Dispersive analysis of $K \to 3\pi$ and cusps*

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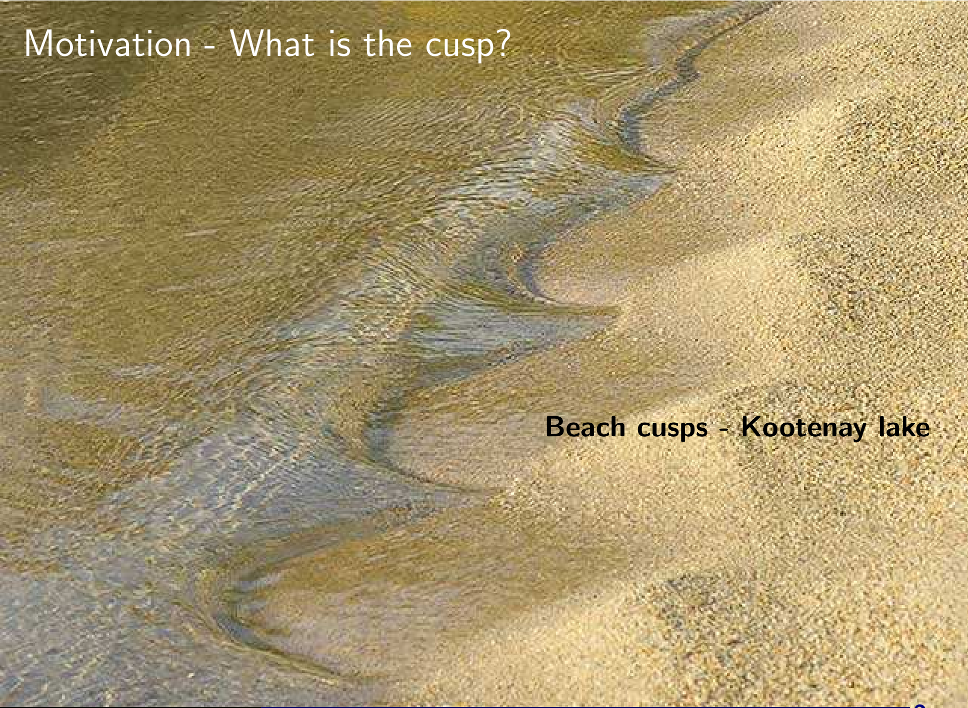
***Work in progress in collaboration with M. Knecht and J. Novotný*

Outline:

- What is the cusp?
- Theoretical approaches - overview of existing approaches
- Dispersive approach
  - introduction to dispersive analysis
  - first iteration - results at $O(p^4)$
  - sketch of second iteration $(O(p^6))$
- Conclusions

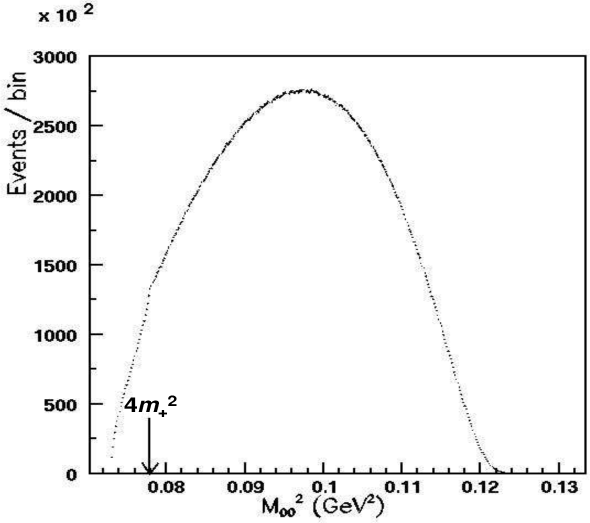
Motivation - What is the cusp?



Beach cusps - Kootenay lake

What is the cusp?

Decay $K^+ \to \pi^+\pi^0\pi^0$ - $6 \cdot 10^8$ reconstructed events at NA48/2



Pictures taken from L. DiLella, Kaon 07

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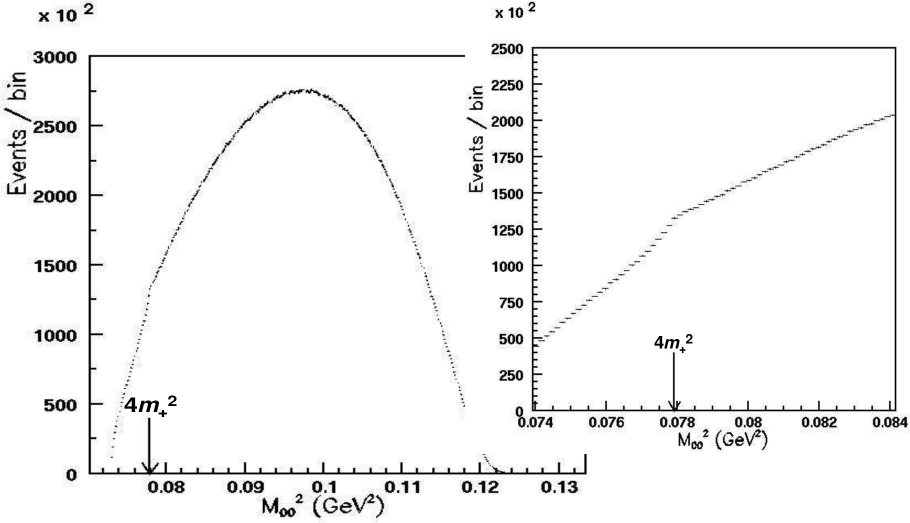

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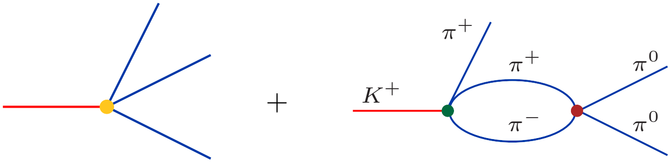
Theory – Why is the cusp?

Cabibbo '04

Amplitude for $K^+ \to \pi^+\pi^0\pi^0$: $\mathcal{M}_0 + \mathcal{M}_1$, schematically:



Theory – Why is the cusp? Cabibbo '04

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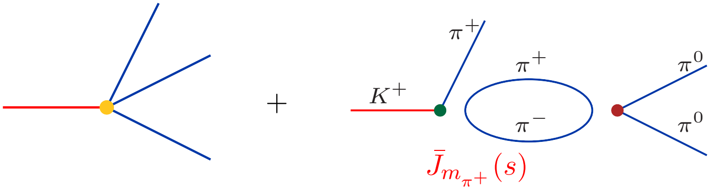
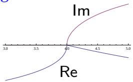


- $\bar{J}_m(s) \sim 2 + v \log \frac{v-1}{v+1} =$  $\approx i\pi v +$ regul. $\Longrightarrow \mathcal{M}_1 \sim i\pi v$, where $v = \sqrt{1-4m^2/s}$

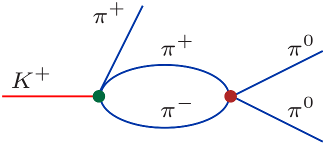
- thus we have square root singularity at $4m_+^2$ above physical threshold $4m_0^2$ and

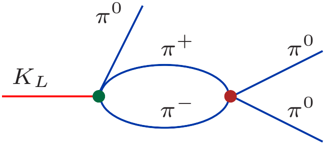
$$|\mathcal{M}|^2 = \begin{cases} (\mathcal{M}_0)^2 + (\mathcal{M}_1)^2 + 2\mathcal{M}_0\mathcal{M}_1 & : \quad s < 4m_+^2 \\ (\mathcal{M}_0)^2 + (i\mathcal{M}_1)^2 & : \quad s > 4m_+^2 \end{cases}$$

- depends on the scattering length of $\pi\pi$ Meißner, Müller, Steininger '97

Where is the cusp?

The same should appear for the $K_L \to \pi^0\pi^0\pi^0$





This second cusp is much weaker – roughly: (Cabibbo, Isidori '05, DiLella – Kaon07)

- Decay $K^+ \to \pi^+\pi^0\pi^0$

$$\frac{\text{“cusp effect”}}{\text{“size of amplitude”}} \sim \left. \frac{A_{+;+-}A_{+;00} + A_{+;+-}A_{+;00}}{|A_{+;00}|^2} \right|_{\text{“branch. point”}} \approx 6$$

- Decay $K_L \to \pi^0\pi^0\pi^0$

$$\frac{\text{“cusp effect”}}{\text{“size of amplitude”}} \sim \left. \frac{A_{L;+-}A_{L;00}}{|A_{L;00}|^2} \right|_{\text{“branch. point”}} \approx 0.5$$

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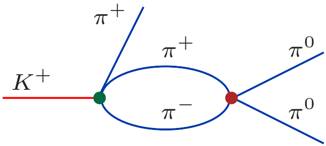

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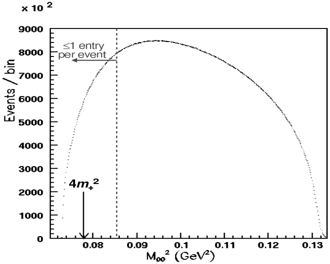
Decay $K_L \to \pi^0\pi^0\pi^0$ - $9 \cdot 10^8$ reconstructed events at NA48/2



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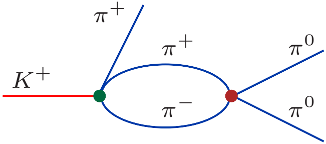

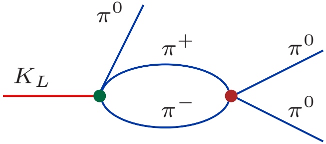

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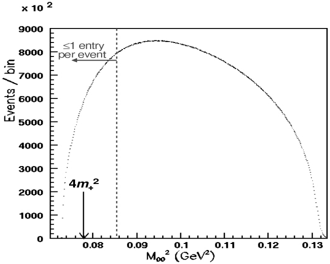

However, it is **seen** both at KTeV and NA48/2!!

— NA48/2: not published yet

— KTeV: arXiv:0806.3535

Pictures taken from L. DiLella, Kaon07

Theoretical approaches - Overview

a direct computation from χPT (with weak part)

- Bijnens, Borg '04, '04, '05
  - up to now just NLO
  - includes isospin breaking and elmag. corrections

b use of analyticity and unitarity

- Cabibbo '04
- Cabibbo, Isidori '05
- Gámiz, Prades, Scimemi '06
- full dispersive approach $\longleftarrow$ this talk

c Nonrelativistic QFT

- Colangelo, Gasser, Kubis, Rusetsky, Bissegger, Fuhrer '06, '07, '08
  - nonrelativistic approach
  - double expansion in velocities and scattering lengths
  - possible to add photons (cf. Gevorkyan, (Madigozhin), Tarasov, Voskresenskaya '06, '07)
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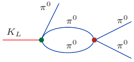
Theoretical approaches - Use of analyticity and unitarity

	Cabibbo 04	Cabibbo, Isidori 05	Gámiz, Prades, Scimemi 06	Dispersive approach 08
Order	$\frac{1}{2}$NLO	NNLO	NNLO	NNLO
Ass'tions (inputs)	$O(p^2)K \to 3\pi =$ first order polynomial	real part of $A_{+;00} \approx$ second order polynomial	real part of $A_{+;00} \approx$ isospin symm. result of χPT	$O(p^2)$ amplitudes = first order polynomial
Parametrization	$A_{+;00} = \begin{cases} A + Bv(s) & s > 4m_+^2, \\ A + iBv(s) & s < 4m_+^2, \end{cases} \quad v(s) = \sqrt{\frac{\lvert s-4m_+^2 \rvert}{s}}$			$A_{+;00} = \mathrm{Re} + \mathrm{Im}$
Method of computation	direct computation: $\pi\pi$ - scattering $\to$ toy model Lagrangian	imaginary part of the amplitude - given by unitarity relations		discontinuity ($\sim$im.part) - given by (generalised) unitarity relations the full amplitude - given by reconstruction theorem from discontinuity
Problem of method	omits diagram 	assumes simple analytic structure of the amplitude $\Rightarrow$ some contributions are missed out		the correct analytic structure of amplitudes and the correct integration contours taken into account $\to$ complicated

Theoretical approaches - Use of analyticity and unitarity

Our dispersive approach

- discontinuity ($\sim$ imaginary part) of $A_{+;00}$ given by (generalised) unitarity relations
- the full amplitude given by reconstruction theorem from this discontinuity
- includes second order rescattering
- the correct analytic structure of the amplitudes and correct integration contours taken into account
- does not take explicitly into account photons

$\Rightarrow$ full-featured approach based just on the unitarity, analyticity (subtracted dispersion relations), crossing symmetry and chiral power-counting

Dispersive approach

- for now we ignore photons, CP violation
- instead of computing $K \to \pi\pi\pi$ amplitudes directly, we use crossing symmetry (analytic continuation) of the $K\pi \to \pi\pi$ amplitudes to the decay region
- partial wave decomposition: $A(s,t,u) = 16\pi\,(f_0(s) + 3f_1(s)\cos\theta) + A_{\ell\geq 2}$,

$$\mathrm{Re}\,A_{\ell\geq 2} \sim O(p^4),\ \mathrm{Im}\,A_{\ell\geq 2} \sim O(p^8),\ \mathrm{Re}\,f_{0,1}(s) \sim O(p^2),\ \mathrm{Im}\,f_{0,1}(s) \sim O(p^4).$$

Reconstruction theorem

Assuming validity of (subtracted) DR's (and further conditions), we can reconstruct the amplitude of the process $AB \to CD$: Stern, Sazdjian, Fuchs '93, M.Z., Novotný '08

$$S(s,t;u) = R + \Phi_0(s) + [s(t-u) + (m_A^2 - m_B^2)(m_C^2 - m_D^2)]\Phi_1(s) + \text{crossed channels} + O(p^8),$$

R - third order polynomial in s,t,u with same symmetries as $S(s,t;u)$,

$$\Phi_0(s) = 16s^3 \int_\Sigma^\infty \frac{dx}{x^3}\frac{\mathrm{Im}\,f_0(x)}{x-s},$$

$$\Phi_1(s) = 48s^3 \int_\Sigma^\infty \frac{dx}{x^3}\frac{\mathrm{Im}\,f_1(x)}{(x-s)\lambda_{AB}^{1/2}(x)\lambda_{CD}^{1/2}(x)},$$

and similar for the $t-$ and $u-$ crossed channel $\left[\lambda_{XY}(s) = (s-(m_X+m_Y)^2)(s-(m_X-m_Y)^2)\right]$

Dispersive approach

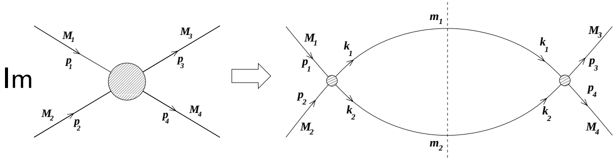
Unitarity relation

Assuming T-invariance and the real analyticity of the amplitude, the unitarity relation gives for the partial waves

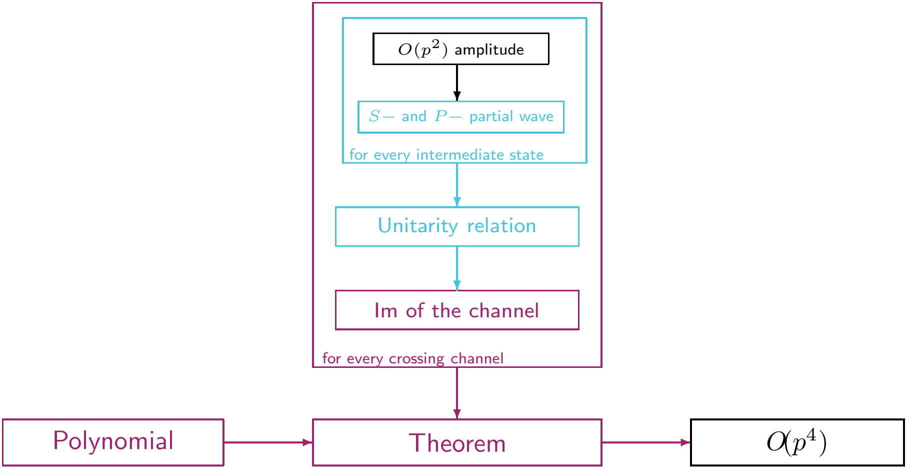
$$\mathrm{Im}\, f_\ell^{i\to f}(s) = \sum_{(1,2)} \frac{1}{S} \frac{\lambda^{1/2}(s,m_1^2,m_2^2)}{s} f_\ell^{i\to(k_1,k_2)}(s) \left[f_\ell^{f\to(k_1,k_2)}(s)\right]^* \theta(s-(m_1+m_2)^2)$$

$S = 1(2)$ for (un)distinguishable states k_1, k_2

- in the low-energy region the intermediate states other than those containing pairs of pseudoscalar mesons are suppressed up to $O(p^8)$
- intermediate states other than $\pi\pi$ induce singularities far from the central region of Dalitz plot of $K \to \pi\pi\pi$ processes $\Rightarrow$ can be expanded in series and included into the polynomial



Application of the dispersive approach



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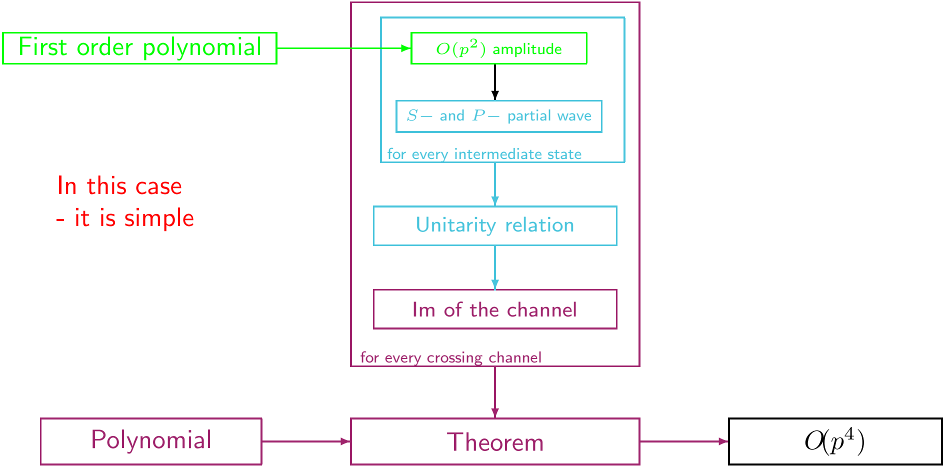
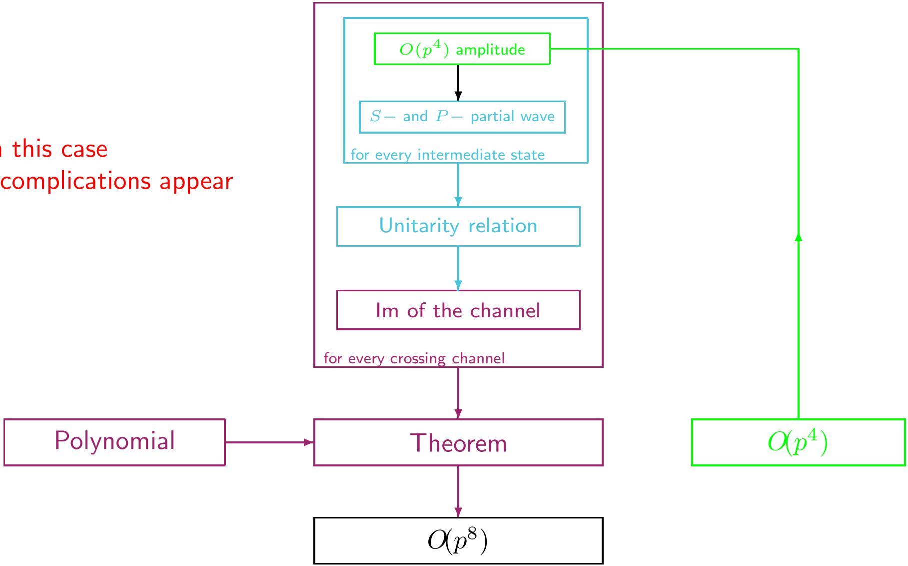


In this case
- it is simple

Application of the dispersive approach

In this case
- complications appear



First iteration - illustration on $K_L \to \pi^0\pi^0\pi^0$

- The reconstruction theorem and crossing symmetry says that amplitude looks like

$$\mathcal{A}_{L;00}(s,t,u) = \mathrm{P}_{L;00} + \Phi_0^{L;00}(s) + \Phi_0^{L;00}(t) + \Phi_0^{L;00}(u) + O(p^8)$$

with the polynomial $\left[s_0^L = 1/3 M_K^2 + m_0^2, \quad C_F = -\frac{3}{5} V_{us}^* V_{ud} \frac{G_F}{\sqrt{2}}\right]$

$$\mathrm{P}_{L;00} = C_F\big(A_{00}^L M_K^2 + \{C_{00}^L[(s-s_0^L)^2] + E_{00}^L[(s-s_0^L)^3]\} + \{s \leftrightarrow t\} + \{s \leftrightarrow u\}\big).$$

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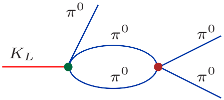
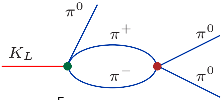
- To compute $O(p^4)$ $\Phi_0^{L;00}(s)$ from the unitarity relation we need $O(p^2)$ intermediate amplitude: $\left[s_\pm^L = 1/3(M_K^2 + m_0^2 + 2m_+^2)\right]$





$$\mathcal{A}_{L;00}^{\mathrm{LO}} = C_F A_{00}^L M_K^2, \qquad \mathcal{A}_{L;+-}^{\mathrm{LO}} = C_F\left[B_{+-}^L(s - s_\pm^L) + A_{+-}^L M_K^2\right]$$

$$\mathcal{A}_{\mathrm{LO}}^{00;00} = \frac{\alpha_{00} m_0^2}{F_\pi^2}, \qquad \mathcal{A}_{\mathrm{LO}}^{+-;00} = -\frac{\beta_{\pm 0}}{F_\pi^2}\left(s - \frac{2}{3}m_+^2 - \frac{2}{3}m_0^2\right) - \frac{\alpha_{\pm 0} m_0^2}{3F_\pi^2}.$$

First iteration - illustration on $K_L \to \pi^0\pi^0\pi^0$

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with the polynomial $\left[s_0^L = 1/3 M_K^2 + m_0^2, \quad C_F = -\frac{3}{5} V_{us}^* V_{ud} \frac{G_F}{\sqrt{2}}\right]$

$$\mathrm{P}_{L;00} = C_F\big(A_{00}^L M_K^2 + \{C_{00}^L[(s-s_0^L)^2] + E_{00}^L[(s-s_0^L)^3]\} + \{s \leftrightarrow t\} + \{s \leftrightarrow u\}\big).$$

- The $O(p^4)$ result: $\left[s_\pm^L = 1/3(M_K^2 + m_0^2 + 2m_+^2)\right]$

$$\begin{aligned}\Phi_0^{L;00}(s) = {} & \frac{C_F}{2F_\pi^2} A_{00}^L M_K^2 \alpha_{00} m_0^2 \bar{J}_0(s) \\ & - \frac{C_F}{F_\pi^2}[\beta_{\pm 0}(s - \frac{2}{3}m_+^2 - \frac{2}{3}m_0^2) + \frac{1}{3}\alpha_{\pm 0} m_0^2][A_{+-}^L M_K^2 + B_{+-}^L(s - s_{+-}^L)]\bar{J}_\pm(s) \\ & + \text{polynomial} + O(p^6),\end{aligned}$$

$$\bar{J}_P(s) = \frac{1}{16\pi^2}\left(2 + \sigma_P \ln\frac{\sigma_P - 1}{\sigma_P + 1}\right), \qquad \sigma_P = \sqrt{1 - \frac{4m_P^2}{s}}.$$

First iteration - illustration on $K_L \to \pi^0\pi^0\pi^0$ (scatt. lengths)

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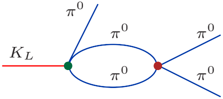
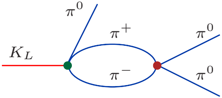
- Another choice of $\pi\pi$ parametrization - scattering lengths and effective range parameters (convergent proper's?, stability of fit?):





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$$\mathcal{A}_{\mathrm{LO}}^{00;00} = \mathrm{a}_{00}\,, \qquad \mathcal{A}_{\mathrm{LO}}^{+-;00} = \mathrm{a}_x - \frac{\beta_{\pm 0}}{F_\pi^2}(s - 4m_+^2)\,.$$

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- Another choice of $\pi\pi$ parametrization - scattering lengths and effective range parameters (convergent proper's?, stability of fit?):

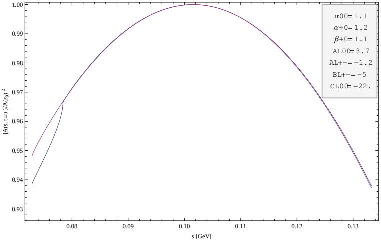
$$\Phi_0^{L;00}(s) = \frac{C_F}{2} A_{00}^L M_K^2 \mathrm{a}_{00} \bar{J}_0(s)$$
$$+ C_F[\mathrm{a}_x - \frac{\beta_{\pm 0}}{F_\pi^2}(s - 4m_+^2)][A_{+-}^L M_K^2 + B_{+-}^L(s - s_{+-}^L)]\bar{J}_\pm(s)$$
$$+ \text{polynomial} + O(p^6),$$

- Moreover, we can retain a_i physical interpretation up to two-loops (by adjustment of the polynomial of the $\pi\pi$ reconstruction theorem) - as in CGKR approach

First iteration - $O(p^4)$ result for $K_L \to \pi^0\pi^0\pi^0$

- One particular choice of the parameters α_{00}, $\alpha_{\pm 0}$, $\beta_{\pm 0}$, A^L_{00}, A^L_{+-}, B^L_{+-}, C^L_{00} (giving similar pictures like Bijnens):

Examples of squared amplitudes along two curves: $u = t$ and $\sqrt{3}(s - s_0) = u - t$



This is prepared for fit.

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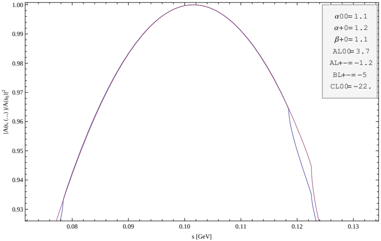
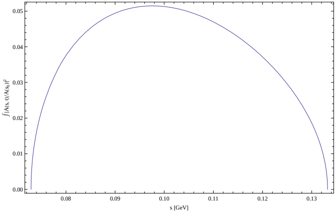


This is prepared for fit. However, integrated this curve we have



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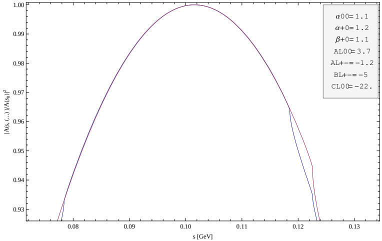
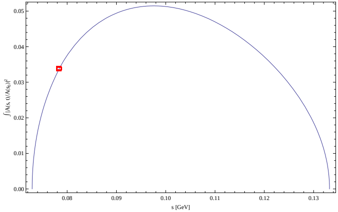
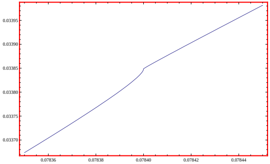
Examples of squared amplitudes along two curves: $u = t$ and $\sqrt{3}(s - s_0) = u - t$



This is prepared for fit. However, integrated this curve we have



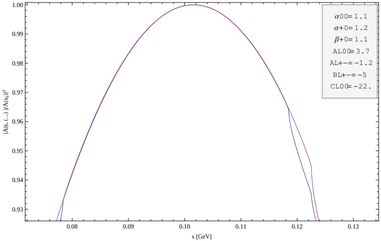
$Zoom$ $\longrightarrow$



First iteration - $O(p^4)$ result for $K_L \to \pi^0\pi^0\pi^0$

- One particular choice of the parameters α_{00}, $\alpha_{\pm 0}$, $\beta_{\pm 0}$, A^L_{00}, A^L_{+-}, B^L_{+-}, C^L_{00} (giving similar pictures like Bijnens):

Examples of squared amplitudes along two curves: $u = t$ and $\sqrt{3}(s - s_0) = u - t$



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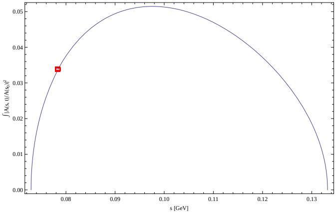
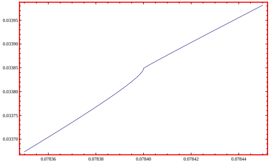


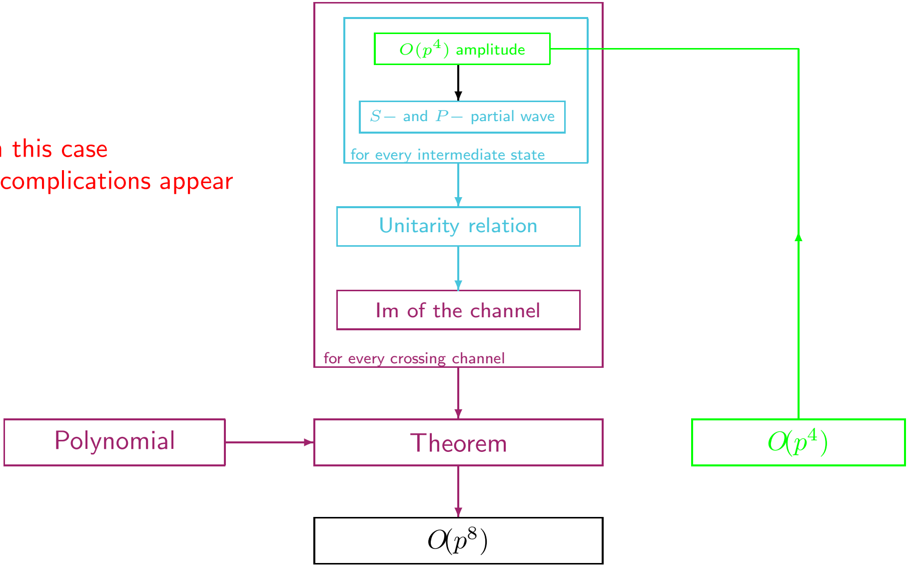
$\overset{Zoom}{\longrightarrow}$



- We have such $O(p^4)$ results for all the other K decays

Second iteration $\longrightarrow$ two-loop expression

In this case
- complications appear



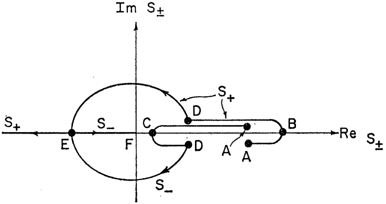
Second iteration - complications - in isospin limit

- we need analytical continuation of unitarity relations to unphysical regions $\Rightarrow$ analytical continuation of $O(p^4)$ partial waves needed
- obtained by careful deformation of integration contour in formula for partial wave projections: Barton; Bronzan; Kacser '61, '63 Anisovich; Anisovich, Ansel'm; Gribov '62, '66, '94

$$\varphi_l^{L;00}(s) = \frac{2}{\lambda_{L0}^{1/2}(s)\sigma_0} \int_{C(t_+,t_-)} \mathrm{d}t\, \mathcal{A}_{L;00}(s,t,3s_0^L - s - t)\, P_l\left(\cos\theta = \frac{2t + s - 3s_0^L}{\lambda_{L0}^{1/2}(s)\sigma_0}\right),$$

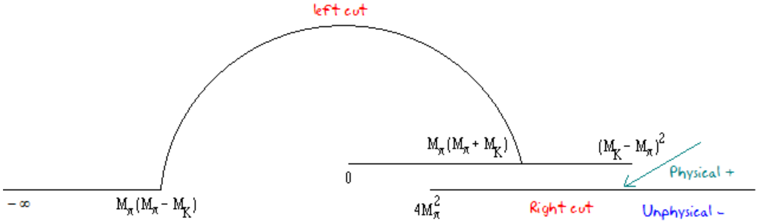
- $C(t_+,t_-)$ has to avoid intersection with branch cut of $\mathcal{A}_{L;00}$ $\Rightarrow$ prescription for the trajectories of the endpoints

$$t_\pm(s) = \frac{1}{2}\left(3s_0^L - s \pm \lambda_{L0}^{1/2}\sigma_0\right) + \mathrm{i}\varepsilon, \qquad \mathrm{sign}\,\varepsilon = \mathrm{sign}\,\frac{\partial t_\pm(s)}{\partial M_K^2}\,.$$



Second iteration - complications - in isospin limit

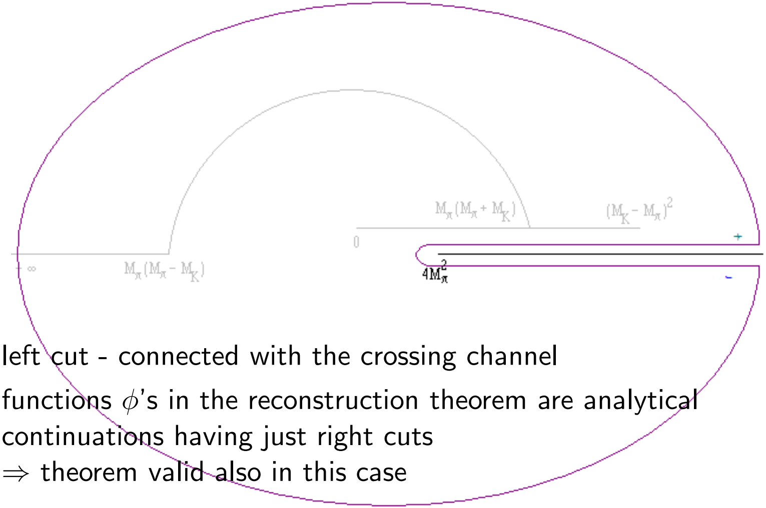
$\Rightarrow$ Analytical structure of partial amplitudes:



- left cut - connected with the crossing channel

Second iteration - complications - in isospin limit

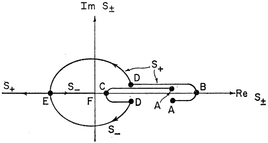
⇒ Analytical structure of partial amplitudes:

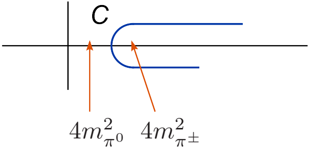


- left cut - connected with the crossing channel
- functions ϕ's in the reconstruction theorem are analytical continuations having just right cuts
  ⇒ theorem valid also in this case

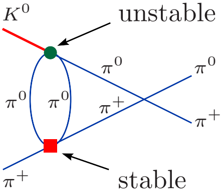
Second iteration - complications - beyond isospin limit

1. point ‘C’ above the threshold





2. existence of Landau anomalous thresholds: cf. also Gasser at Euridice 06 for $K^+ \to \pi^+\pi^0\pi^0$, $K^+ \to \pi^+\pi^-\pi^+$, $\underline{K_L \to \pi^+\pi^-\pi^0}$, e.g.:



The integration contour deformed avoiding the anomalous threshold.

This generalization is straightforward for the process $K_L \to 3\pi^0$.

Second iteration - $O(p^6)$ result for $K_L \to \pi^0\pi^0\pi^0$

$$\mathcal{A}_{L;00}(s,t,u) = \mathrm{P}_{L;00} + \Phi_0^{L;00}(s) + \Phi_0^{L;00}(t) + \Phi_0^{L;00}(u) + O(p^8),$$
$$\mathrm{P}_{L;00} = C_F\big(A_{00}^L M_K^2 + \{C_{00}^L[(s-s_0^L)^2] + E_{00}^L[(s-s_0^L)^3]\} + \{s \leftrightarrow t\} + \{s \leftrightarrow u\}\big).$$

- The $O(p^6)$ result:

$$\Phi_0^{L;00}(s) = 16 s^3 \int_\Sigma^\infty \frac{dx}{x^3} \frac{\mathrm{Im}\, f_0^{L;00}(x)}{x-s},$$

where

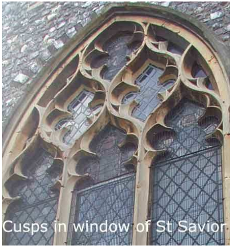
$$\mathrm{Im}\, f_0^{L;00}(s) = \frac{1}{2}\sigma_0\ \theta(s-4m_0^2)\frac{C_F}{F_\pi^4}\sum_k \left(p_k^{L;00;00}(s) + \frac{1}{s} q_k^{L;00;00}(s)\right) K_k^{L;00;00}(s)$$
$$+ \sigma_+\ \theta(s-4m_+^2)\frac{C_F}{F_\pi^4}\sum_j \left(p_j^{L;+-;00}(s) + \frac{1}{s} q_j^{L;+-;00}(s)\right) K_j^{L;+-;00}(s)$$

- $K(s)$ - functions of s $\left(\text{e.g. } \frac{1}{\sigma_0}\ln\frac{1-\sigma_0}{1+\sigma_0}\right)$, $\#k \sim 13, \#j \sim 12$ (we are trying to categorize and integrate them analytically)
- p and q - polynomials in s containing α_{00}, $\alpha_{\pm 0}$, $\beta_{\pm 0}$, λ_{00}, $\lambda_{\pm 0}^{(1)}$, $\lambda_{\pm 0}^{(2)}$, α_{+-}, β_{+-} and A_{00}^L, C_{00}^L, A_{+-}^L, B_{+-}^L, C_{+-}^L, D_{+-}^L

Summary, conclusions and outlook

- fully relativistic approach trying to include all possible nuances for $K \to 3\pi$ to two loops, based only on general principles: analyticity, unitarity, crossing symmetry, chiral counting
- from $K \to 3\pi$ experiments we can obtain some $\pi\pi$ characteristics beyond isospin limit (a_{00}, a_x, ..., subth. par's) $\to$ test of ChPT
- two parameterizations possible - in terms of
  - subthreshold parameters (stable) - same parameterization as other projects - isospin breaking in $\pi\pi$, πK, formfactors (Knecht, Bernard, Oertel, Passemar, Descotes-Genon, ...)
  - scattering lengths - as in existing analyses (CI, CGKR)
- in both cases # of param's reasonable: we can fit

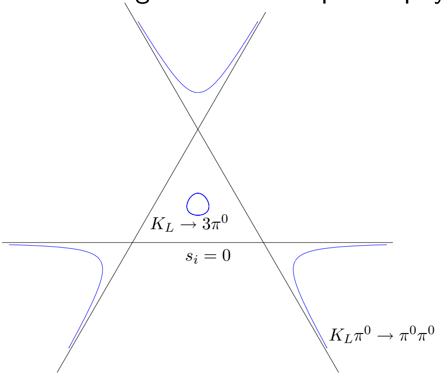
- so far 1^{st} iteration in both parameterizations finished
- 2^{nd} iter'n only for K_L (last integration performed, so far, only numerically we are trying to simplify it and get as most analytical information as possible)
- isospin violation - only via pion mass difference; other EM effects not included



SPARES

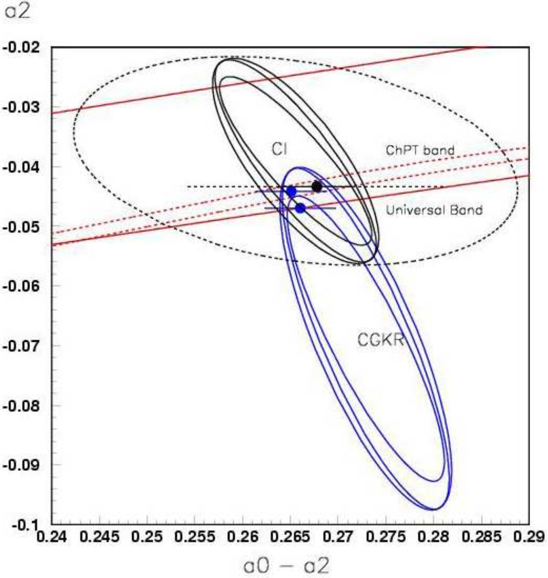
Physical continuation

different regions in Dalitz plot: “physical” and “scattering”



Hwa PR ’64, Aitchison, Pasquier PR ’66: physical amplitude for decay process can be obtained from the scattering process by the continuation where both s and M^2 approach the real axis from above.

Two predictions of scattering lengths



CI = Cabibbo, Isidori
CGKR = Colangelo, Gasser, Kubis, Rusetsky, (Bissegger, Fuhrer)

Taken from D. Madigozhin (NA48/2), Anacapri '08