A Rational Approach to the Resonance Region

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Resonance Saturation

- Phenomenological approximation:

  QCD Green's functions seem to be approximately saturated by a few resonances (VMD and extensions) (Sakurai '69; Bramon et al. '72)

  Low Energy Constants seem to be well saturated by the lowest meson (after constrains in the OPE) (Donoghue '89; Ecker et al. '89)

- Phenomenological support

- Theoretical support: These 2 considerations + Large N_c QCD $\Rightarrow$ **MHA**, keeping only a finite set of resonances, instead of an ∞ in Green's functions. (Knecht, de Rafael '98; Peris, Perrottet, de Rafael '98)

Sum Rule for L_8

- Defining $< SS - PP >$ correlator with a cutoff:

$$\Delta\Pi(q^2) \equiv \frac{1}{2}\left(\Pi_S(q^2) - \Pi'_P(q^2)\right) = \lim_{\Lambda^2\to\infty} \frac{1}{2\pi}\int_0^{\Lambda^2} dt\, \frac{\mathrm{Im}\,\left(\Pi_S(t) - \Pi'_P(t)\right)}{t - q^2 - i\epsilon}\ .$$

$\Delta\Pi(q^2)$ is invariant under $\Lambda^2 \to \Lambda^2 + a^2$ for $a^2 \ll \Lambda^2$

- In the Large-N_c limit:

$$\frac{1}{\pi}\,\mathrm{Im}\,\Pi_S(t)(\leftrightarrow \Pi'_P(t)) = 2\sum_n^{\infty} F^2_{S,P}(n)\ \delta(t - M^2_{S,P}(n))$$

$$\Rightarrow\ 16B^2 L_8 = \Delta\Pi(0) = \lim_{\Lambda^2\to\infty} \sum_n^{N_S(\Lambda^2)} \frac{F_S^2(n)}{M_S^2(n)} - \sum_n^{N_P(\Lambda^2)} \frac{F_P^2(n)}{M_P^2(n)}$$

- Now, $\Delta\Pi(0)$ does not have the same $\Lambda^2 \to \Lambda^2 + a^2$ for $a^2 \ll \Lambda^2$ symmetry.

Sum Rule for L_8

- For example, assuming Regge-like behavior for large n:

$$M^2_{S,P}(n) \sim n\sigma^2 \quad \text{and} \quad F^2_{S,P}(n) \sim \kappa\sigma^2 M^2_{S,P}(n) \ ,$$

$$16B^2 L_8 \sim \kappa\sigma^2 \left(\sum_n^{\infty} 1 - \sum_n^{\infty} 1 \right) = \textbf{??}$$

$$\Downarrow$$

Is this a Large-N_c effect?

(i.e., a $\Gamma_{S,P} \to 0$ effect?)

L_8 at finite N_c in QCD

A Regge-like model at finite N_c:

Golterman, Peris '06

$$\frac{2F_S^2(n)}{z\sigma^2 + M_S^2(n)} \quad \text{with} \quad z = \left(\frac{-q^2 - i\epsilon}{\sigma^2}\right)^\zeta \quad \text{and} \quad \zeta = 1 - \frac{a}{\pi N_c}$$

$$\begin{aligned} M_{S,P}^2(n) &= m_{S,P}^2 + n\sigma^2\,, \qquad n = 0, 1, \ldots\,, \\ F_{S,P}^2(n) &= \kappa\sigma^2 M_{S,P}^2(n)\,, \quad \Gamma_{S,P}(n) \sim a M_{S,P}(n)/N_c \end{aligned}$$

then (first $\sum$, then $\int dt$)

$$16B^2 L_8 = \frac{\kappa}{\pi}\int_0^\infty \frac{dt}{t}\sum_{n=0}^{\infty}\left(\frac{(n+m_S^2)t^\zeta \sin(\zeta\pi)}{\left(t^\zeta\cos(\zeta\pi)+n+m_S^2\right)^2 + \left(t^\zeta \sin(\zeta\pi)\right)^2} - (S \to P)\right)$$

$$= \kappa\left(m_P^2 - m_S^2\right)$$

while (first $\int dt$, then $\sum$) $\Rightarrow$ $16B^2 L_8 = \kappa\sigma^2\left(\sum_{n=0}^{N} 1 - \sum_{n=0}^{N-c} 1\right) = c\kappa\sigma^2$

Weinberg Sum Rule for Π_{LR}

- Second example, $< VV - AA >$ correlator with a cutoff:

$$\Pi_{LR}(q^2) \equiv \frac{1}{2}\left(\Pi_V(q^2) - \Pi_A(q^2)\right) = \lim_{\Lambda^2\to\infty} \frac{1}{2\pi}\int_0^{\Lambda^2} dt\, \frac{\mathrm{Im}\ (\Pi_V(t) - \Pi_A(t))}{t - q^2 - i\epsilon}$$

- Now, taking the expansion $Q^2 \to \infty$ at first order and a Large-N_c limit:

$$\lim_{Q^2\to\infty} Q^2 \Pi_{LR}(Q^2) \sim \lim_{\Lambda^2\to\infty}\left\{-F_0^2 + \sum_{n_V}^{N_V(\Lambda^2)} F_V^2 - \sum_{n_A}^{N_A(\Lambda^2)} F_A^2\right\} + \mathcal{O}\left(\frac{1}{Q^2}\right)$$

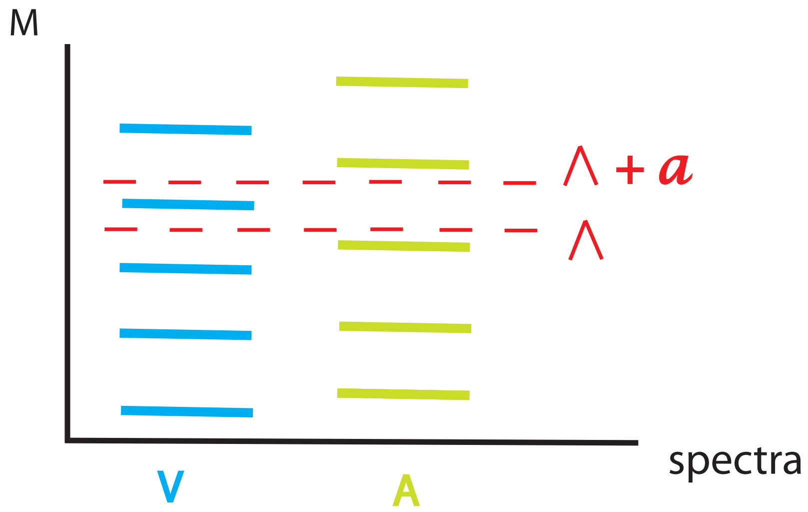
- Or (first Weinberg Sum Rule):

$$F_0^2 = \lim_{\Lambda^2\to\infty} \sum_{n=0}^{N_V(\Lambda^2)} F_V^2 - \sum_{n=0}^{N_A(\Lambda^2)} F_A^2$$

Weinberg Sum Rule for Π_{LR}

In a Regge model ($M^2_{V,A}(n) \sim n$ and $F^2_{V,A}(n) \sim F^2$, parton model o.k.)

$$F_0^2 = \lim_{\Lambda^2 \to \infty} F^2 \left(\sum_{n=0}^{N_V(\Lambda^2)} 1 - \sum_{n=0}^{N_A(\Lambda^2)} 1 \right) = \textbf{??}$$



VV-AA: a Model

Let's see a model for the self-energy function

$$\Pi_{LR}(Q^2) = \frac{1}{2}(\Pi_V(Q^2) - \Pi_A(Q^2))$$

Shifman '00; Cata, Golterman, Peris '05; P.M., Peris '07

$$\Pi_{LR}(q^2) = \frac{F_0^2}{q^2} + \frac{F_\rho^2}{-q^2 + M_\rho^2} + \sum_{n=0}^{\infty} \left\{ \frac{F_V^2}{-q^2 + M_V^2(n)} - \frac{F_A^2}{-q^2 + M_A^2(n)} \right\}$$

where $F_{V,A}^2(n) = F^2 = \text{constant}\ , \qquad M_{V,A}^2(n) = m_{V,A}^2 + n\,\sigma^2$

(according Regge Theory and Large N_c limit of QCD)

- $Q^2 \Pi_{LR}(Q^2)|_{Q^2 \to 0} \approx C_0 + C_2\, Q^2 + C_4\, Q^4 + \ldots$
- $Q^2 \Pi_{LR}(Q^2)|_{Q^2 \to \infty} \approx 0 + \frac{0}{Q^2} + \frac{C_{-4}}{Q^4} + \frac{C_{-6}}{Q^6} + \ldots$

Padé Theory

Let $f(z)$ a function of a complex z with an expansion

$$f(z) \quad = \quad \sum_{n=0}^{\infty} f_n z^n \quad , \quad z \to 0 \; .$$

Define rational function $P_N^M(z)$ such that

$$P_N^M(z) \equiv \frac{Q_M(z)}{R_N(z)} \approx f_0 + f_1 z + f_2 z^2 + \ldots + f_{M+N} z^{M+N} + \mathcal{O}(z^{M+N+1})$$

$P_N^M(z)$ is a Pade Approximant.

Padé Theory II: theorem

Convergence Theorem

(Pommerenke '73)

Let $f(z)$ be meromorphic and analytic at the origin.

Then,

$$\lim_{M,N\to\infty} P_N^M(z) = f(z)$$

for $z \in$ compact set in $\mathbb{C}$, **except** on isolated points.

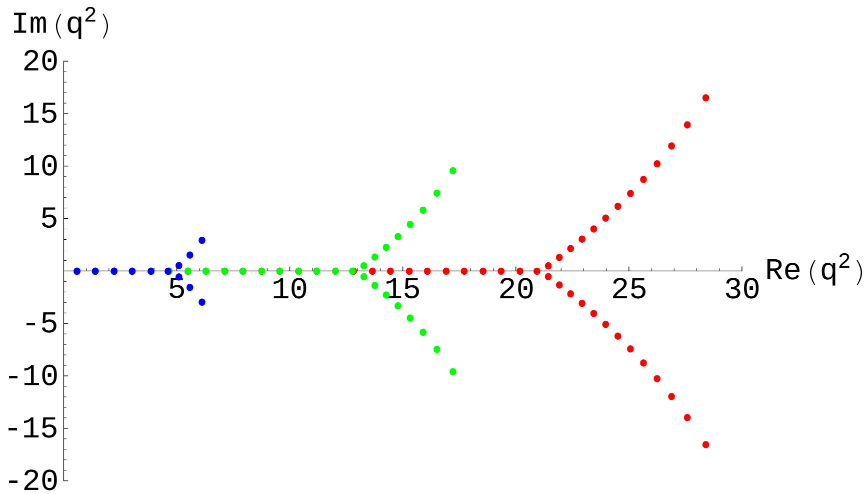
PA with a Model

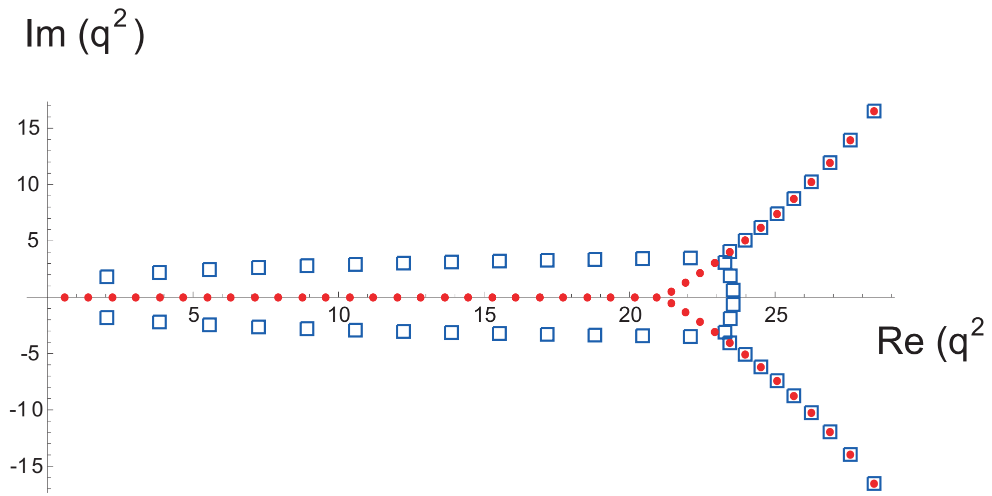
The simplest PA to the function $Q^2\Pi_{LR}(-Q^2)$ with the right fall-off as Q^{-4} at large Q^2 is $P_2^0(Q^2)$:

$$P_2^0(Q^2) = \frac{-\,r_R^2}{(Q^2+z_R)(Q^2+z_R^*)}\ , \begin{cases} r_R^2 = 3.379 \times 10^{-3} \\ z_R = 0.6550 + i\,0.1732\ . \end{cases}$$

- The poles are complex !
- Define $X_i \equiv \frac{\mathsf{C}_i(predicted)}{\mathsf{C}_i(real)}$. We find $X_{-4} = 1.3\ ,\ \ X_6 = 0.97\ , etc...$ (not bad !). And it can be improved.
- Gone up to P_{52}^{50} (with 103 parameters !) and find $X_{-4,-6,-8} = 1 + \mathcal{O}(10^{-52,-48,-45})\ ,\ \ X_{206} = 1 + \mathcal{O}(10^{-192})$!!

PA with a Model: P^N_{N+2}





- Two behaviors for the poles: { as order increase, pole $\longrightarrow \infty$; superposition pole and zero = defect }
- Blue,Green,Red points are the poles of P^{10}_{12}, P^{30}_{32}, P^{50}_{52}.
- Blue squares are the zeros of the numerator of P^{50}_{52}.

Conclusion

- Resonance Saturation (at Large-N_c QCD with a finite number of resonances) can be understood from the theory of Pade Approximants to meromorphic functions.

- large-N_c QCD with only a finite set of resonances $\Rightarrow$ no information on individual mesons -i.e. masses and decay constants- $\Rightarrow$ Euclidean ok, Minkowski no.