on the lower half plane

⇨ select residues with **positive** definite energy

$$L^{(N)}(p_1, \ldots, p_N) = -2\pi i \int_q \sum \mathrm{Res}_{\mathrm{Im}\, q_0 < 0} \left[\prod_{i=1}^{N} G(q_i) \right]$$

$$\mathrm{Res}_{ith-pole} \left[\prod_{j=1}^{N} G(q_j) \right] = \left[\mathrm{Res}_{\; ith-pole} \; G(q_i) \right] \left[\prod_{j \neq i}^{N} G(q_j) \right]_{ith-pole}$$

$$\mathrm{Res}_{ith-pole} \frac{1}{q_i^2 + i0} = \int dq_0 \, \delta_+(q_i^2)$$

- **equivalent to cut that line and set it on-shell**
- **one-loop integral represented as a linear combination of *N* single-cut phase-space integrals**
- **shift $q_i \to q$ in each term ⇔ single phase-space integral over *N* terms**

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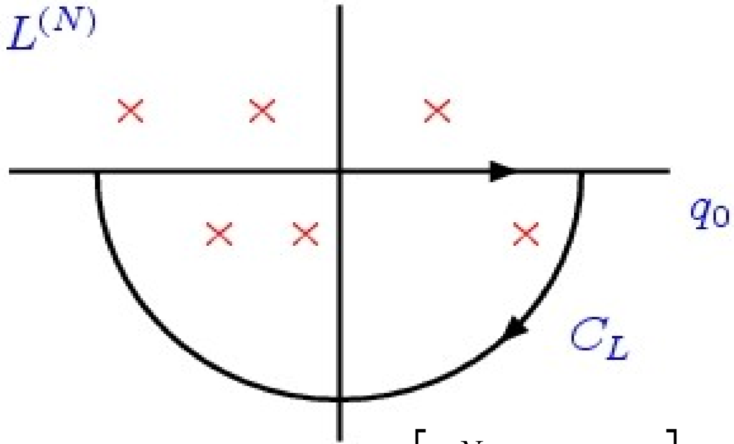

Cauchy residue theorem

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$$\left[\prod_{j \neq i}^{N} \frac{1}{q_j + i0} \right]_{ith-pole} = \prod_{j \neq i}^{N} \frac{1}{q_j^2 - i0\, \eta (q_j - q_i)}$$

- the customary $+i0$ prescription is modified
- ***Lorentz covariant dual prescription***
- ***η is a future-like vector: $\eta_0 > 0$, $\eta^2 \geq 0$***
- analytic continuation: $s_{ij} \to s_{ij} - i0$ ***wrong***

The calculation is elementary, but involves some **subtle points**

$$\left[\frac{1}{(q+k_j)^2+i0}\right]_{q^2=-i0\,,\,q_0=q_0^{(+)}} = \frac{1}{2q_0^{(+)}k_{j0}-2\boldsymbol{q}\cdot\boldsymbol{k_j}+k_j^2}$$

where $q_0^{(+)} = \sqrt{\boldsymbol{q}^2-i0}$

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where $q_0^{(+)} = \sqrt{\boldsymbol{q}^2-i0} \simeq |\boldsymbol{q}|-\frac{i0}{2|\boldsymbol{q}|}+O(i0^2)$

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$$= \frac{1}{2\,qk_j+k_j^2-i0\,k_{j0}/|\boldsymbol{q}|}$$

$$q_0^{(+)} = \sqrt{\boldsymbol{q}^2-i0} \simeq |\boldsymbol{q}|-\frac{i0}{2|\boldsymbol{q}|}$$

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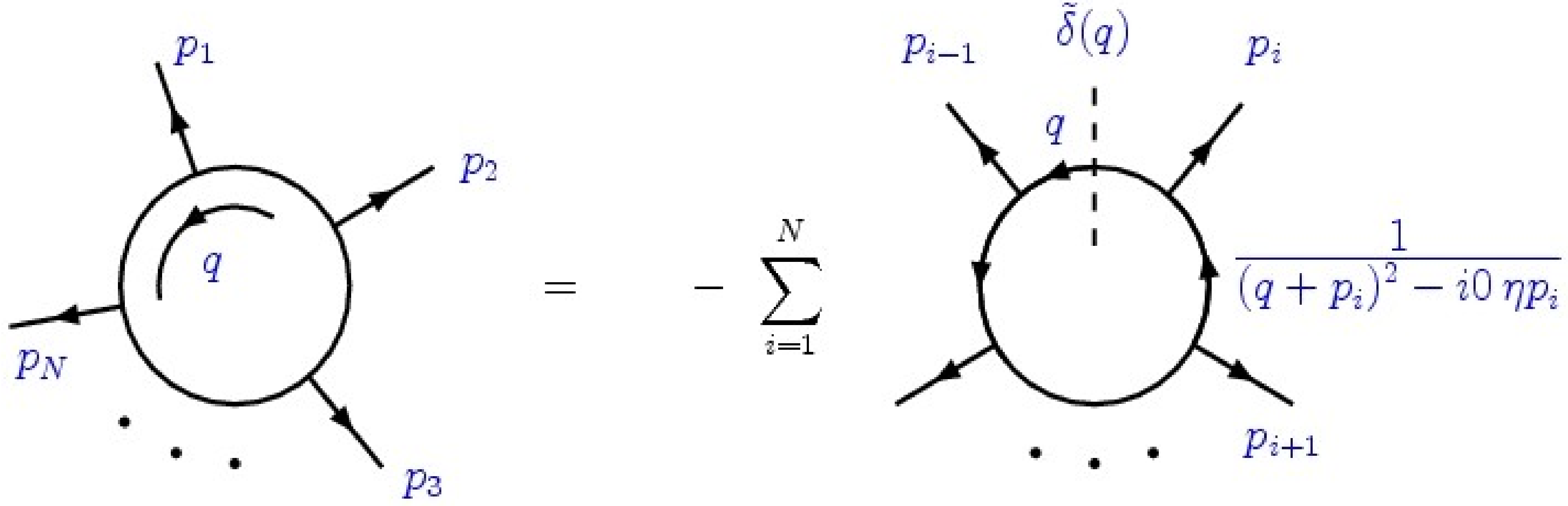
$$q_0^{(+)} = \sqrt{\boldsymbol{q}^2-i0} \simeq |\boldsymbol{q}|-\frac{i0}{2|\boldsymbol{q}|}$$

- only the sign matters:

  $- i0\, k_{j0}/|\boldsymbol{q}| \to - i0\, k_{j0} \to - i0\, \eta k_j$ where $\eta^\mu = (\eta_0, 0)$ with $\eta_0>0$

- *different choices of the future-like vector η are equivalent to different choices of the coordinate system*

Duality theorem



Duality relation between one-loop integrals and single-cut phase-space integrals

$$\begin{aligned} L^{(N)}(p_1,\dots,p_N) &= -\ \tilde{L}^{(N)}(p_1,\dots,p_N) \\ &= -\left[I^{(N-1)}(p_1,p_{12},\dots,p_{1,N-1}) + \textit{cyclic perms.}\right] \end{aligned}$$

where

$$I^{(n)}(k_1,\dots,k_n) = \int_q \tilde{\delta}(q) \prod_{j=1}^{n} \frac{1}{2\,q\,k_j + k_j^2 - i0\,\eta\,k_j}$$

N one-particle phase-space integrals ⇔ one phase-space integral over *N* tree quantities

duality-FTT relation

- multiple-cut contributions (m≥2) are absent in the duality relation, **only single-cut** contributions are involved

- Feynman propagators (+*i*0) replaced by **dual propagators (-*i*0 ηk_i)**

- individual cut integrals depend on the **future-like vector η^μ** (residues are not Lorentz-invariant) it has to be the same for all, then it cancels

- Single-cut contributions have extra **unphysical singularities** in the s_{ij} complex plane

  η^μ correlates the unphysical single-cut singularities
  FTT cancellation among multiple-cut contributions

Relating FTT with duality

an algebraic proof

Feynman and dual propagators are related by

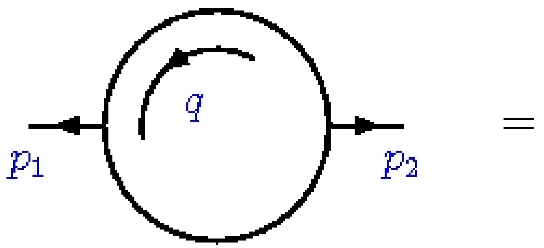
$$\tilde{\delta}(q)\frac{1}{2qk+k^2-i0\eta k} = \tilde{\delta}(q)\left[G(q+k)+\theta(\eta k)\tilde{\delta}(q+k)\right]$$

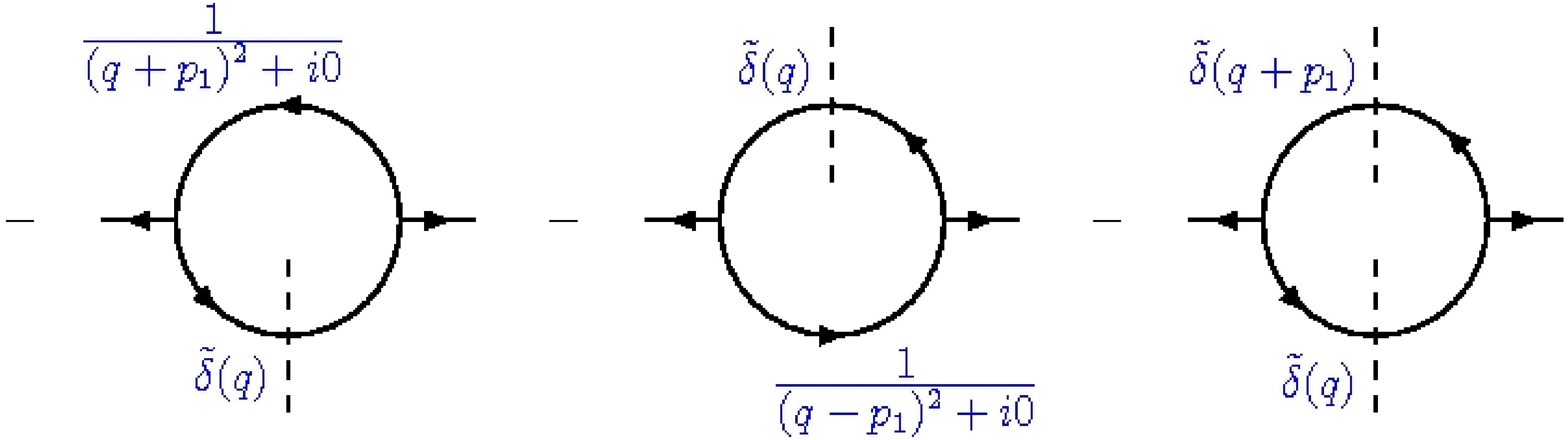
The key ingredient is to proof (e.g. by induction)
the following algebraic identity

$$\theta(\eta p_1)\theta(\eta p_{12}) \cdots \theta(\eta p_{1,N-1}) + \textit{cyclic perms.} = 1$$

which follows from **momentum conservation**

Two-point function from FTT





$$L^{(2)}_{1-cut}(p_1, p_2) \;=\; I^{(1)}_{1-cut}(p_1) \;+\; (p_1 \Leftrightarrow -p_1)$$

$$I^{(1)}_{1-cut}(k) \;=\; -\frac{c_\Gamma}{2}\frac{(-k^2-i0)^{-\epsilon}}{\epsilon(1-2\epsilon)}\left[1-i\;\frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)}\left[\theta(-k^2)+\theta(k^2)\,sign(k_0)\right]\right]$$

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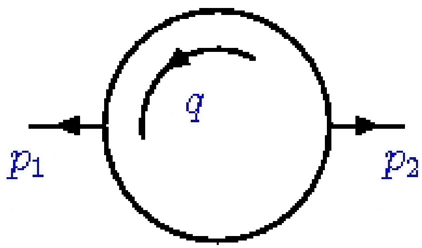

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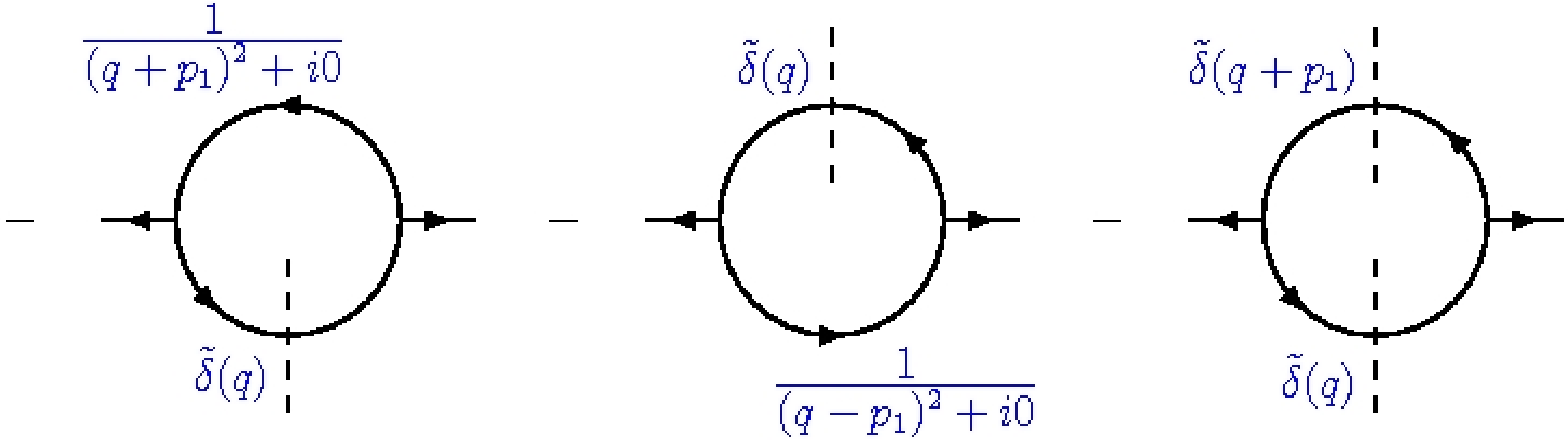

$$L^{(2)}_{1-cut}(p_1,p_2) = I^{(1)}_{1-cut}(p_1) + (p_1 \Leftrightarrow -p_1)$$

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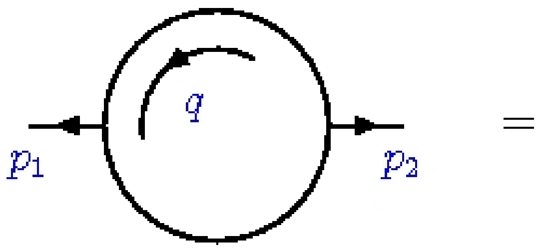

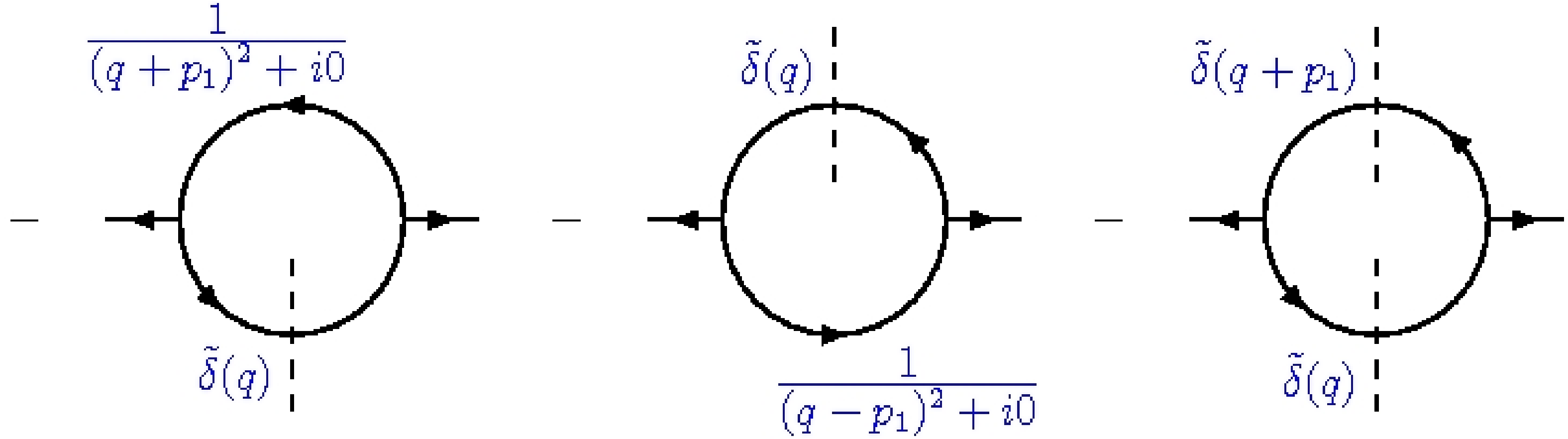

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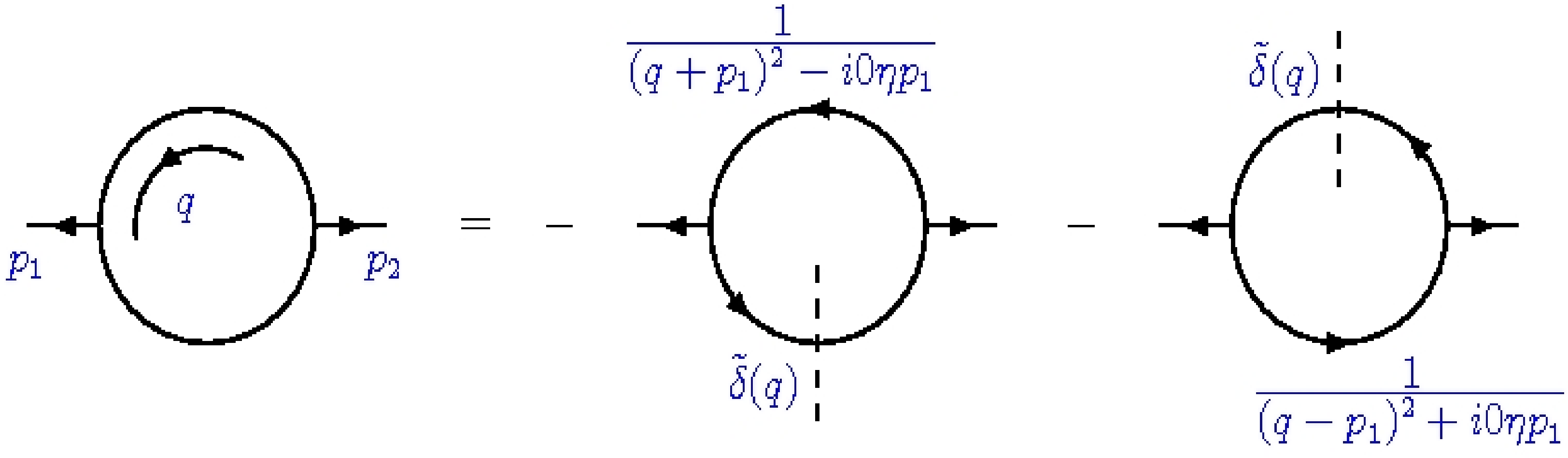
$$L^{(2)}_{2-cut}(p_1,p_2) = -i\,c_\Gamma\frac{(|p_1^2|)^{-\epsilon}}{\epsilon(1-2\epsilon)}\frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)}\theta(-p_1^2)$$

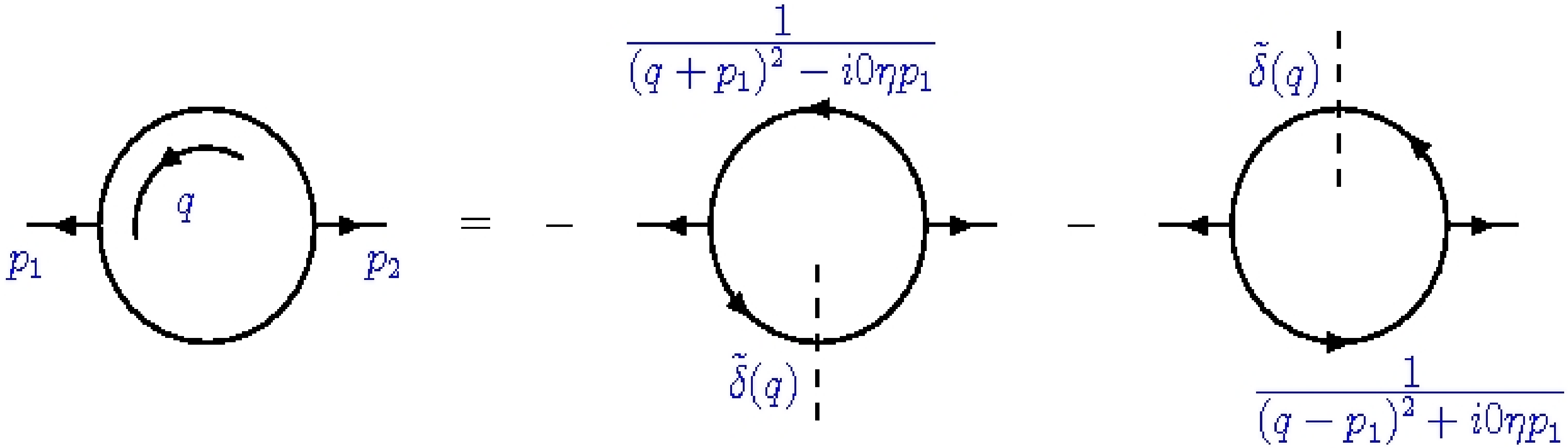
Two-point function from duality



$$\tilde{L}^{(2)}(p_1, p_2) = I^{(1)}(p_1) + (p_1 \Leftrightarrow -p_1)$$

$$I^{(1)}(k) = -\frac{c_\Gamma}{2} \frac{(-k^2 - i0)^{-\epsilon}}{\epsilon(1-2\epsilon)} \left[1 - i \frac{\sin(\pi\epsilon)}{\cos(\pi\epsilon)} sign(k^2 \eta k) \right]$$

Two-point function from duality



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Massive integrals, complex masses and unstable particles

Real masses: $\tilde{\delta}(q_i) \rightarrow \tilde{\delta}(q_i, M_i) = 2\pi\ i\ \delta_+(q_i^2 - M_i^2)$
do not affect the dual prescription

$$\frac{1}{q_j^2 - M_j^2 - i0\,\eta(q_j - q_i)}$$

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Unstable particles: Dyson summation produces finite-width effects that lead to the introduction of finite imaginary contributions in the propagators. In the complex mass scheme

$$G_C(q;s) = \frac{1}{q^2 - s}$$

$s = \mathrm{Re}\ s + i\ \mathrm{Im}\ s$ *with* $\mathrm{Im}\ s < 0 < \mathrm{Re}\ s$

produces poles in the q_0 plane that are located far from the real axis

Gauge poles

Quantization of gauge theories requires a gauge-fixing procedure

fictitious particles: Faddeev-Popov ghosts in unbroken non-Abelian gauge theories, or would-be Goldstone bosons in spontaneously broken gauge theories
⇨ ***cut exactly as physical particles***

gauge bosons: polarization tensor 't Hooft-Feynman gauge ✔

$$d^{\mu\nu} = -g^{\mu\nu} + (\xi - 1)\; l^{\mu\nu}(q)\; G_G(q)$$

$l^{\mu\nu}(q)$ propagates longitudinal polarizations, harmless polynomial dependence on q

- ***Spontaneously-broken gauge theories***

$$G_G(q) = \frac{1}{\xi(q^2 + i0) - M^2}$$

unitary gauge (ξ=0) ✔

- ***Un-broken gauge theories***

covariant gauge $\quad G_G(q) = \frac{1}{q^2 + i0}$ $\quad$ second order pole ✘

physical gauge $\quad G_G(q) = \frac{1}{(n\cdot q)^k} \; , \quad k = 1,2$ $\quad$ if $n\cdot\eta=0$ ✔

duality and FTT for scattering amplitudes

For **relativistic, local and unitary** quantum field theories

$$\mathcal{A}^{(1-loop)} = - \left[\mathcal{A}^{(1-loop)}_{1-cut} + \left(\mathcal{A}^{(1-loop)}_{2-cut} + \ldots \right) \right]$$

$\mathcal{A}^{(1\text{-}loop)}$ is a linear combination of one-loop integrals that differ from $L^{(N)}$ only by the inclusion of interaction vertices and, eventually, particle masses

- **particle masses:** real masses (unitary theories) do not affect the imaginary part of the poles

- **interaction vertices:** introduce numerator factors

  in local theories at worst polynomials in the loop momentum ⇨ no additional singularities (apart from gauge poles)

  unitary constrains the convergence of the q_0 integration at infinity

Loop-tree duality for amplitudes

dual representation of any one-loop quantity

$$\mathcal{A}^{(1-loop)} = - \tilde{\mathcal{A}}^{(1-loop)}$$

- starting from $\mathcal{A}^{(1\text{-loop})}$,consider all single cuts
- replace uncut propagators by dual propagators

$$\mathcal{A}^{(1-loop)} \simeq - \int_q \sum_P \tilde{\delta}(q; M_P) \sum_{dof(P)} \mathcal{A}_P^{(tree)}$$

Green's functions

Off-shell Green's function with *N* external legs

$$\mathcal{A}_N^{(1-loop)}(\ldots) = +\frac{1}{2}\int \frac{d^d q}{(2\pi)^{d-1}} \sum_P \delta_+(q^2 - M_P^2)\,\sigma(P)\,\underbrace{\tilde{\mathcal{A}}_{N+2}^{(tree)}(P(q)\leftarrow P(q),\ldots)}$$

- $\sigma(P)=\pm 1$ Bose-Fermi statistics factor
- $\sum_P$ sums over particles and antiparticles

Tree-level amplitude for the forward scattering process $P(q)\rightarrow P(q)$ in the field of N external legs

(**tadpole-like** cancel: summing over color,
in QED summing over particles and antiparticles)

e.g.

$$\mathcal{A}_{N+2}^{(tree)}(g(q)\leftarrow g(q),\ldots) = \sum_\lambda \sum_{a,b} \left(\epsilon_{a,\mu}^{(\lambda)}(q)\right)^* \left[\mathcal{A}_{N+2}(g(q), g(-q),\ldots)\right]_{ab}^{\mu\nu} \epsilon_{b,\nu}^{(\lambda)}(q)$$

Scattering amplitudes: only relevant point is the on-shell limit of the corresponding Green's function (wave function factors of the external lines)

Summary

- **Duality relation** between one-loop integrals and **single-cut** phase space integrals realized by a modification of the customary **+i0 prescription** of the Feynman propagators.

- The (Lorentz covariant) **dual prescription**, compensates for the absence of multiple-cut contributions that appear in the **FTT**.

- provides **IR** behaviour automatically

- Valid for any relativistic, **local and unitary field theory**, in arbitrary space-time dimensions.

- Suitable for analytical calculations of one-loop scattering amplitudes, and for numerical evaluation of cross-sections at **NLO** (on-going implementation)

- natural extension to **two-loops**

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