The decay $\pi^0 \to \gamma\gamma$ in ChPT

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*Work in progress

Outline:

- Overview picture
- Motivation/Introduction
- $N_f = 2$ vs. $N_f = 3$
- Two-loop calculation
- Conclusions

PDG 2007 about π^0

π^0 DECAY MODES

For decay limits to particles which are not established, see the appropriate Search sections (A^0 (axion) and Other Light Boson (X^0) Searches, etc.).

	Mode		Fraction (Γ_i/Γ)	Scale factor/ Confidence level
Γ_1	2γ		(98.798 ± 0.032) %	S=1.1
Γ_2	$e^+ e^- \gamma$		(1.198 ± 0.032) %	S=1.1
Γ_3	γ positronium		$(1.82 \pm 0.29) \times 10^{-9}$	
Γ_4	$e^+ e^+ e^- e^-$		$(3.14 \pm 0.30) \times 10^{-5}$	
Γ_5	$e^+ e^-$		$(6.46 \pm 0.33) \times 10^{-8}$	
Γ_6	4γ		$< 2 \times 10^{-8}$	CL=90%
Γ_7	$\nu\overline{\nu}$		[a] $< 2.7 \times 10^{-7}$	CL=90%
Γ_8	$\nu_e\overline{\nu}_e$		$< 1.7 \times 10^{-6}$	CL=90%
Γ_9	$\nu_\mu\overline{\nu}_\mu$		$< 1.6 \times 10^{-6}$	CL=90%
Γ_{10}	$\nu_\tau\overline{\nu}_\tau$		$< 2.1 \times 10^{-6}$	CL=90%
Γ_{11}	$\gamma\nu\overline{\nu}$		$< 6 \times 10^{-4}$	CL=90%
	Charge conjugation (*C*) or Lepton Family number (*LF*) violating modes			
Γ_{12}	3γ	*C*	$< 3.1 \times 10^{-8}$	CL=90%
Γ_{13}	$\mu^+ e^-$	*LF*	$< 3.8 \times 10^{-10}$	CL=90%
Γ_{14}	$\mu^- e^+$	*LF*	$< 3.4 \times 10^{-9}$	CL=90%
Γ_{15}	$\mu^+ e^- + \mu^- e^+$	*LF*	$< 1.72 \times 10^{-8}$	CL=90%

Conclusions made at Flavienet meeting 2007 (Orsay)

The 4 most important (in BR) decay modes of π^0 were discussed.
The ordering in this talk reflects the ordering of my study in time

- $\pi \to e^+e^-\gamma$
  published in [KK, Knecht, Novotný '06]
- $\pi \to e^+e^-e^+e^-$
  only preliminary first study
- $\pi \to e^+e^-$
  classification of diagrams finished, painstaking work is now in progress
- $\pi \to \gamma\gamma$
  important process, so far we have calculated double-logarithm correction which is negligible. This process, however, deserves the full two loop calculation...

$\pi^0 \to \gamma\gamma$ in a nut-shell

- π^0 lightest hadron $\Rightarrow$ primary decay mode $\pi^0 \to \gamma\gamma$
- in chiral limit exact due to QCD axial anomaly:

$$\Gamma(\pi^0 \to \gamma\gamma) = \frac{m_{\pi^0}^3}{64\pi}\left(\frac{\alpha N_C}{3\pi F_\pi}\right)^2 = 7.73\, eV$$

- NLO corrections are hidden in $F_\pi \to F_{\pi^0}$ and $O(p^6)$ LECs [Donoghue, Holstein, Lin '85] [Bijnens, Bramon, Cornet '88]
- in 3-flavour case we can study π^0, η, η' mixing, resulting to [Goity, Bernstein, Holstein '02]:

$$\Gamma^{\text{NLO}} = 8.1 \pm 0.08\,\text{eV}$$

  in 2-flavour case EM corrections [Ananth., Moussallam '02]:

$$\Gamma^{\text{NLO}} = 8.06 \pm 0.02 \pm 0.06\,\text{eV}$$

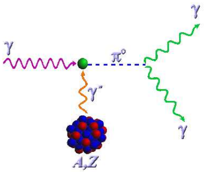
- Quite recently another study based on dispersion relations, QCD sum rules, using only the value $\Gamma(\eta \to \gamma\gamma)$ gives [Ioffe, Oganesian '07]:

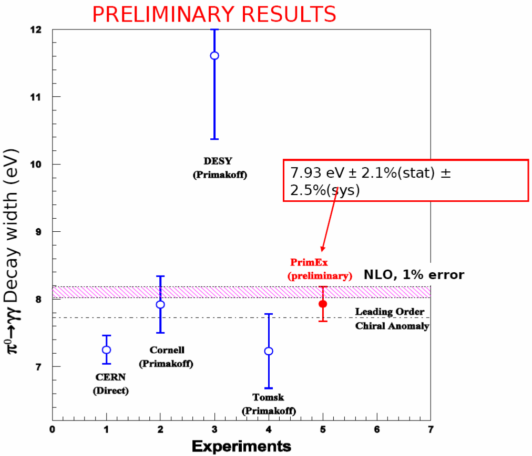
$$\Gamma^{\text{NLO}} = 7.93 \pm 0.11\,\text{eV}$$

$\pi^0 \to \gamma\gamma$: experiments

One can imagine three kinds of measurements:

- direct (time of flight)
- photon collisions
- Primakoff effect (i.e. photopion production in the Coulomb field of nucleus [Primakoff '51])





[PrimEx group April '07]

Calculation within 2-flavour ChPT

Is it possible to study anomaly within $SU(2) \times SU(2)$?

anomaly free theory: $\text{Tr}\big(t^i\{t^j, t^k\}\big) = 0$ → *generators*

$\Longrightarrow$ currents of $SU_R \times SU_L$ are anomaly free.

? Electromagnetic interaction in the case of two flavours:

$$Q = \begin{pmatrix} 2/3 & 0 \\ 0 & -1/3 \end{pmatrix} \notin SU(2)_R \times SU(2)_L$$

$\Longrightarrow$ we need $SU(2)_R \times SU(2)_L \times U(1)_V$

Construction in [R. Kaiser '01]

$$\mathcal{L}_{\text{WZW}} = -\frac{N_C}{32\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\big\{\langle U^\dagger \hat{r}_\mu U \hat{l}_\nu - \hat{r}_\mu \hat{l}_\nu + i\Sigma_\mu(U^\dagger \hat{r}_\nu U + \hat{l}_\nu)\rangle\langle v_{\rho\sigma}\rangle + \tfrac{2}{3}\langle\Sigma_\mu\Sigma_\nu\Sigma_\rho\rangle\langle v_\sigma\rangle\big\}$$

with $\Sigma_\mu = U^\dagger\partial_\mu U$, $\hat{r}_\mu = \hat{v}_\mu + \hat{a}_\mu$, $\hat{l}_\mu = \hat{v}_\mu - \hat{a}_\mu$.

Calculation within 2-flavour ChPT, even sector

[Gasser, Leutwyler '84],[Bijnens, Colangelo, Ecker '99]

$$U = \sigma + i\frac{\phi}{F} \ ,\sigma^2 + \frac{\phi^2}{F^2} = 1 \ ,\phi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} = \phi^i \tau^i \ ,$$

$$u_\mu = iu^\dagger \partial_\mu U u^\dagger \ ,\chi_+ = u^\dagger \chi u^\dagger + u\chi^\dagger u \ ,\chi = 2B(\hat{m}) \ \ ,\hat{m} = \frac{1}{2}(m_u + m_d) \ ,$$

- $O(p^2)$

$$\mathcal{L}_2 = \frac{F^2}{4}\langle u_\mu u^\mu + \chi_+ \rangle \, .$$

- $O(p^4)$

$$\mathcal{L}_4 = \frac{l_1}{4}\langle u_\mu u^\mu \rangle^2 + \frac{l_2}{4}\langle u_\mu u_\nu \rangle \langle u^\mu u^\nu \rangle + \frac{l_3}{16}\langle \chi_+ \rangle^2 + i\frac{l_4}{4}\langle u_\mu \chi_-^\mu \rangle - \frac{l_5}{2}\langle f_{-\mu\nu} f_-^{\mu\nu} \rangle$$
$$+ i\frac{l_6}{2}\langle f_{+\mu\nu} u^\mu u^\nu \rangle - \frac{l_7}{16}\langle \chi_- \rangle^2$$

$$l_i = l_i^r + \gamma_i (c\mu)^{d-4} \Lambda \, ,$$

- $O(p^6)$

$$\mathcal{L}_6 = c_6 \langle \chi_+ h_{\mu\nu} h^{\mu\nu} \rangle + c_7 \langle u_\mu u^\mu \chi_+ \chi_+ \rangle + c_8 \langle u_\mu u^\mu \chi_+ \rangle \langle \chi_+ \rangle + c_9 \langle \chi_+ u_\mu \chi_+ u^\mu \rangle + \ldots$$

$$c_i = \frac{(c\mu)^{2(d-4)}}{F^2}\big(c_i^r - \gamma_i^{(2)} \Lambda^2 - (\gamma_i^{(1)} + \gamma_i^{(L)})\Lambda\big).$$

Calculation within 2-flavour ChPT

NLO odd-intrinsic Lagrangian is given by [Bijnens, Girlanda, Talavera '02]

$$\mathcal{L}_6^W = \sum_{i=1}^{13} c_i^W o_i^W \,, \qquad c_i^W = c_i^{W\,r} + \eta_i (c\mu)^{d-4}\Lambda \,, \tag{1}$$

monomial (o_i^W)	i 2-flavour	$384\pi^2 F^2 \eta_i$
$\epsilon^{\mu\nu\alpha\beta}\langle\chi_+ [f_{-\mu\nu}, u_\alpha u_\beta]\rangle$	1	0
$\epsilon^{\mu\nu\alpha\beta}\langle\chi_- \{f_{+\mu\nu}, u_\alpha u_\beta\}\rangle$	2	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_- f_{+\mu\nu} f_{+\alpha\beta}\rangle$	3	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_- f_{-\mu\nu} f_{-\alpha\beta}\rangle$	4	0
$i\epsilon^{\mu\nu\alpha\beta}\langle\chi_+ [f_{+\mu\nu}, f_{-\alpha\beta}]\rangle$	5	0
$\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle\chi_- u_\alpha u_\beta\rangle$	6	$-5N_C$
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\alpha\beta}\chi_-\rangle$	7	$4N_C$
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\alpha\beta}\rangle\langle\chi_-\rangle$	8	$-2N_C$
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\gamma\mu}\rangle\langle h_{\gamma\nu} u_\alpha u_\beta\rangle$	9	$2N_C$
$i\epsilon^{\mu\nu\alpha\beta}\langle f_{+\gamma\mu}\rangle\langle f_{-\gamma\nu} u_\alpha u_\beta\rangle$	10	$-6N_C$
$\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\gamma\alpha} h_{\gamma\beta}\rangle$	11	$4N_C$
$\epsilon^{\mu\nu\alpha\beta}\langle f_{+\mu\nu}\rangle\langle f_{+\gamma\alpha} f_{-\gamma\beta}\rangle$	12	0
$\epsilon^{\mu\nu\alpha\beta}\langle\nabla_\gamma f_{+\gamma\mu}\rangle\langle f_{+\nu\alpha} u_\beta\rangle$	13	$-4N_C$

n.b. it depends on the form of $\mathcal{L}_4$ (cf. [Ananth., Moussallam '02])

Calculation within 2-flavour ChPT

Chiral logarithms

- NLO $m_\pi^2 \log m_\pi$ no contribution if: $F \to F_\pi$
- NNLO $m_\pi^4 \log m_\pi$ $m_\pi^4 \log^2 m_\pi$

Calculation within 2-flavour ChPT

Chiral logarithms

- NLO $m_\pi^2 \log m_\pi$ no contribution if: $F \to F_\pi$
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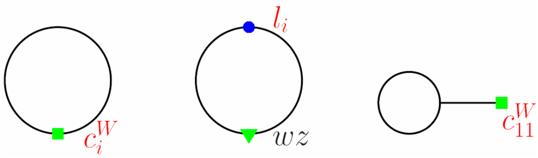
Double logarithms can be calculated using Weinberg consistency relations
i.e. $\log m/\epsilon$ must cancel out because there is no such counterterm

used e.g. for $\pi\pi$ scattering by [Colangelo '95]

schematically for the full $O(p^8)$ contribution ($\log = \frac{L}{(4\pi)^2}$, $\frac{1}{\epsilon} = \frac{\Lambda}{(4\pi)^2}$):

$$a(d)(\Lambda + 1/2L + \ldots)^2 + b_i(d)c_i(\Lambda)(\Lambda + 1/2L + \ldots) + \text{“}ct(p^8)\text{”}$$

heat-kernel expansion



Calculation within 2-flavour ChPT: amplitude

$$
\begin{aligned}
A_{NNLO} = \frac{e^2}{F_\pi}\Big\{ & \frac{1}{4\pi^2} \\
& + \frac{16}{3} m_\pi^2 (-4c_3^{Wr} - 4c_7^{Wr} + c_{11}^{Wr}) + \frac{64}{9} B(m_d - m_u)(5c_3^{Wr} + c_7^{Wr} + 2c_8^{Wr}) \\
& + \frac{128}{3} \frac{m_\pi^4}{F_\pi^2} (c_3^{Wr} + c_7^{Wr}) l_3^r + \frac{64}{3} \frac{m_\pi^4}{F_\pi^2} L_\pi (c_2^{Wr} + 2c_3^{Wr} + c_6^{Wr} + 2c_7^{Wr} - \tfrac{1}{2} c_{11}^{Wr}) \\
& + \frac{2}{\pi^2} \frac{m_\pi^2 B(m_d - m_u)}{F_\pi^2} L_\pi (-2c_2^{Wr} - \tfrac{11}{3} c_3^{Wr} + 2c_4^{Wr} - 4c_5^{Wr} - \tfrac{1}{3} c_7^{Wr} - \tfrac{2}{3} c_8^{Wr}) \\
& - \frac{1}{24\pi^2} \frac{m_\pi^4}{F_\pi^4} L_\pi^2 + \frac{m_\pi^4}{F_\pi^4} (\alpha(\mu) + \gamma L_\pi) + \frac{m_\pi^2 B(m_d - m_u)}{F_\pi^4} \beta(\mu) \Big\}
\end{aligned}
$$

- 4 LECs in 2 combinations of NLO
- additional 4 LECs in 3 combinations of NNLO

Is it at all possible to make some reliable prediction?

$N_f = 2$ vs. $N_f = 3$

Using m_s expansion, similarly to [Gasser; Haefeli, Ivanov, Schmid '07]

$$
\begin{aligned}
c_2^{Wr} &= +c_0 + C_4^{Wr} - \tfrac{1}{2}C_5^{Wr} + \tfrac{3}{2}C_6^{Wr} + O(m_s) \\
c_3^{Wr} &= -\tfrac{3}{2}c_0 + C_7^{Wr} + 3C_8^{Wr} + O(m_s) \\
c_4^{Wr} &= -\tfrac{1}{2}c_0 + C_9^{Wr} + 3C_{10}^{Wr} + O(m_s) \\
c_5^{Wr} &= + C_{11}^{Wr} + \tilde{L} + \tfrac{1}{2F_0^2(64\pi^2)^2} + O(m_s) \\
c_6^{Wr} &= -c_0 + C_5^{Wr} - \tfrac{3}{2}C_6^{Wr} - \tfrac{1}{2}C_{14}^{Wr} - \tfrac{1}{2}C_{15}^{Wr} + O(m_s) \\
c_7^{Wr} &= \tfrac{3}{2}c_0 - 3C_8^{Wr} + \tfrac{1}{4}C_{22}^{Wr} + O(m_s) \\
c_8^{Wr} &= \tfrac{3}{4}c_0 + \tfrac{1}{2}C_7^{Wr} + 3C_8^{Wr} - \tfrac{1}{8}C_{22}^{Wr} + O(m_s) \\
c_{11}^{Wr} &= + C_{22}^{Wr} + O(m_s)
\end{aligned}
$$

where
$$
\begin{aligned}
c_0 &= \frac{-1}{512\pi^2 B\, m_s} + \frac{1}{16\pi^2 F_0^2}(3L_7^r + L_8^r) - \frac{1}{32F_0^2}\tilde{L}, \\
\tilde{L} &= L_K + \tfrac{2}{3}L_\eta.
\end{aligned}
$$

- At NLO: only two combinations of LECs (C_7^W, C_8^W)
- NNLO simplified, if we retain only $1/m_s$ terms $\rightarrow$ modified counting!

$N_f = 2$ vs. $N_f = 3$: modified counting

$$m_u, m_d \sim O(p^2) \quad \text{and} \quad m_s \sim O(p)$$

result:

$$\begin{aligned}
A^{mod}_{NNLO} =& \frac{1}{4\pi^2} + \frac{1}{16\pi^2}\frac{m_d - m_u}{m_s} - \frac{64}{3} m_\pi^2 C_7^W \\
&+ 128B(m_d - m_u)[\tfrac{1}{3}C_7^W + C_8^W + f(L_3^r, L_4^r)] \\
&- \frac{3}{32\pi^2}\frac{m_d - m_u}{m_s}\frac{m_\pi^2}{F_\pi^2}L_\pi \\
&- 384B(m_d - m_u)\frac{m_\pi^2}{F_\pi^2}L_\pi C_8^W - \frac{1}{24\pi^2}\frac{m_\pi^4}{F_\pi^4}L_\pi^2 - \gamma\frac{m_\pi^4}{F_\pi^4}L_\pi \\
&+ O(p^5)\,.
\end{aligned}$$

Amplitude originally with 8 $O(p^6)$ LECs reduced to 2 LECs: C_7^W, C_8^W.

Phenomenology

- resonance saturation model [Ananth., Moussallam '02]

$$C_7^W \simeq \frac{g_{\pi_{1300}\gamma\gamma} d_m}{M^2_{\pi_{1300}}}, \qquad C_8^W \simeq \frac{g_{\eta'\gamma\gamma}\tilde{d}_m}{M'^2_\eta}$$

  this implies $C_7^W \ll C_8^W$

- $\eta \to 2\gamma$ (retain only C_8^W)

$$A_\eta \simeq \frac{e^2}{\sqrt{3}F_\pi}\left(\frac{F_\pi}{4\pi^2 F_\eta} + \frac{512}{3}(m_K^2 - m_\pi^2)C_8^W + O(m_s^2)\right)$$

PDG: $\Gamma_{\eta\to 2\gamma} = 0.510 \pm 0.026$ keV

$$\Rightarrow \qquad C_8^W = (0.6 \pm 0.2) \times 10^{-3}\ \text{GeV}^{-2}$$

- quark mass ratios: [Bijnens, Ghorbani '07], [MILC '07]

Phenomenology

- F_π from [Marciano, Sirlin '93] π_{l2} decay:

$$\Gamma = \frac{G_F^2}{4\pi}|V_{ud}F_\pi|^2 m_l^2 m_\pi (1-z^2)^2 \left(1 + \frac{2\alpha}{\pi}\ln\frac{m_Z}{m_\rho}\right)(1 + C_1 + \ldots)$$

  - C_1 estimate via ChPT
    - [Knecht,Neufeld,Rupertsberger,Talavera '00]

$$C_1 = \frac{Z}{4}\left(3 + 2\ln\frac{m_\pi^2}{m_\rho^2} + \ln\frac{m_K^2}{m_\rho^2}\right) - \frac{1}{2} + f(K_i^r, X_i^r)$$

    - [Ananth.,Moussallam '02] [Descotes, Moussallam '06]

$$C_1 = -2.56 \pm 0.50$$

  - V_{ud} [Towner CKM '08]

$$V_{ud} = 0.97425(23)$$

$\Rightarrow$ $F_\pi = 92.21 \pm 0.06$ MeV

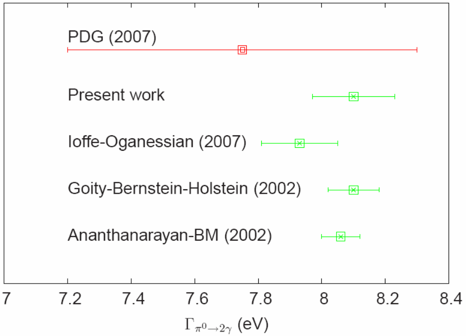
[rem.: *of course if SM is correct. See talk of Passemar*]

(uncomplete) Results

Due to the large-N_C enhancement of C_8^W we can conclude even without chiral logarithms from two loops

ano	$O(p)$	$O(p^2)$	$O(e^2)$	$O(p^3 \log p)$	$O(p^4 \log^2 p)$
7.765	0.089	0.305	-0.05	0.005	-0.004

$$\Gamma_{\pi^0\to 2\gamma}(eV) = 8.11 \pm \underset{m_d-m_u}{0.08} \pm \underset{C_8^W}{0.10} \pm \underset{C_7^W}{0.01} \pm \underset{F_\pi}{0.01}$$



Two loops

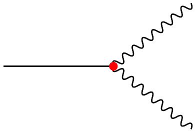
Reasons to calculate full two-loop result

- corrections could be still at the level of percents (rather than permille)
- purely academical reasons: $\pi^0 \to \gamma\gamma$ one of the fundamental process of quantum field theory, full two loop calculation can help us to understand its behaviour (e.g. why there is no chiral log at NLO)
- theory ahead an experiment :-)

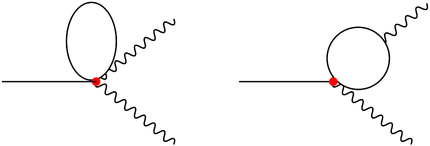
(preliminary) icing on the cake

Two loops

- LO ($O(p^4)$)



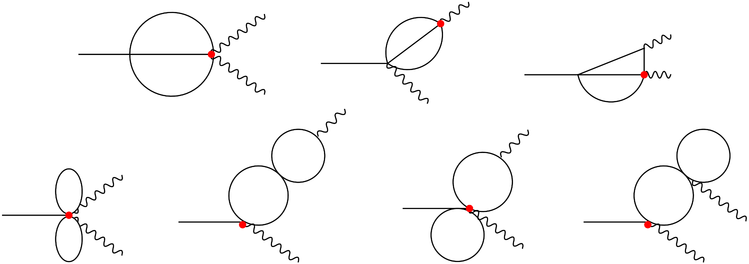
- NLO ($O(p^6)$)



Two loops

- NNLO ($O(p^8)$)

cf. e.g. $\pi\pi$ scattering: [Bijnens, Colangelo, Ecker, Gasser, Sainio '97]



$F \to F_\pi$ [Bijnens, Colangelo, Talavera '98]

$M_0 \to M_\pi$ at NLO

$$Z = 1 - \frac{T_M}{F^2} + \frac{M^4}{F^4}\dot{T}_M\left\{Q^Z\dot{T}_M - \sum_{i=1}^{3} Q_i^Z\, l_i\right\} + \frac{M^4\, r_Z}{F^4} + O(M^6)$$

[Bürgi '96]

Two loops: (preliminary) finished

- NNLO ($O(p^8)$): techniques we have employed
  - IBP relations (Laporta algorithms, program AIR, FIRE), Mellin-Barnes representation (Czakon code), numerical checking (program FIESTA), dispersive representation (used e.g. [Bijnens, Colangelo, Ecker, Gasser, Sainio '97], or [Gasser, Sainio '98]) ...
  - (*see talk of Boughezal*)
- independent verification of Z factor [Bürgi '96] (nontrivial sunset diagram)
- double-logarithms: checked
- single-logarithms: (preliminary!):

$$\gamma = -\frac{43}{768\pi^4}$$

(relatively important, if compared with double log)

Summary

- First full $O(p^8)$ calculation in ChPT (however, still only NNLO)
- Different counting proposed
- LEC C_8^{Wr} enhanced due to large N_C effects
- possible theoretical properties of full two-loop amplitude under investigation