Vacuum polarization function at $O(\alpha_s^2)$ and $O(\alpha_s^3)$ from Padé approximants

Vicent Mateu

Max-Plank-Institute for Physics

Munich

Taskforce: A. H. Hoang, V.M. and M. S. Zebarjad

Euroflavour, Durham, September 2008

FlaviA net

Charm and bottom mass determination from sum rules

Vacuum polarization function at $O(\alpha_s^2)$ and $O(\alpha_s^3)$ from Padé approximants

Vicent Mateu

Max-Plank-Institute for Physics
Munich

Taskforce: A. H. Hoang, V.M. and M. S. Zebarjad

References: Taskforce: arXiv:0807.4173 [hep-ph] [1]

+ w.i.p. [2]

Euroflavour, Durham, September 2008

Outline

- General remarks on heavy quark masses
  - Different schemes
  - High precision…why?
- How to determine it
  - Perturbative errors
  - Sum rules: need of contour improvement
- Padé approximants
  - Predictions
- Preliminary results

Remarks on heavy quark masses

Confinement ⟶ m_q not physical observable

Parameter in QCD Lagrangian ⟶ formal definition (as strong coupling)

Renormalization and scheme dependent

In general running mass $m(\mu)$ (RGE evolution)

$$m_q^{\mathrm{schemeA}} = m_q^{\mathrm{schemeB}}(1 + \alpha_s + \alpha_s^2 + \ldots)$$

Remarks on heavy quark masses

Confinement ⟶ m_q not physical observable

Parameter in QCD Lagrangian ⟶ formal definition (as strong coupling)

Renormalization and scheme dependent

In general running mass $m(\mu)$ (RGE evolution)

$$m_q^{\text{schemeA}} = m_q^{\text{schemeB}}(1 + \alpha_s + \alpha_s^2 + \ldots)$$

Some schemes better than others…

Pole mass M → Infrared renormalon $O(\Lambda_{QCD})$

$\overline{\text{MS}}$ **scheme:**

- short-distance mass
- standard mass for comparison: $\overline{m}_q(\overline{m}_q)$

And free form renormalon ambiguities

MAX-PLANCK-GESELLSCHAFT

FlaviA net

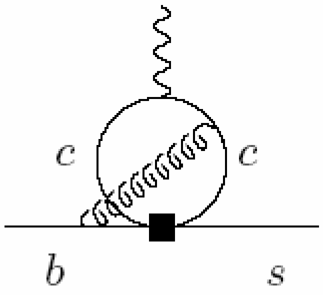
Why high precision?

Strong dependence in flavour processes

Constrains new physics

$B \to X_s \gamma$

strong charm mass (scheme) dependence in NLO matrix elements Misiak, Gambino



$K^+ \to \pi^+ \nu \bar{\nu}$

NNLO QCD computations for charm contributions

See talks by Ceccuci and Gorbahn

Why high precision?

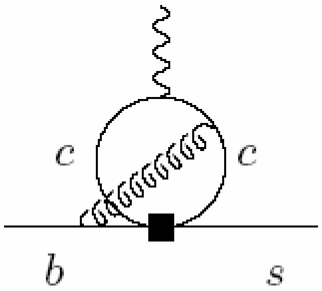
Strong dependence in flavour processes

Constrains new physics

$B \to X_s \gamma$

strong charm mass (scheme) dependence in NLO matrix elements

Misiak, Gambino



$K^+ \to \pi^+ \nu \bar{\nu}$

NNLO QCD computations for charm contributions

See talks by Ceccuci and Gorbahn

$B \to X_u l \nu$

$$\Gamma \sim G_F^2 |V_{\rm ub}|^2 m_{\rm b}^5 \left[1 + \alpha_s + \alpha_s^2 + \ldots\right]$$

$$\frac{\delta V_{\rm ub}}{V_{\rm ub}} \sim 2.5 \frac{\delta m_b}{m_b}$$

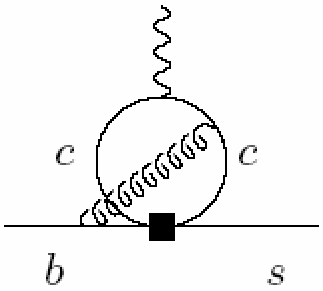
Why high precision?

Strong dependence in flavour processes

Constrains new physics

$B \to X_s \gamma$

strong charm mass (scheme) dependence in NLO matrix elements Misiak, Gambino



$K^+ \to \pi^+ \nu \bar{\nu}$

NNLO QCD computations for charm contributions

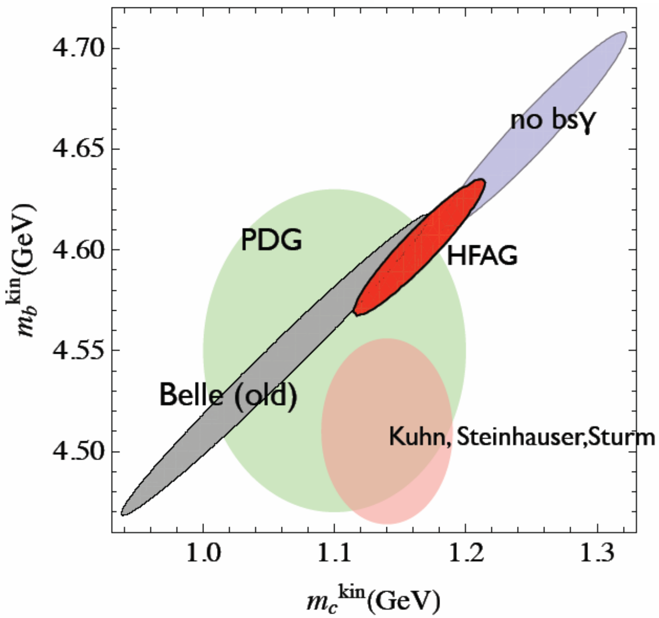
See talks by Ceccuci and Gorbahn

$B \to X_u l \nu$

$$\Gamma \sim G_F^2 |V_{\rm ub}|^2 m_{\rm b}^5 \left[1 + \alpha_s + \alpha_s^2 + \ldots\right]$$

$$\frac{\delta V_{\rm ub}}{V_{\rm ub}} \sim 2.5 \frac{\delta m_b}{m_b}$$

Taken from P. Gambino CKM'08



Determination of m_c

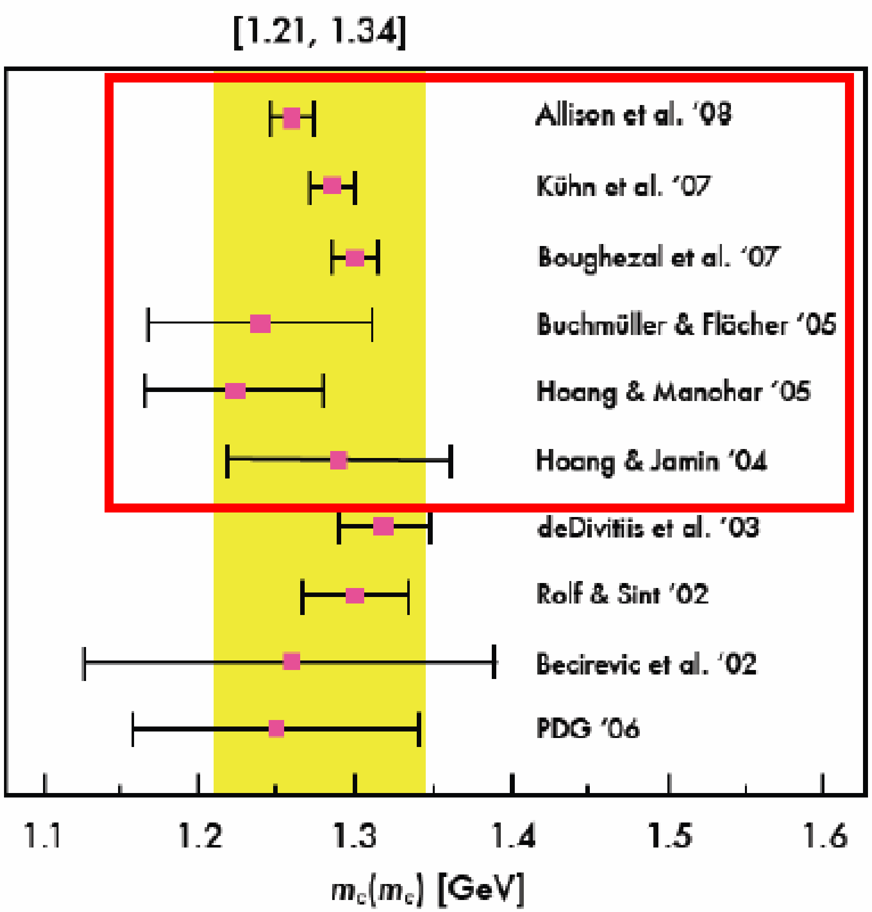
Spectral moments of inclusive B decays (nonrelativistic)

Charmominum sum rules (relativistic)

Lattice

Taken from A. Hoang

Flavor institute CERN 2008



$m_c(m_c)$ [GeV]	method
1.266 ± 0.014	lattice, unquenched, staggered
1.286 ± 0.013	low-momentum sum rules, N^3LO
1.295 ± 0.015	low-momentum sum rules, N^3LO
1.24 ± 0.07	fit to B-decay distribution, $\alpha_s^2\beta_0$
$1.224 \pm 0.017 \pm 0.054$	fit to B-decay data, $\alpha_s^2\beta_0$
1.29 ± 0.07	NNLO moments
1.319 ± 0.028	lattice, quenched
1.301 ± 0.034	lattice, quenched
$1.26 \pm 0.04 \pm 0.12$	lattice, quenched
1.25 ± 0.09	PDG 2006

MAX-PLANCK-GESELLSCHAFT

FlaviA net

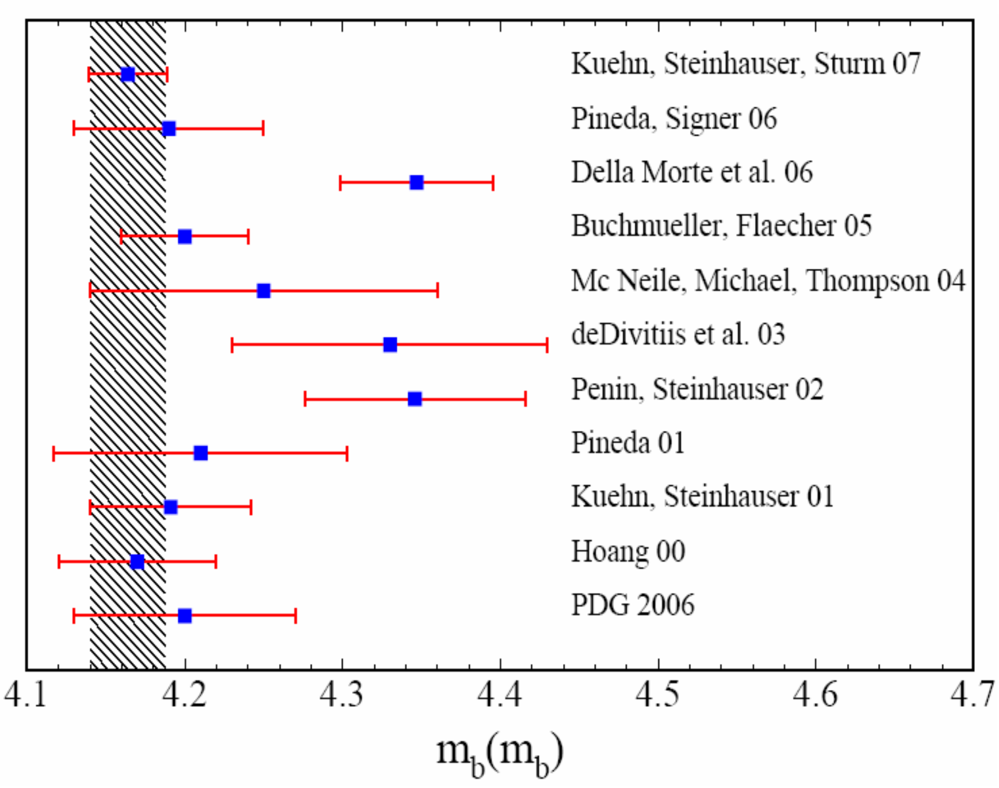
Determination of m_b

Spectral moments of inclusive B decays (nonrelativistic)

Bottomonium sum rules (relativistic)

Lattice

Taken from Kühn et al ’07 [3]



$m_b(m_b)$ (GeV)	Method
4.164 ± 0.025	low-moment sum rules, NNNLO
4.19 ± 0.06	Υ sum rules, NNLL (not complete)
4.347 ± 0.048	lattice (ALPHA), quenched
4.20 ± 0.04	fit to B decay distribution, $\alpha_s^2\beta_0$
$4.25 \pm 0.02 \pm 0.11$	lattice (UKQCD)
4.33 ± 0.10	lattice, quenched
4.346 ± 0.070	$\Upsilon(1S)$, NNNLO
$4.210 \pm 0.090 \pm 0.025$	$\Upsilon(1S)$, NNLO
4.191 ± 0.051	low-moment sum rules, NNLO
4.17 ± 0.05	Υ sum rules, NNLO
4.20 ± 0.07	PDG

Also low-moment sum rules N^3LO Boughezal et al [4]

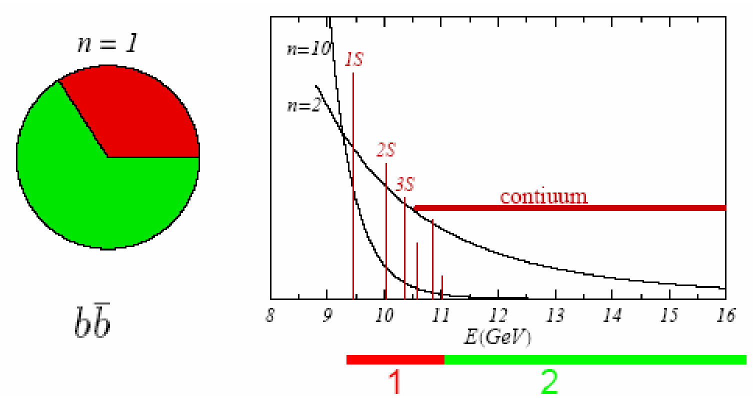
$$m_b(m_b) = 4.205 \pm 0.058$$

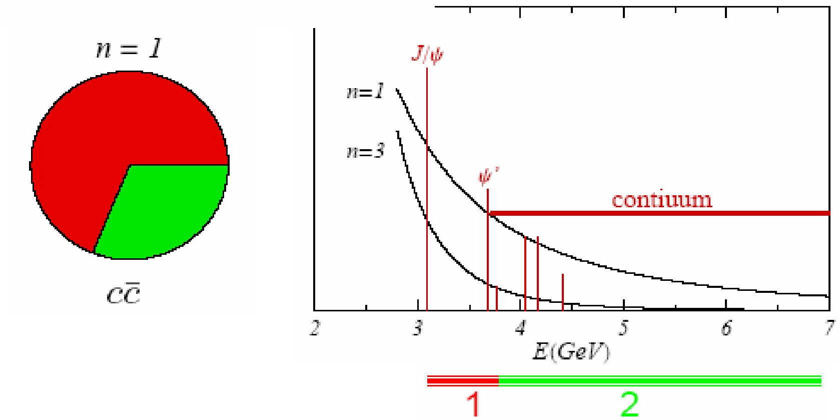
Relativistic sum rules

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma_{\text{pt}}} \Longrightarrow M_n = \int_{4m^2}^{\infty} \frac{\mathrm{d}s}{s^{n+1}} R(s) \overset{z=\frac{q^2}{4m^2}}{=} \frac{1}{(4m^2)^n} \int_1^{\infty} \frac{\mathrm{d}z}{z^{n+1}} R(z)$$

Relativistic sum rules

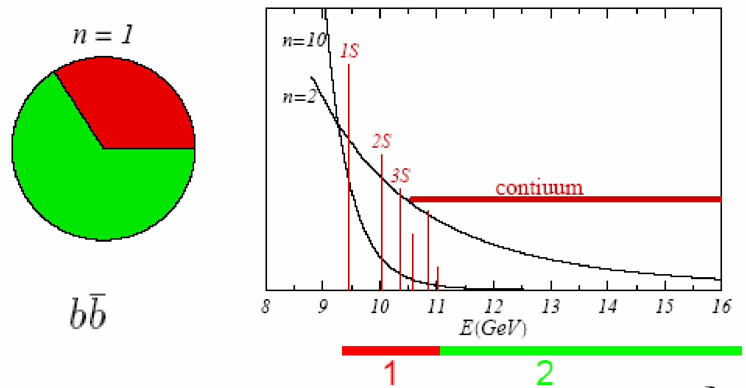
$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma_{\text{pt}}} \quad \Longrightarrow \quad M_n = \int_{4m^2}^{\infty} \frac{ds}{s^{n+1}} R(s) \overset{z=\frac{q^2}{4m^2}}{=} \frac{1}{(4m^2)^n} \int_1^{\infty} \frac{dz}{z^{n+1}} R(z)$$

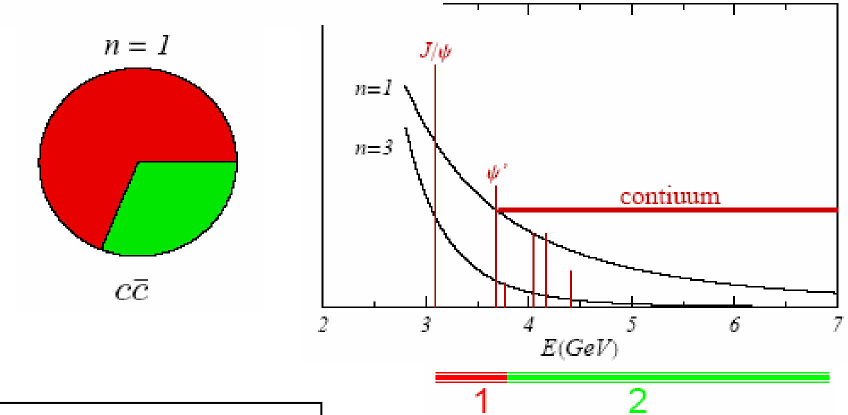




Relativistic sum rules

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma_{\text{pt}}} \quad \Longrightarrow \quad M_n = \int_{4m^2}^{\infty} \frac{ds}{s^{n+1}} R(s) \overset{z=\frac{q^2}{4m^2}}{=} \frac{1}{(4m^2)^n} \int_1^{\infty} \frac{dz}{z^{n+1}} R(z)$$





$$(g_{\mu\nu} q^2 - q_\mu q_\nu)\, \Pi(q^2) = -i \int dx\, e^{iqx} \langle 0 | T\, j_\mu(x) j_\nu(0) | 0 \rangle$$

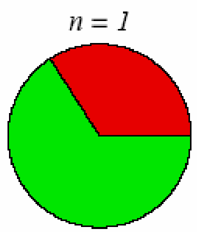
$$R(q^2) = 12\pi Q^2\, \text{Im}\, \Pi(q^2 + i\,0^+)$$

Dispersion relation

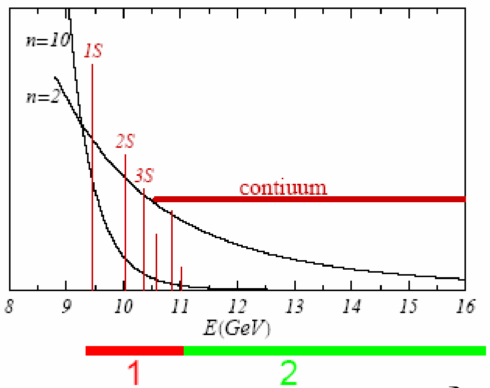
$$\Pi(q^2) - \Pi(0) = \frac{q^2}{12\pi^2 Q^2} \int_{4m^2}^{\infty} dz\, \frac{R(z)}{z(z-q^2)}$$

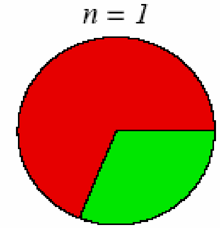
Relativistic sum rules

$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma_{\text{pt}}} \;\Rightarrow\; M_n = \int_{4m^2}^{\infty} \frac{ds}{s^{n+1}} R(s) \overset{z=\frac{q^2}{4m^2}}{=} \frac{1}{(4m^2)^n} \int_1^\infty \frac{dz}{z^{n+1}} R(z)$$

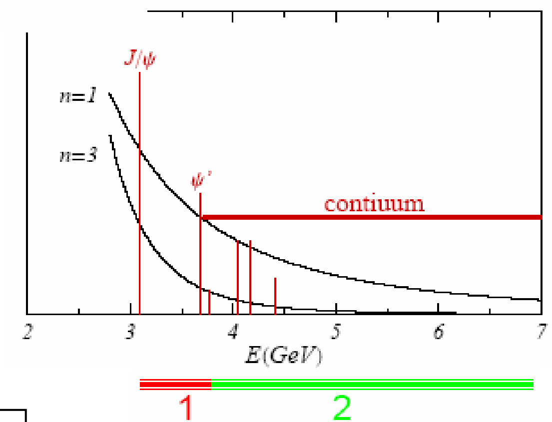


$b\bar{b}$





$c\bar{c}$



$$(g_{\mu\nu} q^2 - q_\mu q_\nu)\, \Pi(q^2) = -i \int dx\, e^{iqx} \langle 0 | T\, j_\mu(x) j_\nu(0) | 0 \rangle$$

$$R(q^2) = 12\,\pi\, Q^2\, \text{Im}\, \Pi(q^2 + i\,0^+)$$

Dispersion relation

$$\Pi(q^2) - \Pi(0) = \frac{q^2}{12\,\pi^2\, Q^2} \int_{4m^2}^{\infty} dz\, \frac{R(z)}{z(z-q^2)}$$

$$\Pi(q^2 \approx 0, m^2) = \frac{1}{12\,\pi^2\, Q_q^2} \sum_{n=1}^{\infty} M_n\, q^{2n} \;\Rightarrow\; M_n = -6\,\pi\, i\, Q^2 \oint ds\, \frac{\Pi(s)}{s^{n+1}}$$

Determination of m_q from sum rules

bottom

Kühn et al ('08) [3] $m(m)=4.149\pm0.020_{exp}\pm0.007_{\alpha}\pm0.002_{\mu}$

Boughezal et al ('08) [4] $4.205\pm0.057_{exp}\pm0.010_{\alpha}\pm0.002_{\mu}$

Maier et al ('08) [5] $4.162\pm0.014_{exp}\pm0.012_{\alpha}\pm0.003_{\mu}$

Only for n = 1 **[3,4]**, 2 **[5]** 3-loops in pert. theory. Updated experimental data

Determination of m_q from sum rules

bottom

Kühn et al ('08) [3]	$m(m)=4.149\pm0.020_{exp}\pm0.007_{\alpha}\pm0.002_{\mu}$
Boughezal et al ('08) [4]	$4.205\pm0.057_{exp}\pm0.010_{\alpha}\pm0.002_{\mu}$
Maier et al ('08) [5]	$4.162\pm0.014_{exp}\pm0.012_{\alpha}\pm0.003_{\mu}$

charm

Kühn et al ('08) [3]	$m(m)=1.286\pm0.009_{exp}\pm0.009_{\alpha}\pm0.002_{\mu}$
Boughez et al ('08) [4]	$1.295\pm0.012_{exp}\pm0.009_{\alpha}\pm0.003_{\mu}$
Maier et al (08) [5]	$1.277\pm0.006_{exp}\pm0.014_{\alpha}\pm0.005_{\mu}$

Only for n = 1 **[3,4]**, 2 **[5]** 3-loops in pert. theory. Updated experimental data

MAX-PLANCK-GESELLSCHAFT

FlaviA net

Determination of m_q from sum rules

bottom

Kühn et al ('08) [3] $m(m)=4.149 \pm 0.020_{exp} \pm 0.007_{\alpha} \pm 0.002_{\mu}$

Boughezal et al ('08) [4] $4.205 \pm 0.057_{exp} \pm 0.010_{\alpha} \pm 0.002_{\mu}$

Maier et al ('08) [5] $4.162 \pm 0.014_{exp} \pm 0.012_{\alpha} \pm 0.003_{\mu}$

Fixed order analysis (s-independent) $\mu \approx m_q^2$

charm

Kühn et al ('08) [3] $m(m)=1.286 \pm 0.009_{exp} \pm 0.009_{\alpha} \pm 0.002_{\mu}$

Boughez et al ('08) [4] $1.295 \pm 0.012_{exp} \pm 0.009_{\alpha} \pm 0.003_{\mu}$

Maier et al (08) [5] $1.277 \pm 0.006_{exp} \pm 0.014_{\alpha} \pm 0.005_{\mu}$

Only for n = 1 [3,4], 2 [5] 3-loops in pert. theory. Updated experimental data

Determination of m_q from sum rules

bottom

Kühn et al ('08) [3] $m(m) = 4.149 \pm 0.020_{exp} \pm 0.007_{\alpha} \pm 0.002_{\mu}$

Boughezal et al ('08) [4] $4.205 \pm 0.057_{exp} \pm 0.010_{\alpha} \pm 0.002_{\mu}$

Maier et al ('08) [5] $4.162 \pm 0.014_{exp} \pm 0.012_{\alpha} \pm 0.003_{\mu}$

Fixed order analysis

(s-independent)

$\mu \approx m_q^2$

charm

Kühn et al ('08) [3] $m(m) = 1.286 \pm 0.009_{exp} \pm 0.009_{\alpha} \pm 0.002_{\mu}$

Boughez et al ('08) [4] $1.295 \pm 0.012_{exp} \pm 0.009_{\alpha} \pm 0.003_{\mu}$

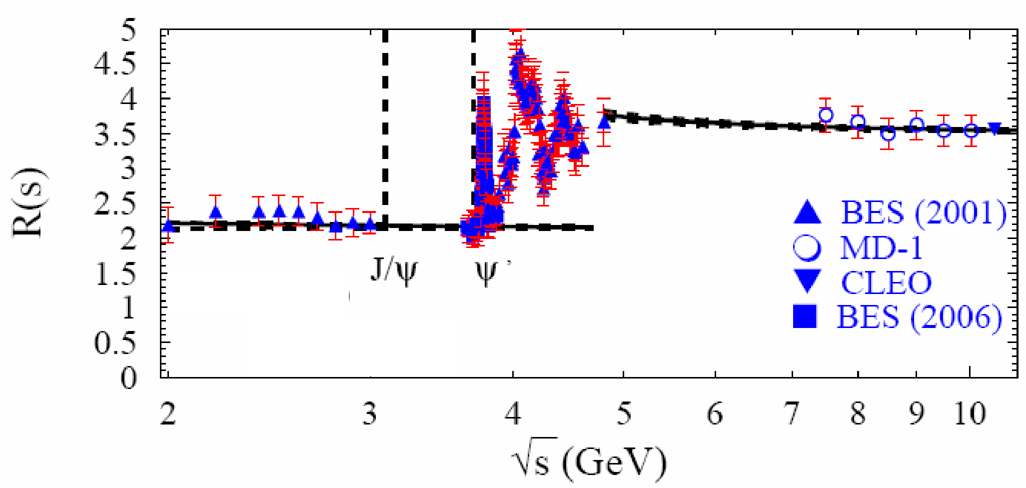
Maier et al (08) [5] $1.277 \pm 0.006_{exp} \pm 0.014_{\alpha} \pm 0.005_{\mu}$

Only for n = 1 [3,4], 2 [5] 3-loops in pert. theory. Updated experimental data

Tiny errors! (underestimated ?)

Need for contour improved analysis

A critical view on sum rules



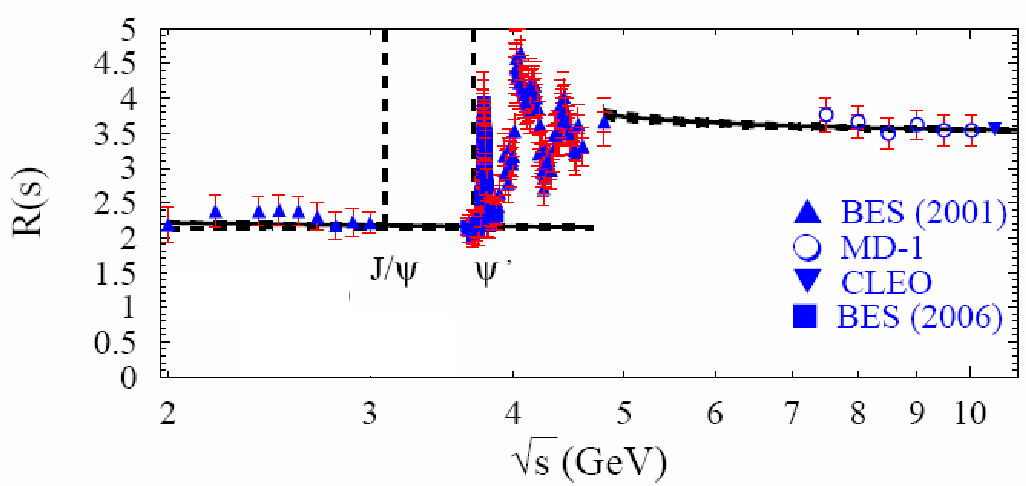
Experimental input

- Resonances region ☺
- Threshold region 😐
- Continuum region ☹

MAX-PLANCK-GESELLSCHAFT

FlaviA net

A critical view on sum rules



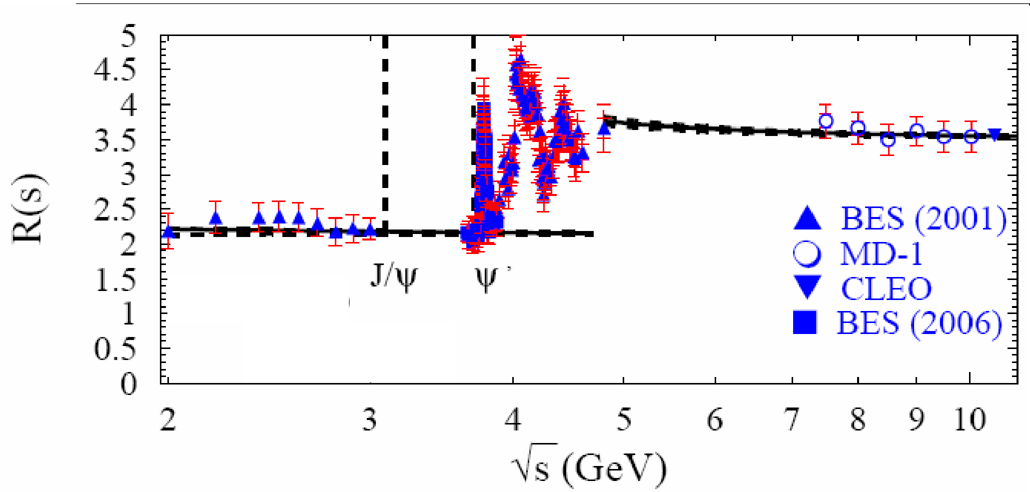
Experimental input

- Resonances region ☺
- Threshold region 😐
- Continuum region ☹

n	$1(10^0)$	$2(10^1)$	$3(10^2)$	$4(10^3)$
Kühn;Maier	0.2166(31)	0.1497(27)	0.1312(27)	0.1249(27)
Boughezal	0.2087(42)	–	–	–

charm

A critical view on sum rules



Experimental input

- Resonances region ☺
- Threshold region 😐
- Continuum region ☹

charm

n	$1(10^0)$	$2(10^1)$	$3(10^2)$	$4(10^3)$
Kühn;Maier	0.2166(31)	0.1497(27)	0.1312(27)	0.1249(27)
Boughezal	0.2087(42)	–	–	–

Large experimental errors because of the continuum

Refs [3,5] substitute by theory, that is why they have small errors

bottom

n	$1(10^3)$	$2(10^5)$	$3(10^7)$	$4(10^9)$
Kühn;Maier	4.601(43)	2.881(37)	2.370(34)	2.178(32)
Boughezal	4.456(121)	–	–	–

MAX-PLANCK-GESELLSCHAFT

FlaviA net

A critical view on sum rules

Determine m(1,2,3 GeV) and then run down to m(m)

$$[4m(\mu)^2]^n M_n^{pert} = \bar{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi}\left(\bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_{m_c}\right) + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left(\bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_{m_c} + \bar{C}_n^{(22)} l_{m_c}^2\right)$$

$$+ \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \left(\bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_{m_c} + \bar{C}_n^{(32)} l_{m_c}^2 + \bar{C}_n^{(33)} l_{m_c}^3\right) + \ldots .$$

$$l_{m_c} \equiv \ln(m_c^2(\mu)/\mu^2)$$

A critical view on sum rules

Determine m(1,2,3 GeV) and then run down to m(m)

$$[4m(\mu)^2]^n M_n^{pert} = \bar{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi}\left(\bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_{m_c}\right) + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left(\bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_{m_c} + \bar{C}_n^{(22)} l_{m_c}^2\right)$$
$$+ \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \left(\bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_{m_c} + \bar{C}_n^{(32)} l_{m_c}^2 + \bar{C}_n^{(33)} l_{m_c}^3\right) + \ldots .$$

$l_{m_c} \equiv \ln(m_c^2(\mu)/\mu^2)$

Let us repeat Boughezal analysis with their input, and fixed order, but determining directly m(m)

$$[4\bar{m}(\bar{m})^2]^n M_n^{pert} = f_n^0 + \frac{\alpha_s(\mu)}{\pi} f_n^1 + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left[f_n^2 + \frac{\beta_0}{4} f_n^1 \log\left(\frac{\mu^2}{\overline{m}^2}\right)\right]$$
$$+ \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \left[f_n^3 + \left(\frac{\beta_0}{2} f_n^2 + \frac{\beta_1}{16} f_n^1\right) \log\left(\frac{\mu^2}{\overline{m}^2}\right) + \frac{\beta_0^2}{16} f_n^1 \log^2\left(\frac{\mu^2}{\overline{m}^2}\right)\right]$$

A critical view on sum rules

Determine m(1,2,3 GeV) and then run down to m(m)

$$[4m(\mu)^2]^n M_n^{pert} = \bar{C}_n^{(0)} + \frac{\alpha_s(\mu)}{\pi}\left(\bar{C}_n^{(10)} + \bar{C}_n^{(11)} l_{m_c}\right) + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left(\bar{C}_n^{(20)} + \bar{C}_n^{(21)} l_{m_c} + \bar{C}_n^{(22)} l_{m_c}^2\right)$$
$$+ \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \left(\bar{C}_n^{(30)} + \bar{C}_n^{(31)} l_{m_c} + \bar{C}_n^{(32)} l_{m_c}^2 + \bar{C}_n^{(33)} l_{m_c}^3\right) + \ldots .$$

$l_{m_c} \equiv \ln(m_c^2(\mu)/\mu^2)$ Let us repeat Boughezal analysis with their input, and fixed order, but determining directly m(m)

$$[4\bar{m}(\bar{m})^2]^n M_n^{pert} = f_n^0 + \frac{\alpha_s(\mu)}{\pi} f_n^1 + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left[f_n^2 + \frac{\beta_0}{4} f_n^1 \log\left(\frac{\mu^2}{\bar{m}^2}\right)\right]$$
$$+ \left(\frac{\alpha_s(\mu)}{\pi}\right)^3 \left[f_n^3 + \left(\frac{\beta_0}{2} f_n^2 + \frac{\beta_1}{16} f_n^1\right)\log\left(\frac{\mu^2}{\bar{m}^2}\right) + \frac{\beta_0^2}{16} f_n^1 \log^2\left(\frac{\mu^2}{\bar{m}^2}\right)\right]$$

$$1.286 \pm 0.009_{exp} \pm 0.011_{\mu} \pm 0.012_{exp} \longleftrightarrow 1.295 \pm 0.009_{exp} \pm 0.003_{\mu} \pm 0.012_{exp}$$

Different central values, different μ variation. Similarly for bottom.

Contour improved analysis

First applied to hadronic tau decays Liberder Pich ('92) (Previous speaker's talk)

Now μ depends on s $\rightarrow$ rearangement of higher order contributions

Similar to fixed order $\longrightarrow$ $\mu^2 = \xi^2 M^2 z$

Reweighs threshold versus continuum effects $\longrightarrow$ $\mu^2 = \xi^2 M^2 (1-z)$

$$\left.\begin{array}{l} \mu^2 = \xi^2 M^2 z \\ \mu^2 = \xi^2 M^2 (1-z) \end{array}\right\} \rightarrow z = \frac{q^2}{4m^2}$$

2 - loops

Hoang, Jamin (2004) [6]

Contour improved analysis

First applied to hadronic tau decays Liberder Pich ('92) (Previous speaker's talk)

Now μ depends on s $\rightarrow$ rearangement of higher order contributions

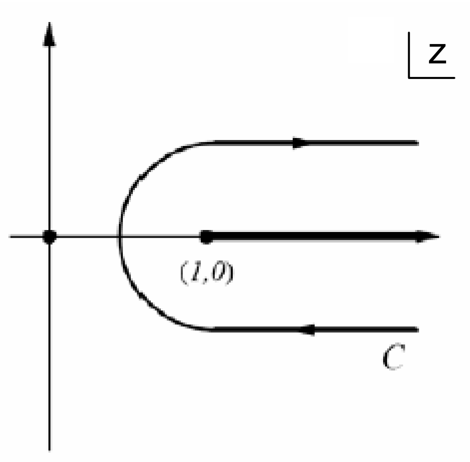
Similar to fixed order $\longrightarrow$ $\mu^2 = \xi^2 M^2 z$

Reweighs threshold versus continuum effects $\longrightarrow$ $\mu^2 = \xi^2 M^2 (1-z)$

$$\left.\begin{array}{l}\mu^2 = \xi^2 M^2 z \\ \mu^2 = \xi^2 M^2 (1-z)\end{array}\right\} \rightarrow z = \frac{q^2}{4m^2}$$

2 - loops
Hoang, Jamin
(2004) [6]

Calculations $\longrightarrow$ more convenient through the vacuum polarization function



- Solve the RGE for strong coupling in the complex plane
- Need full dependence of polarization function $\longrightarrow$ Padé
- Difference between two methods much higher that individual errors due to μ variation

Construction of $\Pi^{(2,3)}$

$$\Pi = \Pi^{(0)} + C_F\left(\frac{\alpha_S}{\pi}\right)\Pi^{(1)} + \left(\frac{\alpha_S}{\pi}\right)^2\Pi^{(2)} + \left(\frac{\alpha_S}{\pi}\right)^3\Pi^{(3)} + \cdots$$

For $\Pi^{(0)}$ and $\Pi^{(1)}$ the full q^2 and mas dependence is exactly known

Construction of $\Pi^{(2,3)}$

$$\Pi = \Pi^{(0)} + C_F \left(\frac{\alpha_S}{\pi}\right)\Pi^{(1)} + \left(\frac{\alpha_S}{\pi}\right)^2 \Pi^{(2)} + \left(\frac{\alpha_S}{\pi}\right)^3 \Pi^{(3)} + \cdots$$

For $\Pi^{(0)}$ and $\Pi^{(1)}$ the full q^2 and mas dependence is exactly known

30 moments known Maier et al (2008)

↑

For $\Pi^{(2)}$ and $\Pi^{(3)}$ →

- $q^2 \sim 0$ (fixed order moments) — Regular (no logs)
- $q^2 \sim \infty$ (expansion for small mass) — $\log^n(-4z)$
- $q^2 \sim 4m^2$ (threshold expansion) → $\log^n(1-z)$ + Coulomb singularity

↓

Only two moments known 1st [3,4], 2nd [5]

Construction of $\Pi^{(2,3)}$

$$\Pi = \Pi^{(0)} + C_F \left(\frac{\alpha_S}{\pi}\right)\Pi^{(1)} + \left(\frac{\alpha_S}{\pi}\right)^2 \Pi^{(2)} + \left(\frac{\alpha_S}{\pi}\right)^3 \Pi^{(3)} + \cdots$$

For $\Pi^{(0)}$ and $\Pi^{(1)}$ the full q^2 and mas dependence is exactly known

30 moments known Maier et al (2008)

For $\Pi^{(2)}$ and $\Pi^{(3)}$ →

- $q^2 \sim 0$ (fixed order moments) — Regular (no logs)
- $q^2 \sim \infty$ (expansion for small mass) — $\log^n(-4z)$
- $q^2 \sim 4m^2$ (threshold expansion) → $\log^n(1-z)$ + Coulomb singularity

Only two moments known 1st [3,4], 2nd [5]

The three regimes can be matched into a single function ⟶ Padé

$\Pi^{(2)}$ → Chetyrkin et al ('96)

$\Pi^{(2)}$, $\Pi^{(3)}$ → Taskforce ('08), [1]

Prediction of moments and constants

Renders R-ratio at all energies

Reliable estimation of errors

Construction of $\Pi^{(2,3)}$

$$\Pi^{(2,3)}(z) = \Pi^{(2,3)}_{\rm reg}(z) + \Pi^{(2,3)}_{\log}(z)$$

Construction of $\Pi^{(2,3)}$

$$\Pi^{(2,3)}(z) = \Pi^{(2,3)}_{\text{reg}}(z) + \Pi^{(2,3)}_{\text{log}}(z)$$

→ Accounts for logs at threshold and infinity

Construction of $\Pi^{(2,3)}$

$$\Pi^{(2,3)}(z) = \Pi^{(2,3)}_{\rm reg}(z) + \Pi^{(2,3)}_{\rm log}(z)$$

$\Pi^{(2,3)}_{\rm log}(z)$ → Accounts for logs at threshold and infinity

$\Pi^{(2,3)}_{\rm reg}(z)$ → Will be approximated by a Padé

Construction of $\Pi^{(2,3)}$

$$\Pi^{(2,3)}(z) = \Pi^{(2,3)}_{\text{reg}}(z) + \Pi^{(2,3)}_{\log}(z)$$

$\Pi^{(2,3)}_{\log}(z)$ → Accounts for logs at threshold and infinity

$\Pi^{(2,3)}_{\text{reg}}(z)$ → Will be approximated by a Padé

General form of a Padé →

$$P_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{1 + \sum_{j=1}^{m} b_j x^j}$$

Construction of $\Pi^{(2,3)}$

$$\Pi^{(2,3)}(z) = \Pi^{(2,3)}_{\text{reg}}(z) + \Pi^{(2,3)}_{\text{log}}(z)$$

$\Pi^{(2,3)}_{\text{log}}$ → Accounts for logs at threshold and infinity

$\Pi^{(2,3)}_{\text{reg}}$ → Will be approximated by a Padé

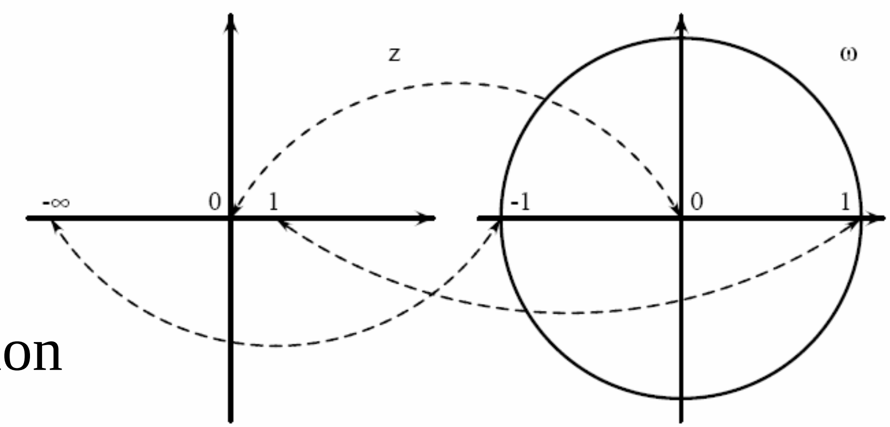
General form of a Padé →

$$P_{n,m}(x) = \frac{\sum_{i=0}^{n} a_i x^i}{1 + \sum_{j=1}^{m} b_j x^j}$$

It would be unwise using $P_{m,n}(z)$, we better do a conformal mapping

$$\omega = \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}$$

and use $P_{m,n}(\omega)$



ω has a cut for z > 1, much as Π

so the Padé contributes to the R-function

Construction of $\Pi_{\log}^{(2,3)}$

$$\Pi_{\log}^{(2,3)}(z) = \Pi_{\mathrm{thr}}^{(2,3)}(z) + \Pi_{\mathrm{inf}}^{(2,3)}(z) + \Pi_{\mathrm{zero}}^{(2,3)}(z)$$

Construction of $\Pi^{(2,3)}_{\log}$

$$\Pi^{(2,3)}_{\log}(z) = \Pi^{(2,3)}_{\rm thr}(z) + \Pi^{(2,3)}_{\rm inf}(z) + \Pi^{(2,3)}_{\rm zero}(z)$$

up to $1/z^2$

$\ln^m(1-z)$	$[\Pi^{(1)}(z)]^m$
$(1-z)^{-n/2}\ln^m(1-z)$	$[G(z)]^n[\Pi^{(1)}(z)]^m$
$(1-z)^{n/2}\ln^m(1-z)$	$(1-z)^n[G(z)]^n[\Pi^{(1)}(z)]^m$

$$G(z) = \frac{2\,u\,\ln u}{u^2-1}, \quad \text{with} \quad u \equiv \frac{\sqrt{1-1/z}-1}{\sqrt{1-1/z}+1}$$

G(z) is the scalar equal-mass one-loop function

Construction of $\Pi^{(2,3)}_{\log}$

$$\Pi^{(2,3)}_{\log}(z) = \Pi^{(2,3)}_{\rm thr}(z) + \Pi^{(2,3)}_{\rm inf}(z) + \Pi^{(2,3)}_{\rm zero}(z)$$

up to $1/z^2$

$\ln^m(1-z)$	$[\Pi^{(1)}(z)]^m$
$(1-z)^{-n/2}\ln^m(1-z)$	$[G(z)]^n[\Pi^{(1)}(z)]^m$
$(1-z)^{n/2}\ln^m(1-z)$	$(1-z)^n[G(z)]^n[\Pi^{(1)}(z)]^m$

up to $O(v)$ for $\Pi^{(2)}$ and $O(v^0)$ for $\Pi^{(3)}$

$\ln^n(-4z)$	$(1-z)^n\,[G(z)]^n$
$\frac{1}{z}\ln^n(-4z)\,,(n>1)$	$(1-z)^{n-1}\,[G(z)]^n$
$\frac{1}{z}\ln(-4z)$	$\frac{1-z}{z}\,G(z)$
$\frac{1}{z^2}\ln(-4z)$	$\frac{1-z}{z^2}\,G(z)$
$\frac{1}{z^2}\ln^2(-4z)$	$\frac{1-z}{z}\,[G(z)]^2$
$\frac{1}{z^2}\ln^3(-4z)$	$\frac{(1-z)^2}{z}\,[G(z)]^3$
$\frac{1}{z^2}\ln^4(-4z)$	$(1-z)^2\,[G(z)]^4$

$$G(z) = \frac{2\,u\ln u}{u^2-1}\,, \quad \text{with} \quad u \equiv \frac{\sqrt{1-1/z}-1}{\sqrt{1-1/z}+1}$$

G(z) is the scalar equal-mass one-loop function

Construction of $\Pi^{(2,3)}_{\text{log}}$

$$\Pi^{(2,3)}_{\text{log}}(z) = \Pi^{(2,3)}_{\text{thr}}(z) + \Pi^{(2,3)}_{\text{inf}}(z) + \Pi^{(2,3)}_{\text{zero}}(z) \longrightarrow S_0^{(2,3)} + \frac{S_1^{(2,3)}}{z} + \frac{S_2^{(2,3)}}{z^2}$$

up to $1/z^2$

$\ln^m(1-z)$	$[\Pi^{(1)}(z)]^m$
$(1-z)^{-n/2}\ln^m(1-z)$	$[G(z)]^n[\Pi^{(1)}(z)]^m$
$(1-z)^{n/2}\ln^m(1-z)$	$(1-z)^n[G(z)]^n[\Pi^{(1)}(z)]^m$

up to $O(v)$ for $\Pi^{(2)}$ and $O(v^0)$ for $\Pi^{(3)}$

$\ln^n(-4z)$	$(1-z)^n\,[G(z)]^n$
$\frac{1}{z}\ln^n(-4z)\,,(n>1)$	$(1-z)^{n-1}\,[G(z)]^n$
$\frac{1}{z}\ln(-4z)$	$\frac{1-z}{z}\,G(z)$
$\frac{1}{z^2}\ln(-4z)$	$\frac{1-z}{z^2}\,G(z)$
$\frac{1}{z^2}\ln^2(-4z)$	$\frac{1-z}{z}\,[G(z)]^2$
$\frac{1}{z^2}\ln^3(-4z)$	$\frac{(1-z)^2}{z}\,[G(z)]^3$
$\frac{1}{z^2}\ln^4(-4z)$	$(1-z)^2\,[G(z)]^4$

$$G(z) = \frac{2\,u\ln u}{u^2-1}\,, \quad \text{with} \quad u \equiv \frac{\sqrt{1-1/z}-1}{\sqrt{1-1/z}+1}$$

G(z) is the scalar equal-mass one-loop function

Construction of $\Pi^{(2,3)}_{\log}$

$$\Pi^{(2,3)}_{\log}(z) = \Pi^{(2,3)}_{\rm thr}(z) + \Pi^{(2,3)}_{\rm inf}(z) + \Pi^{(2,3)}_{\rm zero}(z) \longrightarrow S_0^{(2,3)} + \frac{S_1^{(2,3)}}{z} + \frac{S_2^{(2,3)}}{z^2}$$

up to $1/z^2$

$\ln^m(1-z)$	$[\Pi^{(1)}(z)]^m$
$(1-z)^{-n/2}\ln^m(1-z)$	$[G(z)]^n[\Pi^{(1)}(z)]^m$
$(1-z)^{n/2}\ln^m(1-z)$	$(1-z)^n[G(z)]^n[\Pi^{(1)}(z)]^m$

Higher logs tuned by $\frac{1+a\,z}{z}$

Handle for estimating errors!

up to $O(v)$ for $\Pi^{(2)}$ and $O(v^0)$ for $\Pi^{(3)}$

$\ln^n(-4z)$	$(1-z)^n\,[G(z)]^n$
$\frac{1}{z}\ln^n(-4z)\,,(n>1)$	$(1-z)^{n-1}\,[G(z)]^n$
$\frac{1}{z}\ln(-4z)$	$\frac{1-z}{z}\,G(z)$
$\frac{1}{z^2}\ln(-4z)$	$\frac{1-z}{z^2}\,G(z)$
$\frac{1}{z^2}\ln^2(-4z)$	$\frac{1-z}{z}\,[G(z)]^2$
$\frac{1}{z^2}\ln^3(-4z)$	$\frac{(1-z)^2}{z}\,[G(z)]^3$
$\frac{1}{z^2}\ln^4(-4z)$	$(1-z)^2\,[G(z)]^4$

The choice of "log-removers"

is not unique

$$G(z) = \frac{2\,u\ln u}{u^2-1}\,,\quad \text{with}\quad u \equiv \frac{\sqrt{1-1/z}-1}{\sqrt{1-1/z}+1}$$

G(z) is the scalar equal-mass one-loop function

Construction of $\Pi^{(2,3)}_{reg}$

Leading Coulomb singularity has no log(1 - z)

$\rightarrow \Pi^{(2,3)}_{reg}$

$\rightarrow \Pi^{(2,3)}_{\log}$

New handle for estimating uncertainties

Construction of $\Pi^{(2,3)}_{reg}$

Leading Coulomb singularity has no log(1 - z) → $\Pi^{(2,3)}_{reg}$, $\Pi^{(2,3)}_{\log}$

New handle for estimating uncertainties

$\Pi^{(2,3)}_{reg}$ Contains all known moments plus

non-logarithmic information at threshold and infinity

If unknown, it can be predicted (unknown moments as well)

Construction of $\Pi_{reg}^{(2,3)}$

Leading Coulomb singularity has no log(1 - z) → $\Pi_{reg}^{(2,3)}$, $\Pi_{\log}^{(2,3)}$

New handle for estimating uncertainties

$\Pi_{reg}^{(2,3)}$ Contains all known moments plus non-logarithmic information at threshold and infinity

If unknown, it can be predicted (unknown moments as well)

For example:

$$\sim \begin{cases} \sqrt{1-z} & \text{threshold} \\ z & \text{infinity} \end{cases}$$

$$P(\omega) = \frac{1-\omega}{(1+\omega)^2}\left[\Pi_{\rm reg}^{(2)}(z) - \Pi_{\rm reg}^{(2)}(-\infty)\right] \longrightarrow \Pi^{(2)}(z) = \frac{(1+\omega)^2}{1-\omega}P(\omega) - P(0) + \Pi_{\log}^{(2)}(z)$$

Construction of $\Pi_{reg}^{(2,3)}$

Leading Coulomb singularity has no log(1 - z) → $\Pi_{reg}^{(2,3)}$, $\Pi_{\log}^{(2,3)}$

New handle for estimating uncertainties

$\Pi_{reg}^{(2,3)}$ Contains all known moments plus non-logarithmic information at threshold and infinity

If unknown, it can be predicted (unknown moments as well)

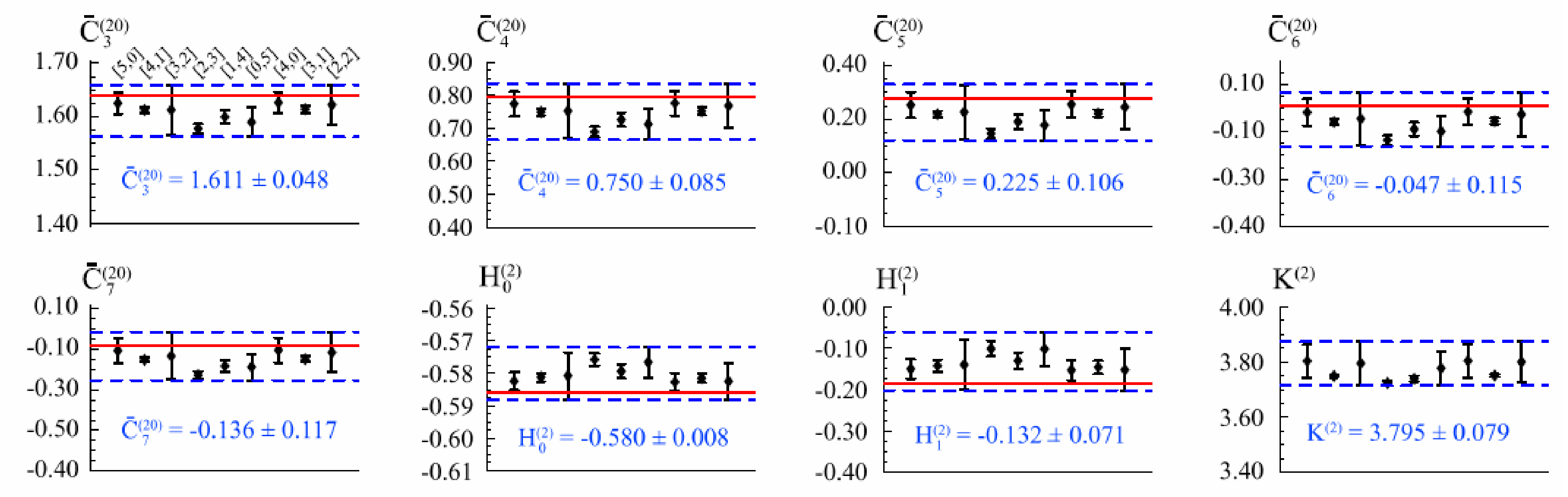
For example:

$$\sim \begin{cases} \sqrt{1-z} & \text{threshold} \\ z & \text{infinity} \end{cases}$$

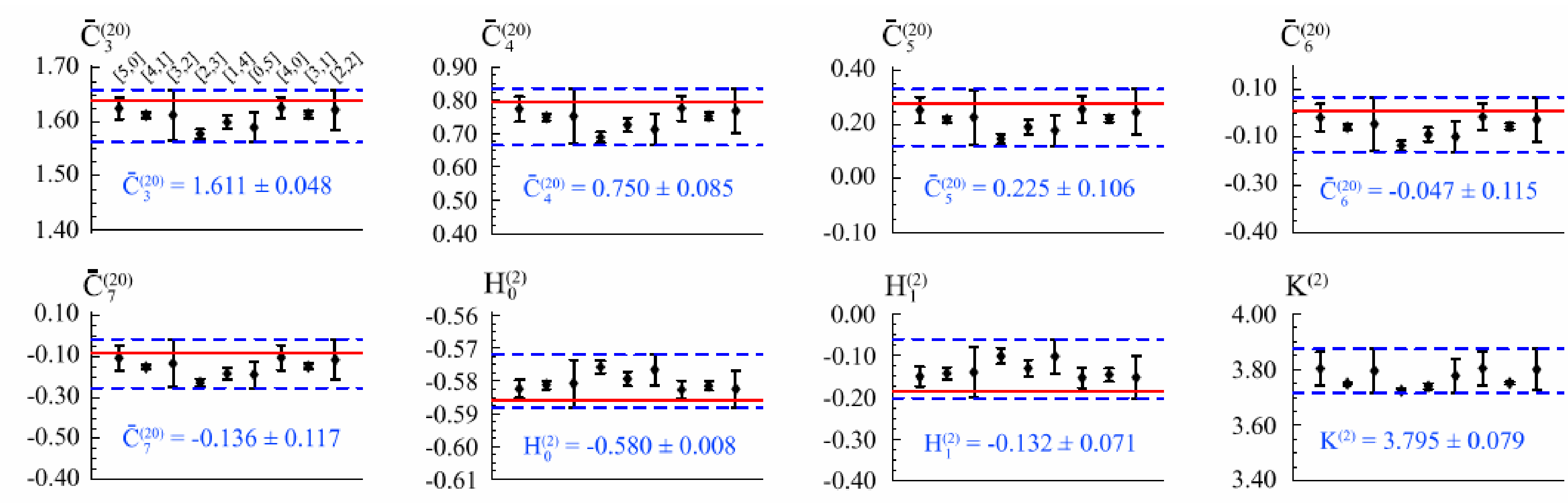
$$P(\omega) = \frac{1-\omega}{(1+\omega)^2}\left[\Pi_{\rm reg}^{(2)}(z) - \Pi_{\rm reg}^{(2)}(-\infty)\right] \longrightarrow \Pi^{(2)}(z) = \frac{(1+\omega)^2}{1-\omega}P(\omega) - P(0) + \Pi_{\log}^{(2)}(z)$$

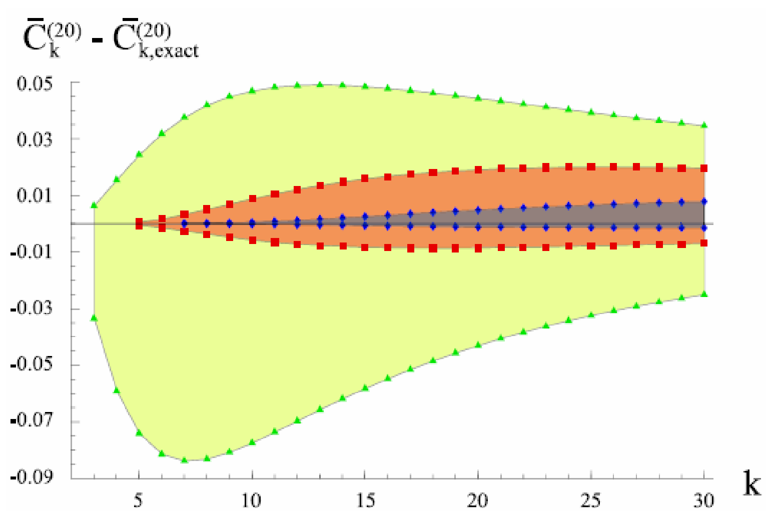
We also demand that P(ω) has no $1/z^{(2n+1)/2}$ terms at infinity up to the order considered

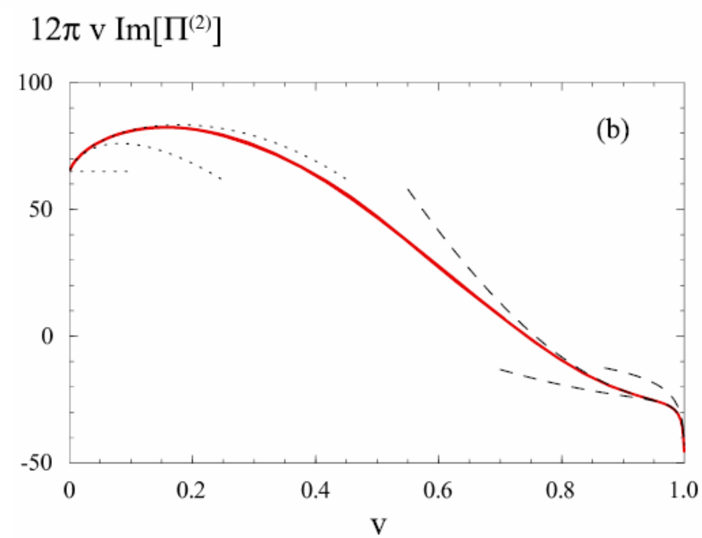
Padé predictions for $\Pi^{(2)}$



Padé predictions for $\Pi^{(2)}$



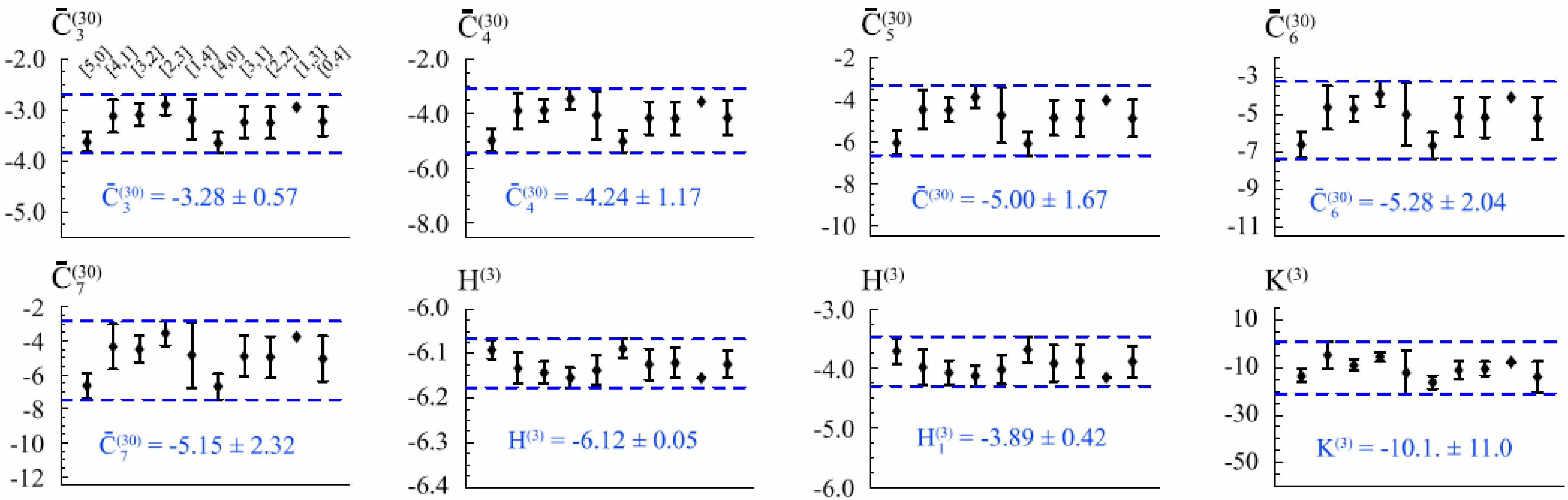




MAX-PLANCK-GESELLSCHAFT

FlaviA net

Padé predictions for $\Pi^{(3)}$



	$n_f = 4$	$n_f = 5$
$\bar{C}_1^{(30)}$	-5.6404	-7.7624
$\bar{C}_2^{(30)}$	-3.4937	-2.6438
$\bar{C}_3^{(30)}$	-3.279 ± 0.573	-1.457 ± 0.579
$\bar{C}_4^{(30)}$	-4.238 ± 1.171	-1.935 ± 1.201
$\bar{C}_5^{(30)}$	-4.996 ± 1.666	-2.507 ± 1.732
$\bar{C}_6^{(30)}$	-5.280 ± 2.045	-2.809 ± 2.150
$\bar{C}_7^{(30)}$	-5.151 ± 2.321	-2.847 ± 2.467
$H_0^{(3)}$	-6.122 ± 0.054	-4.989 ± 0.053
$H_1^{(3)}$	-3.885 ± 0.417	-3.180 ± 0.405
$K^{(3)}$	-10.09 ± 11.00	-5.97 ± 10.09

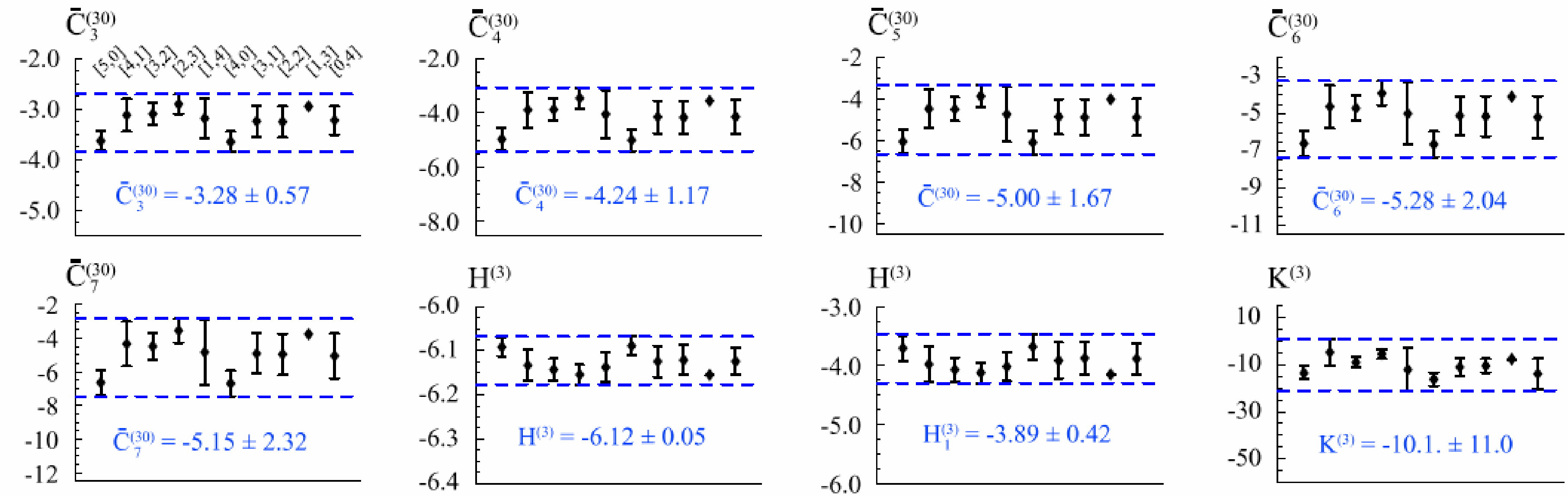
Agreement with Chetyrkin !

$$H_1^{(3)} = -4.33306$$

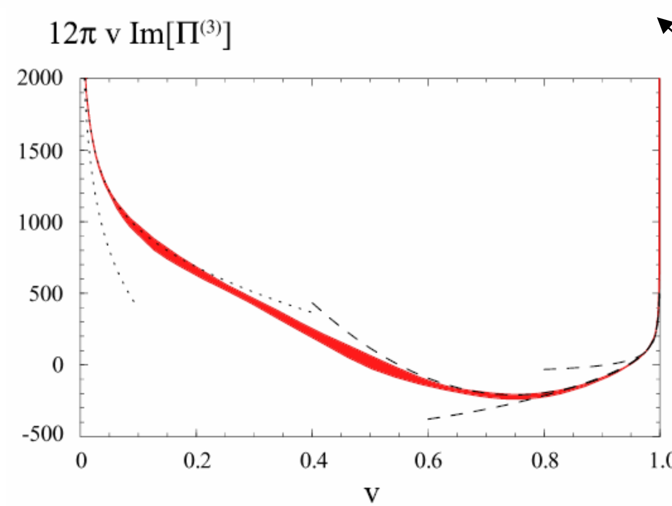
MAX-PLANCK-GESELLSCHAFT

FlaviA net

Padé predictions for $\Pi^{(3)}$



	$n_f = 4$	$n_f = 5$
$\bar{C}_1^{(30)}$	-5.6404	-7.7624
$\bar{C}_2^{(30)}$	-3.4937	-2.6438
$\bar{C}_3^{(30)}$	-3.279 ± 0.573	-1.457 ± 0.579
$\bar{C}_4^{(30)}$	-4.238 ± 1.171	-1.935 ± 1.201
$\bar{C}_5^{(30)}$	-4.996 ± 1.666	-2.507 ± 1.732
$\bar{C}_6^{(30)}$	-5.280 ± 2.045	-2.809 ± 2.150
$\bar{C}_7^{(30)}$	-5.151 ± 2.321	-2.847 ± 2.467
$H_0^{(3)}$	-6.122 ± 0.054	-4.989 ± 0.053
$H_1^{(3)}$	-3.885 ± 0.417	-3.180 ± 0.405
$K^{(3)}$	-10.09 ± 11.00	-5.97 ± 10.09



Agreement with Chetyrkin !

$$H_1^{(3)} = -4.33306$$

Preliminary (!) results: charm

It constitutes an update of Hoang Jamin [6]. Only the theoretical input is updated

We use more precise two loop results, with the updated Padé's

We use new results for three loop, from Padé approximants

Experimental input still not updated! Do not focus on central values.

Preliminary (!) results: charm

It constitutes an update of Hoang Jamin [6]. Only the theoretical input is updated

We use more precise two loop results, with the updated Padé's

We use new results for three loop, from Padé approximants

Experimental input still not updated! Do not focus on central values.

two loops [6] $1.283 \pm 0.040_{\exp} \pm 0.016_{\mu} \pm 0.014_{method} \pm 0.001_{n}$ $m_c\ (\,\delta \exp + \delta\mu\,)$

n	1	2	3	4
Method 1	1.277(71+23)	1.272(42+17)	1.267(36+11)	1.264(37+8)
Method 2	1.268(68+24)	1.264(41+15)	1.263(35+10)	1.260(36+7)
Method 3	1.277(74+24)	1.290(45+20)	1.295(37+18)	1.295(38+14)
Convined	$1.274(69+24_{\mu}+5_{method})$	1.282(43+18+13)	1.284(36+14+16)	1.282(38+10+18)

Preliminary (!) results: charm

It constitutes an update of Hoang Jamin [6]. Only the theoretical input is updated

We use more precise two loop results, with the updated Padé's

We use new results for three loop, from Padé approximants

Experimental input still not updated! Do not focus on central values.

two loops [6] $1.283 \pm 0.040_{\exp} \pm 0.016_{\mu} \pm 0.014_{method} \pm 0.001_{n}$ $m_c\,(\,\delta \exp + \delta\mu\,)$

n	1	2	3	4
Method 1	1.277(71+23)	1.272(42+17)	1.267(36+11)	1.264(37+8)
Method 2	1.268(68+24)	1.264(41+15)	1.263(35+10)	1.260(36+7)
Method 3	1.277(74+24)	1.290(45+20)	1.295(37+18)	1.295(38+14)
Convined	$1.274(69+24_{\mu}+5_{method})$	1.282(43+18+13)	1.284(36+14+16)	1.282(38+10+18)

three loops [w.i.p.] $1.279 \pm 0.040_{\exp} \pm 0.003_{\mu} \pm 0.012_{method} \pm 0_{n}$

n	1	2	3	4
Method 1	1.285(71+11)	1.277(42+9)	1.271(36+6)	1.264(37+4)
Method 2	1.272(68+8)	1.267(41+5)	1.264(35+3)	1.259(36+2)
Method 3	1.278(74+6)	1.287(45+4)	1.290(37+2)	1.289(38+1)
Convined	1.282(73+7+6)	1.279(43+4+10)	1.279(36+2+13)	1.274(37+1+15)

MAX-PLANCK-GESELLSCHAFT

FlaviA net

Preliminary (!) results: charm

It constitutes an update of Hoang Jamin [6]. Only the theoretical input is updated

We use more precise two loop results, with the updated Padé's

We use new results for three loop, from Padé approximants

Experimental input still not updated! Do not focus on central values.

two loops [6] $1.283 \pm 0.040_{\exp} \pm 0.016_{\mu} \pm 0.014_{method} \pm 0.001_{n}$

$m_c(\delta \exp + \delta\mu)$

Central values for each method approach

n	1	2	3	4
Method 1	1.277(71+23)	1.272(42+17)	1.267(36+11)	1.264(37+8)
Method 2	1.268(68+24)	1.264(41+15)	1.263(35+10)	1.260(36+7)
Method 3	1.277(74+24)	1.290(45+20)	1.295(37+18)	1.295(38+14)
Convined	$1.274(69+24_{\mu}+5_{method})$	1.282(43+18+13)	1.284(36+14+16)	1.282(38+10+18)

three loops [w.i.p.] $1.279 \pm 0.040_{\exp} \pm 0.003_{\mu} \pm 0.012_{method} \pm 0_{n}$

n	1	2	3	4
Method 1	1.285(71+11)	1.277(42+9)	1.271(36+6)	1.264(37+4)
Method 2	1.272(68+8)	1.267(41+5)	1.264(35+3)	1.259(36+2)
Method 3	1.278(74+6)	1.287(45+4)	1.290(37+2)	1.289(38+1)
Convined	1.282(73+7+6)	1.279(43+4+10)	1.279(36+2+13)	1.274(37+1+15)

Preliminary (!) results: charm

It constitutes an update of Hoang Jamin [6]. Only the theoretical input is updated

We use more precise two loop results, with the updated Padé's

We use new results for three loop, from Padé approximants

Experimental input still not updated! Do not focus on central values.

has decreased factor 2

$$m_c \,(\,\delta \exp + \delta\mu\,)$$

Central values for each method approach

two loops [6] $1.283 \pm 0.040_{\exp} \pm 0.016_{\mu} \pm 0.014_{method} \pm 0.001_{n}$

n	1	2	3	4
Method 1	1.277(71+23)	1.272(42+17)	1.267(36+11)	1.264(37+8)
Method 2	1.268(68+24)	1.264(41+15)	1.263(35+10)	1.260(36+7)
Method 3	1.277(74+24)	1.290(45+20)	1.295(37+18)	1.295(38+14)
Convined	$1.274(69+24_{\mu}+5_{method})$	1.282(43+18+13)	1.284(36+14+16)	1.282(38+10+18)

three loops [w.i.p.] $1.279 \pm 0.040_{\exp} \pm 0.003_{\mu} \pm 0.012_{method} \pm 0_{n}$

n	1	2	3	4
Method 1	1.285(71+11)	1.277(42+9)	1.271(36+6)	1.264(37+4)
Method 2	1.272(68+8)	1.267(41+5)	1.264(35+3)	1.259(36+2)
Method 3	1.278(74+6)	1.287(45+4)	1.290(37+2)	1.289(38+1)
Convined	1.282(73+7+6)	1.279(43+4+10)	1.279(36+2+13)	1.274(37+1+15)

Preliminary (!) results: charm

It constitutes an update of Hoang Jamin [6]. Only the theoretical input is updated

We use more precise two loop results, with the updated Padé's

We use new results for three loop, from Padé approximants

Experimental input still not updated! Do not focus on central values.

$m_c(\delta\exp + \delta\mu)$

$\delta\mu$ has decreased factor 2

Central values for each method approach

$\delta\exp$ will decreased substantially when including updated experimental data

two loops [6] $1.283 \pm 0.040_{\exp} \pm 0.016_{\mu} \pm 0.014_{method} \pm 0.001_{n}$

n	1	2	3	4
Method 1	1.277(71+23)	1.272(42+17)	1.267(36+11)	1.264(37+8)
Method 2	1.268(68+24)	1.264(41+15)	1.263(35+10)	1.260(36+7)
Method 3	1.277(74+24)	1.290(45+20)	1.295(37+18)	1.295(38+14)
Convined	$1.274(69+24_{\mu}+5_{method})$	1.282(43+18+13)	1.284(36+14+16)	1.282(38+10+18)

three loops [w.i.p.] $1.279 \pm 0.040_{\exp} \pm 0.003_{\mu} \pm 0.012_{method} \pm 0_{n}$

n	1	2	3	4
Method 1	1.285(71+11)	1.277(42+9)	1.271(36+6)	1.264(37+4)
Method 2	1.272(68+8)	1.267(41+5)	1.264(35+3)	1.259(36+2)
Method 3	1.278(74+6)	1.287(45+4)	1.290(37+2)	1.289(38+1)
Convined	1.282(73+7+6)	1.279(43+4+10)	1.279(36+2+13)	1.274(37+1+15)

Even more preliminary: bottom

Experimental input taken from Kühn et al [3], and (for now) ignore errors.

Many things need to be checked and experimental input reconsidered.

Difference of results for various n's, enhances the error

Even more preliminary: bottom

Experimental input taken from Kühn et al [3], and (for now) ignore errors.

Many things need to be checked and experimental input reconsidered.

Difference of results for various n's, enhances the error

two loops $4.152 \pm 0.011_{\mu} \pm 0.012_{method} \pm 0.009_{n}$

n	1	2	3	4
Method 1	4.125(22)	4.146(15)	4.159(10)	4.168(06)
Method 2	4.117(26)	4.134(15)	4.154(10)	4.165(06)
Method 3	4.125(22)	4.162(13)	4.179(07)	4.188(03)
Combined	4.119(24+4)	4.147(14+14)	4.165(8+10)	4.175(4+12)

Even more preliminary: bottom

Experimental input taken from Kühn et al [3], and (for now) ignore errors.

Many things need to be checked and experimental input reconsidered.

Difference of results for various n's, enhances the error

two loops $4.152 \pm 0.011_{\mu} \pm 0.012_{method} \pm 0.009_{n}$

n	1	2	3	4
Method 1	4.125(22)	4.146(15)	4.159(10)	4.168(06)
Method 2	4.117(26)	4.134(15)	4.154(10)	4.165(06)
Method 3	4.125(22)	4.162(13)	4.179(07)	4.188(03)
Combined	4.119(24+4)	4.147(14+14)	4.165(8+10)	4.175(4+12)

3 loops [w.i.p.] $4.153 \pm 0.003_{\mu} \pm 0.009_{method} \pm 0.016_{n}$

n	1	2	3	4
Method 1	4.136(07)	4.154(06)	4.164(04)	4.170(02)
Method 2	4.123(07)	4.147(04)	4.159(03)	4.167(02)
Method 3	4.131(04)	4.163(01)	4.178(02)	4.185(03)
Combined	4.130(7+6)	4.136(3+8)	4.168(2+10)	4.177(3+9)

Even more preliminary: bottom

Experimental input taken from Kühn et al [3], and (for now) ignore errors.

Many things need to be checked and experimental input reconsidered.

Difference of results for various n's, enhances the error

two loops $4.152 \pm 0.011_{\mu} \pm 0.012_{method} \pm 0.009_{n}$

n	1	2	3	4
Method 1	4.125(22)	4.146(15)	4.159(10)	4.168(06)
Method 2	4.117(26)	4.134(15)	4.154(10)	4.165(06)
Method 3	4.125(22)	4.162(13)	4.179(07)	4.188(03)
Combined	4.119(24+4)	4.147(14+14)	4.165(8+10)	4.175(4+12)

3 loops [w.i.p.] $4.153 \pm 0.003_{\mu} \pm 0.009_{method} \pm 0.016_{n}$

n	1	2	3	4
Method 1	4.136(07)	4.154(06)	4.164(04)	4.170(02)
Method 2	4.123(07)	4.147(04)	4.159(03)	4.167(02)
Method 3	4.131(04)	4.163(01)	4.178(02)	4.185(03)
Combined	4.130(7+6)	4.136(3+8)	4.168(2+10)	4.177(3+9)

But…

Things go in the right direction!

Perturbative errors go down when more loops are included

Errors due to Padé approximants well under control

Promising results once the analysis is finished

Conclusions and outlook

- It is essential to have a reliable error estimate for charm and bottom masses
- Concerning relativistic sum rules, a contour improved analysis is mandatory.
  - For that we need to know the exact vacuum polarization function.
  - Experimental input must be treated with care (secondary radiation, singlet …)
- The Padé method is the best hope we can have for a semi analytical solution for arbitrary q^2 and masses for three and four loops.
- It can be systematically improved if more individual pieces (moments) are known.
  - It can predict constant pieces, but unfortunately cannot predict logs ☹.
  - It has proven to be useful and predicts known pieces with high accuracy.
- The difference between the center values of methods 2 and 3 has gone down, but still is much bigger that the individual errors due to scale variation.
- Errors will go further down when updating exp. input, but still larger than in [3,4,5]
- The analysis can be easily extended to other correlators ⟶ connection to lattice