Kinematics of string junctions

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Outline

Motivation: Strings with junctions – abelian and non-abelian

Kinematics of string junctions:
- e.o.m. for junctions of 3 Nambu-Goto strings
- general solution

Collisions of straight strings: can they intercommute?
- kinematical constraints on angle and velocity
- possibility of ‘locking’ for non-abelian strings

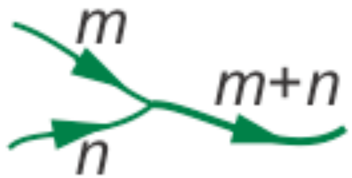
Rates of change of string lengths
- light strings tend to grow at the expense of heavy ones (? important for network evolution)

Average string velocity
- reduced near a junction

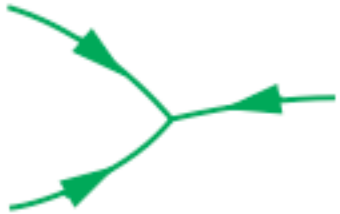
Strings with junctions

- Type-I strings

$\mu_2 < 2\mu_1$

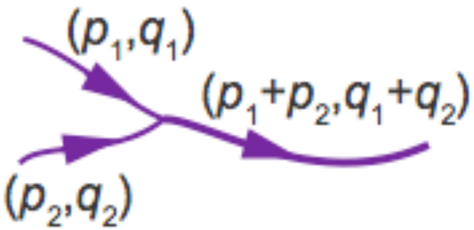


- $\mathbb{Z}_3$ strings

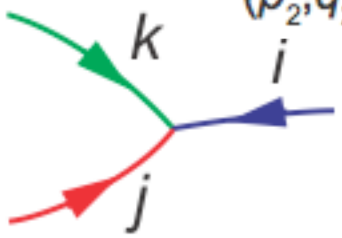


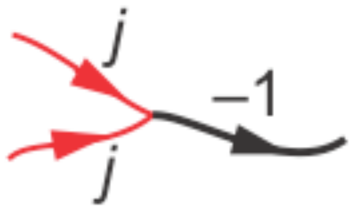
- 3-colour strings

- (*p*,*q*) strings: composites of *p* F-strings and *q* D-strings



- non-abelian strings, e.g. biaxial nematics (brick-shaped molecules) classified by $\{\pm 1, \pm i, \pm j, \pm k\}$ where $i^2 = -1,\ ij = k,$ etc.





[Toulouse & Poénaru 1976, 1977, Volovik & Mineev 1977]

Junctions of Nambu–Goto strings

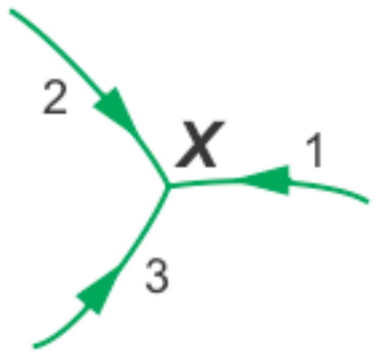
Use gauge conditions: $\dot{x}^2 + x'^2 = 0, \;\; \dot{x} \cdot x' = 0$

with $\dot{x} = \partial_\tau x, \;\; x' = \partial_\sigma x$

(conformal gauge) and $\tau = t = x^0(\sigma, \tau),$

$\Rightarrow x(\sigma, t) = (t, \boldsymbol{x}(\sigma, t)), \qquad \dot{\boldsymbol{x}}^2 + \boldsymbol{x}'^2 = 1, \;\; \dot{\boldsymbol{x}} \cdot \boldsymbol{x}' = 0$

Take σ on each leg j to increase towards the vertex, position $\boldsymbol{X}(t)$



$$S = -\sum_j \mu_j \int dt\, d\sigma\, \theta(s_j(t) - \sigma) \sqrt{\boldsymbol{x}_j'^2 (1 - \dot{\boldsymbol{x}}_j^2)}$$
$$+ \sum_j \int dt\, \boldsymbol{f}_j(t) \cdot [\boldsymbol{x}_j(s_j(t), t) - \boldsymbol{X}(t)]$$

$\mu_j =$ string tension

Solving equations of motion

Varying $\boldsymbol{x}_j \Rightarrow \ddot{\boldsymbol{x}}_j - \boldsymbol{x}_j'' = \mathbf{0} \Rightarrow$

$$\boldsymbol{x}_j(\sigma,t) = \tfrac{1}{2}[\boldsymbol{a}_j(\sigma+t) + \boldsymbol{b}_j(\sigma-t)] \quad \text{with} \quad \boldsymbol{a}_j'^2 = \boldsymbol{b}_j'^2 = 1$$

Boundary terms $\Rightarrow \mu_j(\boldsymbol{x}_j' + \dot{s}_j\dot{\boldsymbol{x}}_j) = \boldsymbol{f}_j$ at $(s_j(t),t)$

Varying $\boldsymbol{f}_j \Rightarrow \boldsymbol{X}(t) = \boldsymbol{x}_j(s_j(t),t) = \tfrac{1}{2}[\boldsymbol{a}_j(s_j+t) + \boldsymbol{b}_j(s_j-t)]$

Initial conditions at $t=0 \Rightarrow$ values of $\boldsymbol{b}_j'(s_j(t)-t)$

(ingoing waves) are known, but not those of $\boldsymbol{a}_j'(s_j(t)+t)$

(outgoing waves); use $(1+\dot{s}_j)\boldsymbol{a}_j' = (1-\dot{s}_j)\boldsymbol{b}_j' + 2\dot{\boldsymbol{X}}$

Varying $\boldsymbol{X} \Rightarrow \sum_j \boldsymbol{f}_j = \mathbf{0} \Rightarrow (\mu_1+\mu_2+\mu_3)\dot{\boldsymbol{X}} = -\sum_j \mu_j(1-\dot{s}_j)\boldsymbol{b}_j'$

To find $\dot{s}_j$, we impose $\boldsymbol{a}_j'^2 = 1$

Solution for $\dot{s}_j$

$\boldsymbol{a}_j'^2 = 1 \;\Rightarrow\;$ eqs for $\dot{s}_j$ in terms of $c_{ij} = \boldsymbol{b}_i'(s_i - t)\cdot \boldsymbol{b}_j'(s_j - t)$

Solution (differential equations for s_j):

$$\frac{\mu_1(1-\dot{s}_1)}{\mu_1+\mu_2+\mu_3} = \frac{M_1(1-c_{23})}{M_1(1-c_{23})+M_2(1-c_{31})+M_3(1-c_{12})} \quad \text{etc.}$$

where $M_1 = \mu_1^2 - (\mu_2 - \mu_3)^2$ etc.

Corollary: since $\dot{s}_j < 1$ and $c_{ij} < 1$ all $M_j > 0$

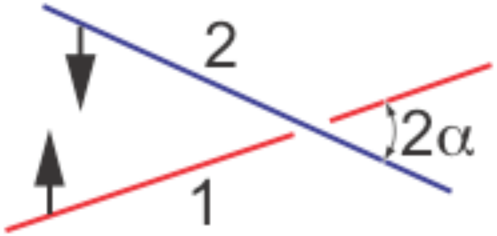
i.e. μ_j satisfy **triangle inequalities** (obvious if $\dot{\boldsymbol{X}} = \boldsymbol{0}$)

— e.g. if $\mu_3 > \mu_1 + \mu_2$ string 3 is unstable

Note: summing 3 eqs. $\Rightarrow \mu_1\dot{s}_1 + \mu_2\dot{s}_2 + \mu_3\dot{s}_3 = 0$

(energy conservation)

Collision of straight strings $(\mu_1 = \mu_2)$



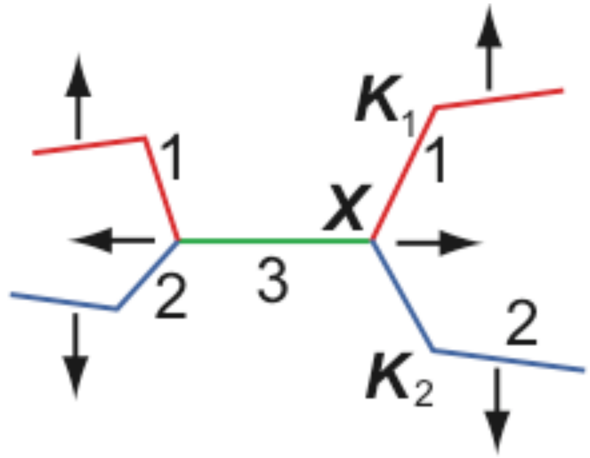
Take $\mu_1 = \mu_2$ and, for $t < 0$,

$$\boldsymbol{x}_{1,2}(\sigma,t) = (-\gamma^{-1}\sigma\cos\alpha, \mp\gamma^{-1}\sigma\sin\alpha, \pm vt)$$

$$\gamma^{-1} = \sqrt{1-v^2}$$

$$\Rightarrow \boldsymbol{a}'_{1,2} = (-\gamma^{-1}\cos\alpha, \mp\gamma^{-1}\sin\alpha, \pm v)$$

$$\boldsymbol{b}'_{1,2} = (-\gamma^{-1}\cos\alpha, \mp\gamma^{-1}\sin\alpha, \mp v)$$



If 1,2 exchange partners, and are joined by 3, it must lie on x or y axis (for small α or large α, resp.?)

Assume x-axis. Then for $t > 0$,

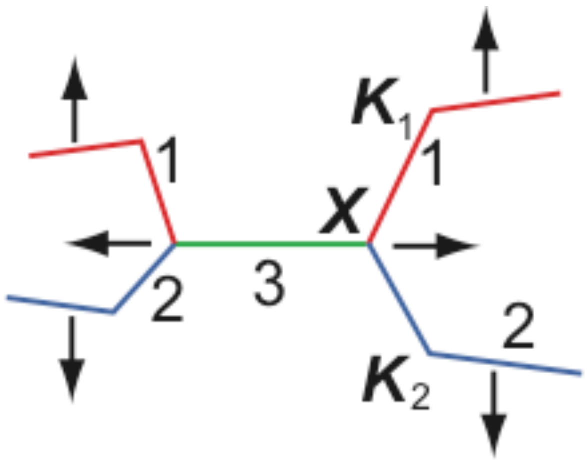
$$\boldsymbol{x}_3(\sigma,t) = (\sigma,0,0), \qquad \boldsymbol{a}'_3 = \boldsymbol{b}'_3 = (1,0,0)$$

Consider vertex $\boldsymbol{X}$ on right

Collision of straight strings (continued)

$$\boldsymbol{X}(t) = s_3(t)(1,0,0)$$

$$\boldsymbol{K}_{1,2}(t) = t(\gamma^{-1}\cos\alpha, \pm\gamma^{-1}\sin\alpha, \pm v)$$



$$\mu_1\dot{s}_1 + \mu_1\dot{s}_2 + \mu_3\dot{s}_3 = 0 \Rightarrow$$

$$\dot{s}_1 = \dot{s}_2 = -\frac{\mu_3}{2\mu_1}\dot{s}_3$$

Now $\quad c_{12} = \boldsymbol{b}_1' \cdot \boldsymbol{b}_2' = \gamma^{-2}\cos 2\alpha - v^2$

$$c_{13} = \boldsymbol{b}_1' \cdot \boldsymbol{b}_3' = -\gamma^{-1}\cos\alpha = c_{23}$$

$$\Rightarrow \dot{s}_3 = \frac{2\mu_1\gamma^{-1}\cos\alpha - \mu_3}{2\mu_1 - \mu_3\gamma^{-1}\cos\alpha}$$

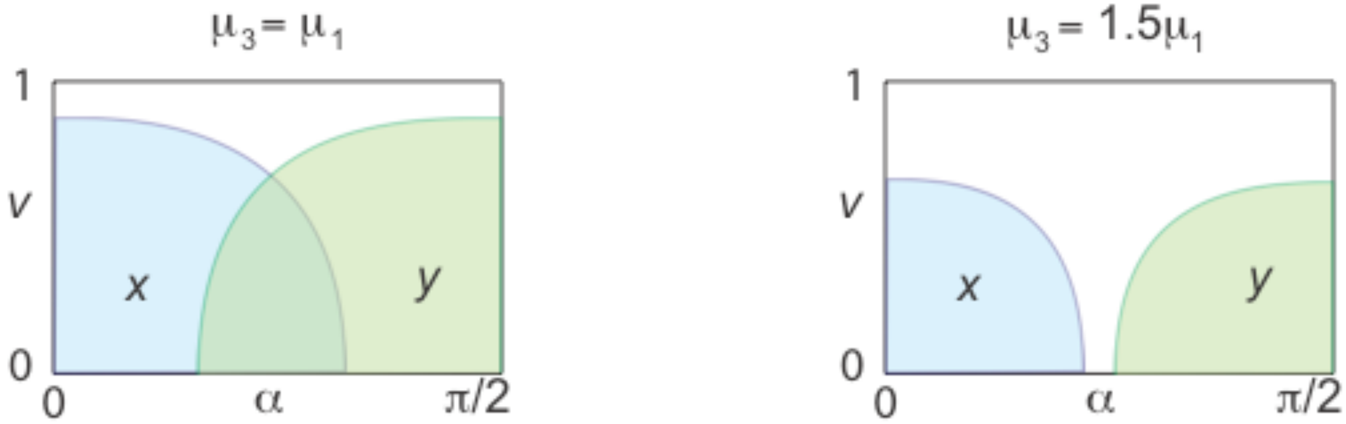
Implications

$$\dot{s}_3 = \frac{2\mu_1\gamma^{-1}\cos\alpha - \mu_3}{2\mu_1 - \mu_3\gamma^{-1}\cos\alpha} \qquad \text{with} \qquad \mu_3 < 2\mu_1$$

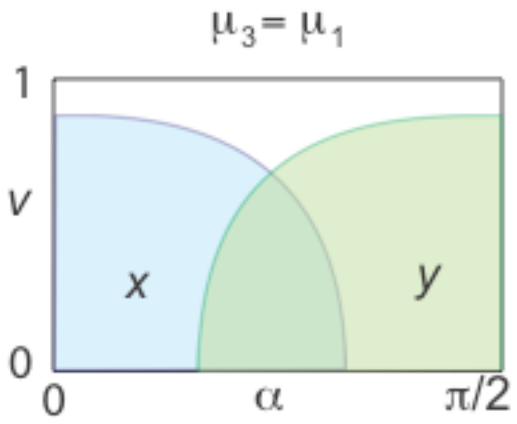
But $\dot{s}_3 > 0,$ so for 3 along x axis, $\alpha < \arccos\left(\dfrac{\mu_3\gamma}{2\mu_1}\right)$

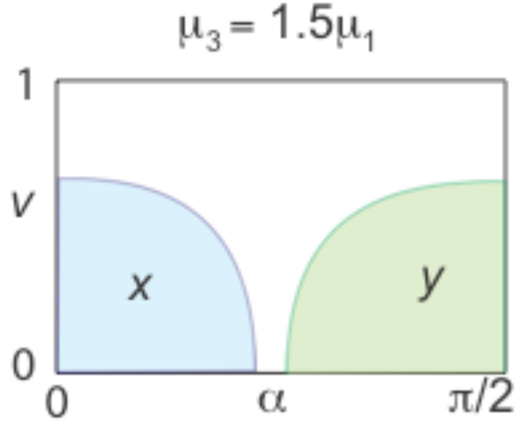
Similarly, for 3 along y axis, $\alpha > \arcsin\left(\dfrac{\mu_3\gamma}{2\mu_1}\right)$

Kinematically allowed regions are:



Implications (continued)





Note: **neither** is possible unless $\gamma < \dfrac{2\mu_1}{\mu_3}$

e.g., if $\mu_3 = \mu_1$, we require $v < \dfrac{\sqrt{3}}{2}$

What happens if this limit is violated?

For **abelian** strings, the only possibility is that they pass through each other without exchanging partners.

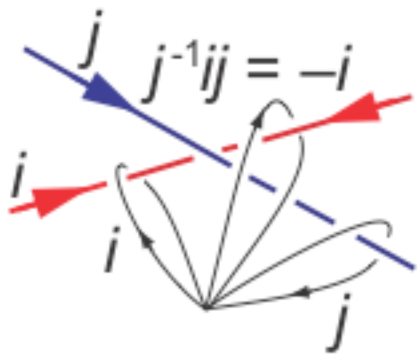
Non-abelian strings

Gauge potential ***A*** is a one-form with values in Lie algebra of *G*

Quantum number or flux on a string is $\gamma = \int_C \boldsymbol{A}$

For non-abelian strings, γ is path-dependent, e.g.:

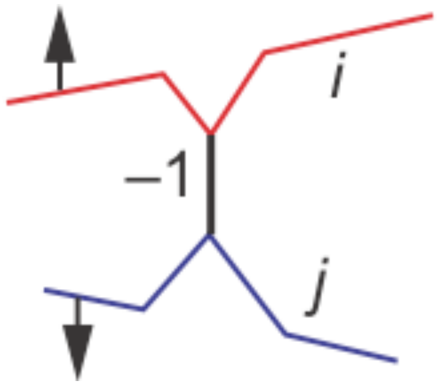
$$\gamma \in \mathrm{Q} = \{\pm 1, \pm i, \pm j, \pm k\}$$
$$i^2 = -1, \; ij = k,$$



If strings with $\gamma_1 = i, \gamma_2 = j$ pass through each other without exchanging partners, they **must** become linked by a string with $\gamma_3 = -1$

Linkage in z direction

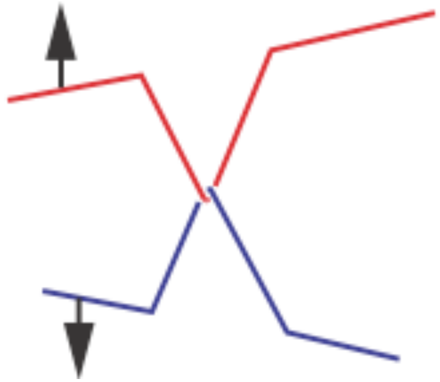
Non-abelian strings with $[\gamma_1, \gamma_2] \neq 0$ cannot pass through one another, and may become linked by a string along z axis.



Here $\quad c_{12} = 2v^2 - 1, \; c_{13} = c_{23} = -v$

$$\Rightarrow \dot{s}_3 = \frac{2\mu_1 v - \mu_3}{2\mu_1 - \mu_3 v} \quad \Rightarrow v > \frac{\mu_3}{2\mu_1}$$

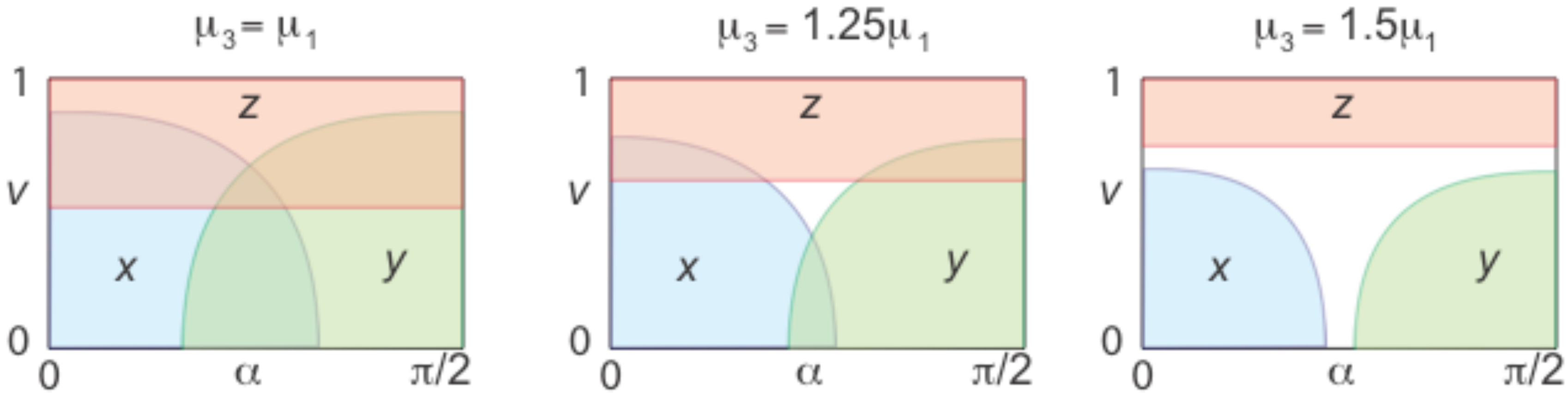
Linking in x or y dir. required $\gamma < \dfrac{2\mu_1}{\mu_3}$



So if $\mu_3 > \sqrt{2}\mu_1$ there is a range of velocities for which the strings cannot move apart, linked in **any** direction; they become locked.

Kinematic constraints

Allowed regions of the α-v plane for links along 3 axes:

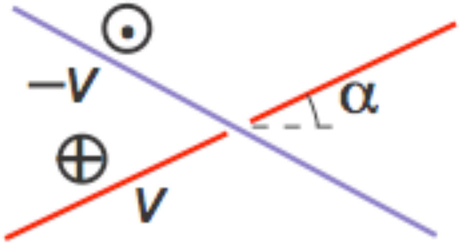


$$\frac{\mu_3}{\mu_1} < \frac{2}{\sqrt{3}} \qquad \frac{2}{\sqrt{3}} < \frac{\mu_3}{\mu_1} < \sqrt{2} \qquad \frac{\mu_3}{\mu_1} > \sqrt{2}$$

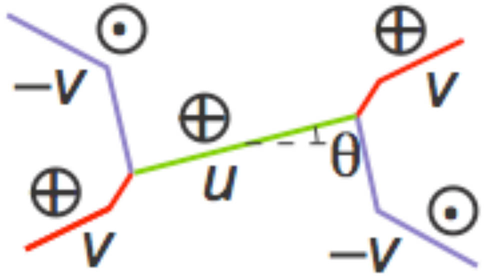
Abelian strings, in white or *z* region, must pass through one another.

Non-abelian-strings, in *z* region, may be linked along the *z* axis; in white region, they will be locked.

Collisions with $\mu_1 \neq \mu_2$



$$\mathbf{x}_{1,2}(\sigma,t) = (-\gamma^{-1}\sigma\cos\alpha, \mp\gamma^{-1}\sigma\sin\alpha, \pm vt)$$



$$\Rightarrow \mathbf{a}'_{1,2} = (-\gamma^{-1}\cos\alpha, \mp\gamma^{-1}\sin\alpha, \pm v)$$

$$\mathbf{b}'_{1,2} = (-\gamma^{-1}\cos\alpha, \mp\gamma^{-1}\sin\alpha, \mp v)$$

$$\mathbf{x}_3(\sigma,t) = (\gamma_u^{-1}\sigma\cos\theta, \gamma_u^{-1}\sigma\sin\theta, ut),$$

Unknowns u and θ are given by $\dfrac{\tan\theta}{\tan\alpha} = \dfrac{u}{v}$ and

$$\mu_-^2(\sin^2\alpha)u^4 + [\mu_3^2(1-v^2) + \mu_-^2(v^2\cos^2\alpha - \sin^2\alpha)]u^2 - \mu_-^2 v^2\cos^2\alpha = 0$$

where $\mu_- = \mu_1 - \mu_2$

Constraints found as before from $\dot{s}_3 \geq 0$

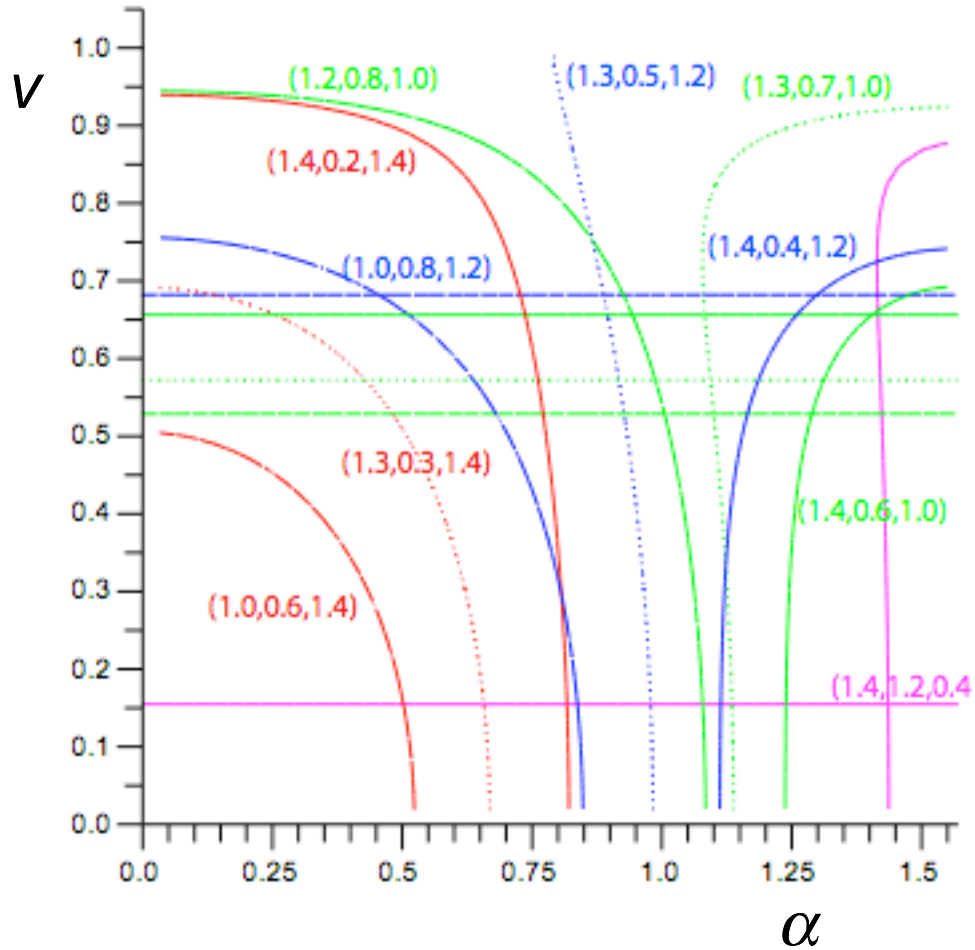
Kinematic constraints with $\mu_1 \neq \mu_2$

Labels are (μ_1, μ_2, μ_3) with

$$\mu_1 + \mu_2 + \mu_3 = 3$$

Allowed regions for an *x*-link are to the **left** of the curves

Those for a *z*-link are **above** the horizontal lines



Regions expand if
μ_3 decreases or $|\mu_1 - \mu_2|$ increases

Distribution of $\dot{s}_j$

Let us assume that the configurations of **incoming** waves on the three strings meeting at a junction are uncorrelated, i.e. $\boldsymbol{b}_1', \boldsymbol{b}_2', \boldsymbol{b}_3'$ are independently randomly distributed on the unit sphere.

What is the distribution of $\dot{s}_1$?

$$P(\dot{s}_1) = \frac{1}{8\pi} \int_{-1}^{1} dc_{13} \int_{-1}^{1} dc_{23} \int_{0}^{2\pi} d\varphi \, \delta\left(\dot{s}_1 - 1 + \frac{\mu M_1 (1 - c_{23})}{\mu_1 \mathcal{M}} \right)$$

with $\mu = \mu_1 + \mu_2 + \mu_3$,

$$\mathcal{M} = M_1(1 - c_{23}) + M_2(1 - c_{31}) + M_3(1 - c_{12})$$

$$c_{12} = c_{13} c_{23} + \sqrt{1 - c_{13}^2} \sqrt{1 - c_{23}^2} \cos\varphi$$

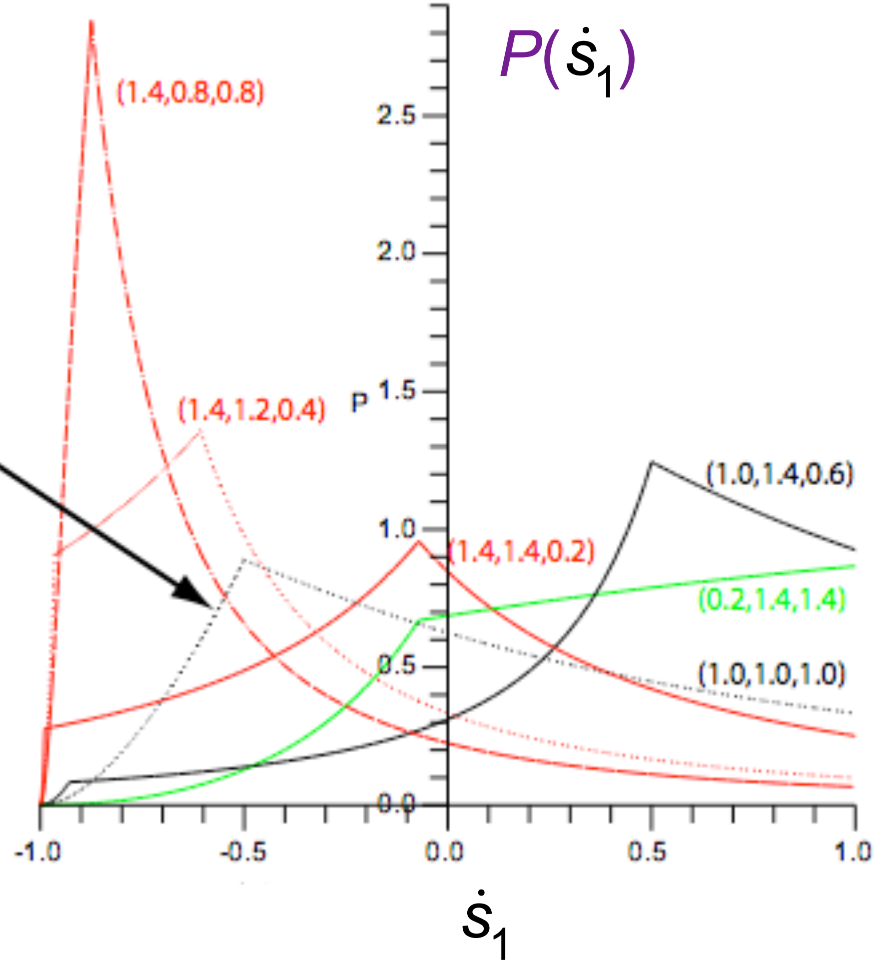
Distribution of $\dot{s}_1$

• The distribution has two kinks (one if $\mu_2 = \mu_3$)

— e.g. at $\dot{s}_1 = -\frac{1}{2}$ if $\mu_1 = \mu_2 = \mu_3$

• Even for equal μ_j, the distribution is **not symmetric**

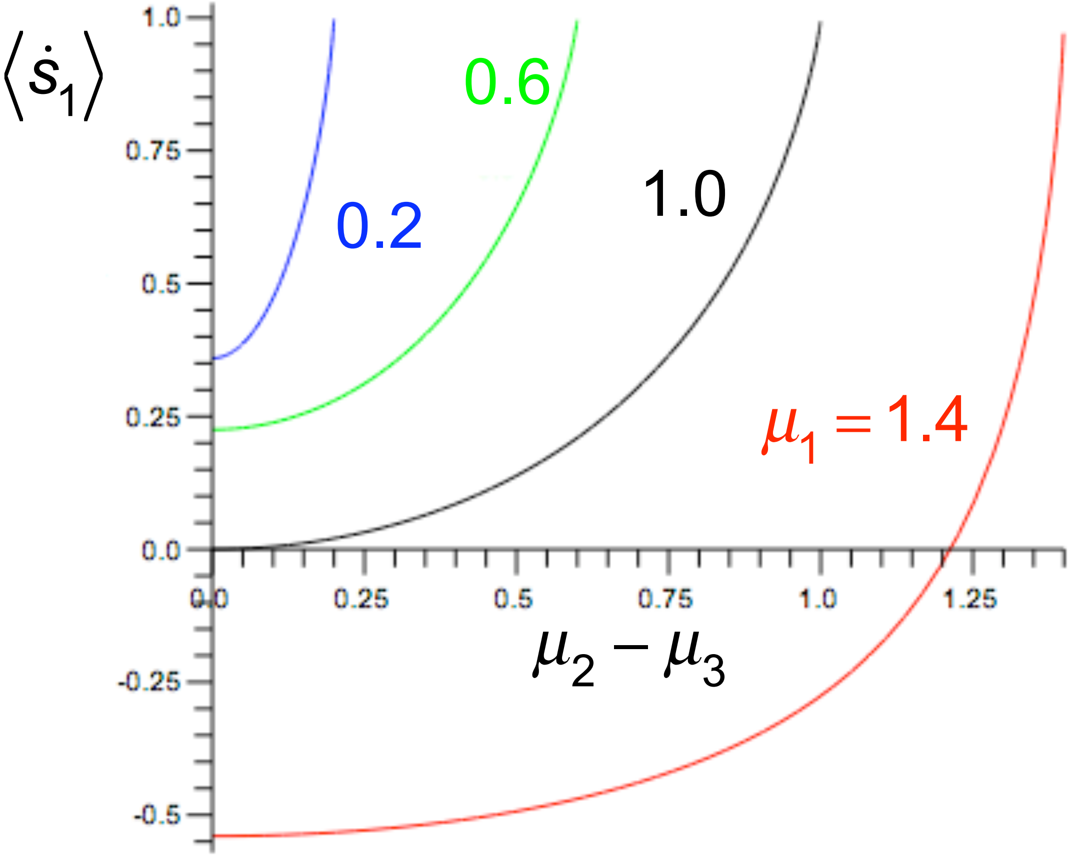
— it is most probable that one leg is growing while two shrink (slowly)



Average of $\dot{s}_1$

What is $\langle \dot{s}_1 \rangle$?

If all μ_j are equal, then $\langle \dot{s}_1 \rangle = 0$
but **not** usually otherwise



Note:

$\langle \dot{s}_1 \rangle$ is large

- if μ_1 is small
- or if $\mu_2 - \mu_3$ is large

— Light strings tend to grow at the expense of heavier strings

This may be important for evolution of a network

Average string velocity

For Nambu-Goto strings in flat space, $\left\langle \boldsymbol{v}^2 \right\rangle = \frac{1}{2}$.

For a string near a junction, we have $\left\langle \dot{\boldsymbol{x}}_1^2 \right\rangle = \frac{1}{2}\left(1 + \left\langle \boldsymbol{a}_1' \cdot \boldsymbol{b}_1' \right\rangle\right)$ which gives

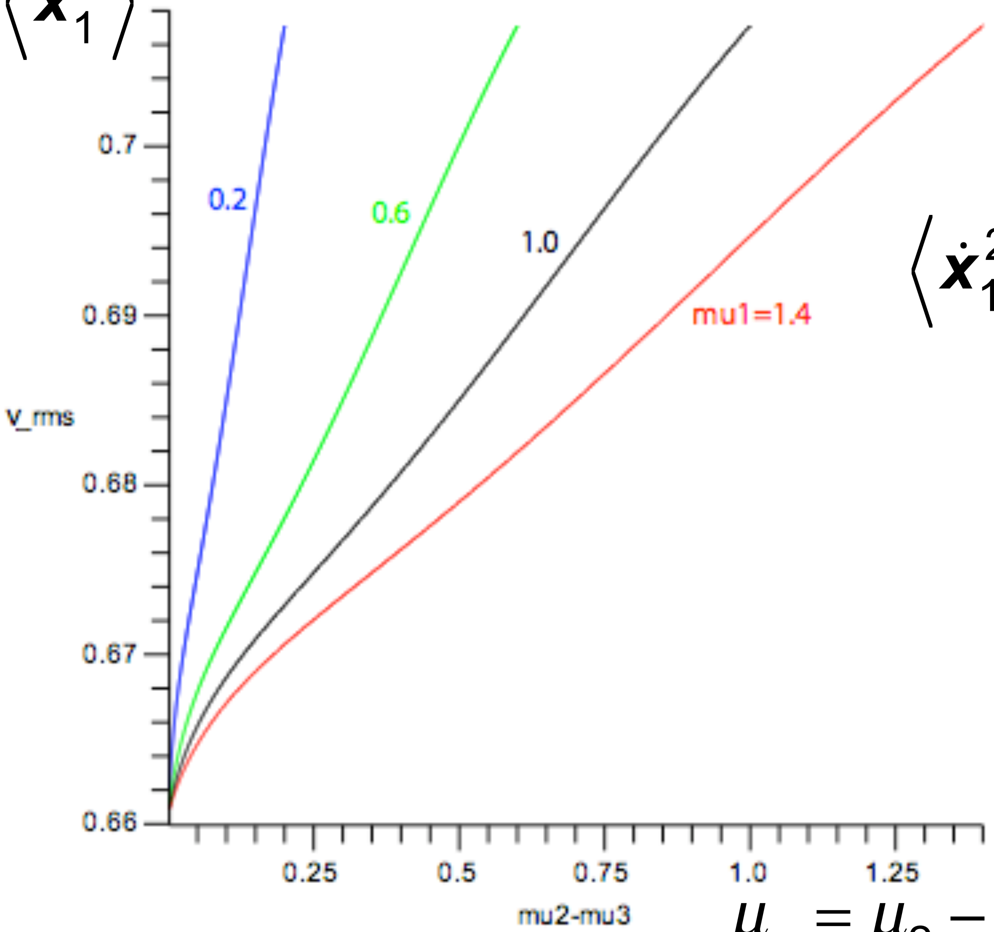
$$\left\langle \dot{\boldsymbol{x}}_1^2 \right\rangle = \frac{1}{15}\left(\frac{\mu_1^2 - 13\mu_-^2}{\mu_1^2 - \mu_-^2} + \frac{(4\mu_1 - \mu_-)(\mu_1 + \mu_-)^2}{\mu_1(\mu_1 - \mu_-)^2} \ln \frac{2\mu_1}{\mu_1 + \mu_-} + \right.$$
$$\left. + \frac{(4\mu_1 + \mu_-)(\mu_1 - \mu_-)^2}{\mu_1(\mu_1 + \mu_-)^2} \ln \frac{2\mu_1}{\mu_1 - \mu_-} \right)$$

where $\mu_- = \mu_2 - \mu_3$

Note: $\left\langle \dot{\boldsymbol{x}}_1^2 \right\rangle$ depends only on μ_1 and μ_-, not $\mu_+ = \mu_2 + \mu_3$

Average string velocity

$\sqrt{\langle \dot{\boldsymbol{x}}_1^2 \rangle}$



$\mu_- = \mu_2 - \mu_3$

$\langle \dot{\boldsymbol{x}}_1^2 \rangle$ has minimum at $\mu_- = 0$

$$= \frac{1+8\ln 2}{15} = 0.436$$

and maximum = 0.5 at

$\mu_- = \pm\mu_1$

Lower rms velocity may affect the network evolution

Conclusions

Equations of motion and general solution for 3-string junctions

Collisions of straight strings: kinematic constraints on angle and velocity for which exchange of partners is possible
— less severe if joining string is light or
— if colliding strings have very different tension

Rates of growth: light strings tend to grow at the expense of heavier strings; even for equal tensions, it is most likely that one string grows while two shrink

Average velocity: rms velocity of string near a junction is (modestly) reduced

These results may be important in studies of evolution of a network of strings with junctions