Determination of Low Energy Constants and Testing Chiral Perturbation Theory at Next to Next to Leading Order

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Outline

1. Chiral Perturbation Theory
2. Relations in ChPT
3. Summary and Future Steps

Outline

$$\mathcal{L}_{QCD} = \sum_{q=1}^{n_f} [i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R - m_q(\bar{q}_R q_L + \bar{q}_L q_R)]$$

(n_f =number of flavours)
If $m_q = 0$ then $SU(n_f)_L \times SU(n_f)_R$ (chiral symmetry)$\Rightarrow$ parity doublets in the spectrum.
They do not exist! $\Rightarrow SU(n_f)_L \times SU(n_f)_R \to SU(n_f)_V$ Spontaneous Symmetry Breaking

- $n_f = 3 \to 8$ Goldstone bosons
- $n_f = 2 \to 3$ Goldstone bosons

$m_q \neq 0$ (but small) $\Rightarrow$ chiral symmetry is also explicitly broken

- Goldstone bosons are not massless
- Energy gap in the spectrum

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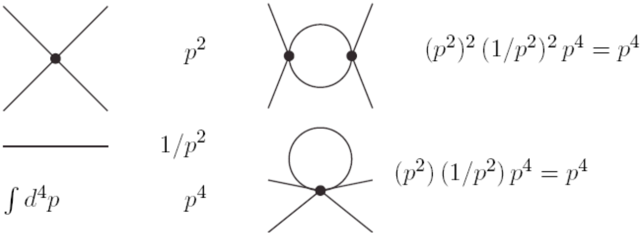
Degrees of freedom Goldstone Bosons (lightest mesons in the QCD spectrum)

Power counting Dimensional counting in momenta and masses (p^2)



Expected breakdown scale Resonances (M_ρ)

Construction of Lagrangians

$n_f = 3$
$U(\Phi) = \exp(i\sqrt{2}\Phi/F_0)$ parametrizes Goldstone Bosons:

$$\Phi = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}} \end{pmatrix}$$

LO Lagrangian: $\mathcal{L}_2 = \frac{{F_0}^2}{4}(\langle D_\mu U^\dagger D^\mu U\rangle + \langle \chi^\dagger U + \chi U^\dagger\rangle), \quad \chi = 2B_0\mathcal{M}$

$D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu, \quad r_\mu(l_\mu) = v_\mu + (-)a_\mu =$ external currents

$F_0, B_0 =$ Low Energy Constants

Low Energy Constants

	2 flavour		3 flavour	
p^2	F, B	2	F_0, B_0	2
p^4	l_i^r, h_i^r	7+3	L_i^r, H_i^r	10+2
p^6	c_i^r	52+4	C_i^r	90+4

Determination of LECs is important:

- to have precise predictions of ChPT
- to check its convergence
- to study the underlying QCD

PROBLEMS:

1. large number of phenomenological constants
2. strong correlations among them
3. many of the observables calculated in ChPT have not been measured yet. (But dispersion relations and lattice results can be used)

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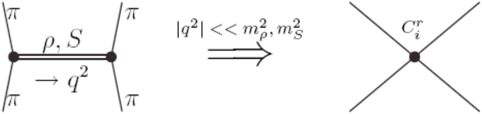
Knowledge of LECs so far

- L_i: existing fit NNLO (Amoros, Bijnens, Talavera, Nucl. Phys. B 602 (2001) 87 [hep-ph/0101127])
- C_i: some knowledge obtained through Resonance Estimates



But now we have a lot of processes and observables calculated in ChPT at NNLO which could be used all together to perform a global fit

Processes in ChPT

In literature you can find many processes calculated to NNLO in ChPT (see Bijnens, hep-ph/0604043 Prog. Part. Nucl. Phys. 58 (2007) 521 for a review and references)

1. $n_f = 2$
   - m_π and decay constant f_π
   - $\pi\pi$ scattering
   - Pion form factors
   - ...
2. $n_f = 3$
   - m_π, m_K, m_η and decay constants F_π, F_K
   - $\pi\pi$ scattering
   - πK scattering
   - Pion and Kaon scalar/vector form factors
   - Vector, Scalar, Axial-Vector two-point functions
   - K_{l4}
   - K_{l3}
   - ...

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Why are we looking for relations between observables?

Chiral Perturbation Theory $\rightarrow$ every observable can be written as a sum of terms of increasing importance in the Chiral expansion.

$$O = O^{(2)} + O^{(4)} + O^{(6)}$$

The p^6 part can be split in as

$$O^{(6)} = O_{C_i(\text{tree level})} + O_{L_i(\text{one loop})} + O_{F_0(\text{two loops})}$$

If we have a relation such that the first contribution cancels out we can

- stop worrying about C_is and perform the fit of the L_is at NNLO
- check how large is the loop contribution and test ChPT convergence

Processes and Quantities considered so far

- $\pi\pi$ scattering
- πK scattering
- $K_{l4}(K \to \pi\pi e\nu)$
- The scalar form factors $F_S^{\pi/K}(t)$
- Meson masses

$\pi\pi$ scattering

- $A(\pi^a\pi^b \to \pi^c\pi^d) = \delta^{a,b}\delta^{c,d}A(s,t,u) + \delta^{cd}\delta^{bd}A(t,u,s) + \delta^{ad}\delta^{bc}A(u,t,s)$
- The isospin amplitudes $T^I(s,t)$ $(I = 0,1,2)$ are written in terms of the function $A(s,t,u)$ via

$$\begin{aligned} T^0(s,t) &= 3A(s,t,u) + A(t,u,s) + A(u,s,t) \\ T^1(s,t) &= A(s,t,u) - A(u,s,t) \\ T^2(s,t) &= A(t,u,s) + A(u,s,t) \end{aligned}$$

  where $t = -\frac{1}{2}(s-4m_\pi^2)(1-\cos\theta), \quad u = -\frac{1}{2}(s-4m_\pi^2)(1+\cos\theta)$
- and then expanded in partial waves:

$$T^I(s,t) = 32\pi \sum_{\ell=0}^{+\infty}(2\ell+1)P_\ell(\cos\theta)t_\ell^I(s)$$

$$\text{Near threshold} \to t_\ell^I(s) = q^{2\ell}(a_\ell^I + b_\ell^I q^2 + \mathcal{O}(q^4))$$

$$q^2 = \frac{1}{4}(s-4m_\pi^2) \qquad a_\ell^I, b_\ell^I \cdots = \text{scattering lengths}, \ldots$$

- We studied only those observables where a dependence on the C_is shows up

$\pi\pi$ scattering relations

As a consequence of the expression of $A(s,t,u)$ as

$$\begin{aligned} A(s,t,u) &= b_1 + b_2 s + b_3 s^2 + b_4 (t-u)^2 + b_5 s^3 + b_6 s (t-u)^2 \\ &\quad + \text{non polynomial part} \end{aligned}$$

there are 5 relations among the scattering lengths:

$$\begin{aligned} \left[3b_1^1 + 25a_2^2\right]_{C_i} &= 10\left[a_2^0\right]_{C_i} \\ \left[5b_0^2 - 2b_0^0\right]_{C_i} + 9\left[2b_1^1 - 3a_1^1\right]_{C_i} &= 3\left[5a_0^2 - 2a_0^0\right]_{C_i} \\ \left[-5b_2^2 + 2b_2^0\right]_{C_i} &= 21\left[a_3^1\right]_{C_i} \\ 20\left[b_2^2 - b_2^0 - a_2^2 + a_2^0\right]_{C_i} &= \left[3a_1^1 + b_0^2\right]_{C_i} \\ -10\left[b_0^2 - 18b_2^0 + 18a_2^0\right]_{C_i} &= \left[2b_0^0 + 18a_1^1\right]_{C_i}; \end{aligned}$$

$a_\ell^I(b_\ell^I)$ expressed in unit of $m_\pi^{2\ell}(m_\pi^{2\ell+2})$

These relations hold for $n_f = 2,\ 3$, both at NLO and NNLO: not only the p^6 LECs cancel out, but also the tree level part involving the p^4 LECs does. Still there is L_is or l_is dependence through the non polynomial part.

πK scattering

- $T^I(s,t,u)$ = scattering amplitude in isospin channel $I = \frac{1}{2}, \frac{3}{2}$
- As for the $\pi\pi$ scattering, it's possible to define scattering lengths a^I_ℓ, b^I_ℓ:

$$T^I(s,t,u) = 16\pi \sum_{\ell=0}^{+\infty} (2\ell+1) P_\ell(\cos\theta) t^I_\ell(s)$$

$$\text{Near threshold} \rightarrow t^I_\ell = \frac{1}{2}\sqrt{s}\, q_{\pi K}^{2\ell}\left(a^I_\ell + b^I_\ell q^2_{\pi K} + \mathcal{O}(q^4_{\pi K})\right)$$

$$q^2_{\pi K} = \frac{s}{4}\left(1 - \frac{(m_K+m_\pi)^2}{s}\right)\left(1 - \frac{(m_K-m_\pi)^2}{s}\right)$$

$$t = -2q^2_{\pi K}(1-\cos\theta), \quad u = -s - t + 2m_K^2 + 2m_\pi^2$$

- Again we studied only those scattering lengths where a dependence on the C_is shows up

πK scattering relations

The isospin amplitudes $T^I(s,t,u)$ are written in terms of the crossing symmetric amplitudes $T^{\pm}(s,t,u)$ which can be expanded around t=0, s=u $(\nu = \frac{s-u}{4m_\pi})$ (subthreshold expansion):

$$T^+(s,t,u) = \sum_{i,j=0}^{\infty} c^+_{ij} t^i \nu^{2j} \qquad T^-(s,t,u) = \sum_{i,j=0}^{\infty} c^-_{ij} t^i \nu^{2j+1}$$

In $\left[c^-_{01}\right]_{C_i}$ and $\left[c^-_{20}\right]_{C_i}$ the same combination $-C_1 + 2C_3 + 2C_4$ appears⇒2 relations between the scattering lengths:

Preliminary

- $$m_\pi^3 m_K^3 (m_\pi + m_K)^2 \left[b_1^{\frac{1}{2}} - b_1^{\frac{3}{2}}\right]_{C_i} - \frac{1}{3} m_\pi^2 m_K^2 (m_\pi^2 + m_K^2) \left[b_0^{\frac{1}{2}} - b_0^{\frac{3}{2}}\right]_{C_i} =$$
$$-\frac{1}{12}[2(m_\pi^4 + m_K^4) + (m_\pi + m_K)^6 4] \left[a_0^{\frac{1}{2}} - a_0^{\frac{3}{2}}\right]_{C_i}$$
$$+\frac{1}{2} m_\pi m_K [(m_\pi + m_K)^4 - m_\pi m_K (m_\pi^2 + m_K^2 + 8 m_\pi m_K)] \left[a_1^{\frac{1}{2}} - a_1^{\frac{3}{2}}\right]_{C_i}$$
- . . . and 1 more relation

These relations hold only in the p^6 case. They also get a dependence on the L_is from the NLO contribution.

$\pi\pi$ scattering and πK scattering

- Considering the scattering lengths for $\pi\pi$ and πK scattering together five more relations appear:

Preliminary

♠
$$
\begin{aligned}
0 = & \left[-b_1^{\frac{1}{2}} - 5a_2^{\frac{3}{2}}\right]_{C_i} (m_\pi + m_K)^2 + \frac{1}{(18 m_\pi m_K)} (m_\pi^2 + 14 m_\pi m_K + m_K^2) \left[b_0^{\frac{3}{2}}\right]_{C_i} \\
& + \frac{(m_K + m_\pi)^3}{2m_\pi^2} \left(\left[5a_2^2\right]_{C_i} + \frac{1}{9m_\pi^2} \left[b_0^0 - 4b_0^2 + 9a_1^1\right]_{C_i} + \frac{1}{6m_\pi^4} \left[5a_0^2 - 2a_0^0\right]_{C_i} \right) \\
& + \frac{1}{6(m_\pi m_K)^2} (2m_\pi^4 + m_\pi^3 m_K - 14 m_\pi^2 m_K^2 + m_\pi m_K^3 + 2m_K^4) \left[a_1^{\frac{1}{2}} - a_1^{\frac{3}{2}}\right]_{C_i} \\
& + \frac{1}{36(m_K m_\pi)^3} (3m_\pi^4 - 5m_\pi^3 m_K - 4m_\pi^2 m_K^2 - 5 m_\pi m_K^3 + 3m_K^4) \left[a_0^{\frac{3}{2}}\right]_{C_i} \\
& - \frac{1}{36(m_K m_\pi)^3} (3m_\pi^4 - 8m_\pi^3 m_K - 10 m_\pi^2 m_K^2 - 8 m_\pi m_K^3 + 3m_K^4) \left[a_0^{\frac{1}{2}}\right]_{C_i} \\
& + \frac{1}{9 m_\pi m_K} (m_\pi^2 - 4 m_\pi m_K + m_K^2) \left[b_0^{\frac{1}{2}}\right]_{C_i}
\end{aligned}
$$

♠ . . . plus 4 more relations

- These are due to the polynomial expression of the amplitudes, thus they hold both for p^4 and for p^6.

In the transition amplitude 4 form factors appear: F, G, H, R (R in K_{e4} is suppressed→only in $K_{\mu 4}$)
Using partial wave expansion + neglecting d wave terms:

$$\begin{aligned} F_s &= f_s + f_s' q^2 + f_s'' q^4 + f_e' s_e/4m_\pi^2 + \ldots \quad (S \text{ wave}) \\ F_p &= f_p + f_p' q^2 + \ldots \quad (P \text{ wave}) \\ G_p &= g_p + g_p' q^2 + \ldots \quad (P \text{ wave}) \\ H_p &= h_p + h_p' q^2 + \ldots \quad (P \text{ wave}) \end{aligned}$$

$s_\pi (s_e)$ =invariant mass of dipion (dilepton) $\quad q^2 = (s_\pi/(4m_\pi^2) - 1)$

Relation between $\pi\pi$, πK scattering lengths and f_s''

Preliminary

$$\begin{aligned} & 3\left[-a_1^{\frac{1}{2}} + a_1^{\frac{3}{2}}\right]_{C_i} + \left[-b_0^{\frac{1}{2}} + b_0^{\frac{3}{2}}\right]_{C_i} + \frac{m_\pi^2 + m_K^2 + m_\pi m_K}{2(m_\pi m_K)^2}\left[a_0^{\frac{1}{2}} - a_0^{\frac{3}{2}}\right]_{C_i} \\ & = -(m_\pi + m_K)\left(-\frac{2}{3} m_\pi \left[f_s''\right]_{C_i} - 5\frac{m_K}{m_\pi}\left[a_2^2\right]_{C_i} + \frac{m_K}{3m_\pi^3}\left[b_0^0 - 2b_0^2 + 3a_1^1\right]_{C_i} \right. \\ & \left. + \frac{5}{18}\frac{m_K}{m_\pi^5}\left[5a_0^2 - 2a_0^0\right]_{C_i}\right) \end{aligned}$$

$F_{Sj}^{\pi/K}$ and masses

The scalar form factors for the pions and the kaons are defined as

$$F_{ij}^{M_1 M_2}(t) = \langle M_2(p)|\bar{q}_i q_j|M_1(q)\rangle$$

$t = p - q, \quad i, j =$ flavour indices $\quad M_i =$ meson state
Indipendent quantities $\to F_S^\pi, F_{Ss}^\pi, F_S^K, F_{Ss}^K, F_S^{\pi K}$

There are two relations between $F_S(t = 0)$ and the ChPT expansion of the masses M_π^2, M_K^2:

$$\begin{aligned} 2B_0 \left[M_\pi^2\right]_{C_i} &= \frac{1}{3}\left\{(2m_K^2 - m_\pi^2)\left[F_{Ss}^\pi(0)\right]_{C_i} + m_\pi^2\left[F_S^\pi(0)\right]_{C_i}\right\} \\ 2B_0 \left[M_K^2\right]_{C_i} &= \frac{1}{3}\left\{(2m_K^2 - m_\pi^2)\left[F_{Ss}^K(0)\right]_{C_i} + m_\pi^2\left[F_S^K(0)\right]_{C_i}\right\} \end{aligned}$$

They are due to the Feynman-Hellmann Theorem (see next)

Feynman-Hellmann Theorem in ChPT

In $\mathcal{L}_{QCD}$ appear $m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s$.

$$\langle\pi|m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s|\pi\rangle = m_\pi^2$$

The Feynman-Hellmann Theorem implies

$$\begin{aligned}
F_{Su}^{\pi}(t=0) = \langle\pi|\bar{u}u|\pi\rangle &= \frac{\partial m_\pi^2}{\partial m_u} \\
F_{Sd}^{\pi}(t=0) = \langle\pi|\bar{d}d|\pi\rangle &= \frac{\partial m_\pi^2}{\partial m_d} \\
F_{Ss}^{\pi}(t=0) = \langle\pi|\bar{s}s|\pi\rangle &= \frac{\partial m_\pi^2}{\partial m_s}
\end{aligned}$$

On the other hand ChPT leads to

$$\left[M_\pi^2\right]_{C_i} = \sum_i C_i(m_q)^3 = f(m_u, m_d, m_s) : \text{homogeneous of 3rd order}$$

$\rightarrow M_\pi^2$ can be written in terms of its derivatives: $f(\boldsymbol{x}) = \frac{1}{3}\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}x_i \quad \boldsymbol{x} \in R^n$

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Summary and Future Steps

- Many observables at NNLO but depending on many correlated LECs!
- On the other hand we found relations among observables not depending of the NNLO constants $\rightarrow$ starting point to perform a fit of the NLO constants

Future steps:

- evaluate the relations at the loop level
- consider also other observables (e.g. F_V, F_π, F_K) to look for more relations
- perform a fit of the L_is with a better treatment of the C_is$\rightarrow$let's start fitting the relations which cancel p^6 contributions first using exp data available, then dispersive analysis and lattice results
- main sources of uncertainties: numerics + not complete exp knowledge
- (not so near future: add isospin breaking corrections)