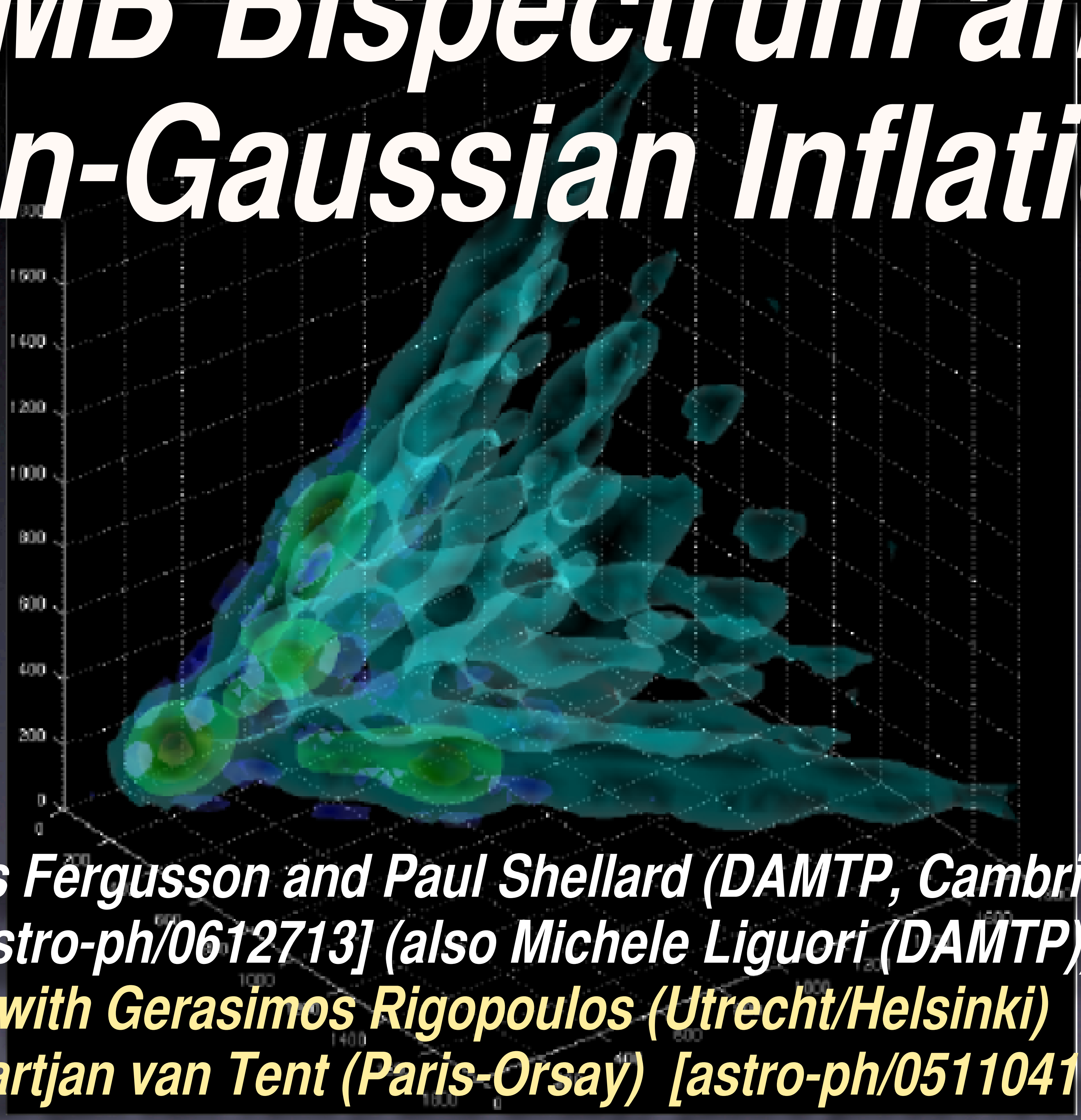
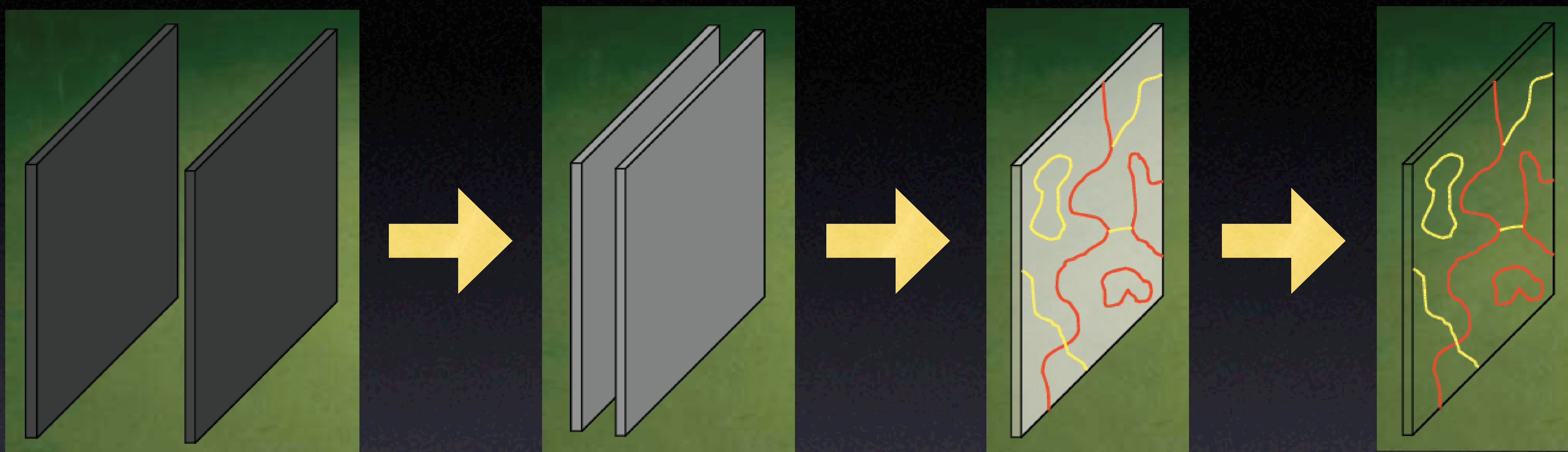
CMB Bispectrum and non-Gaussian Inflation



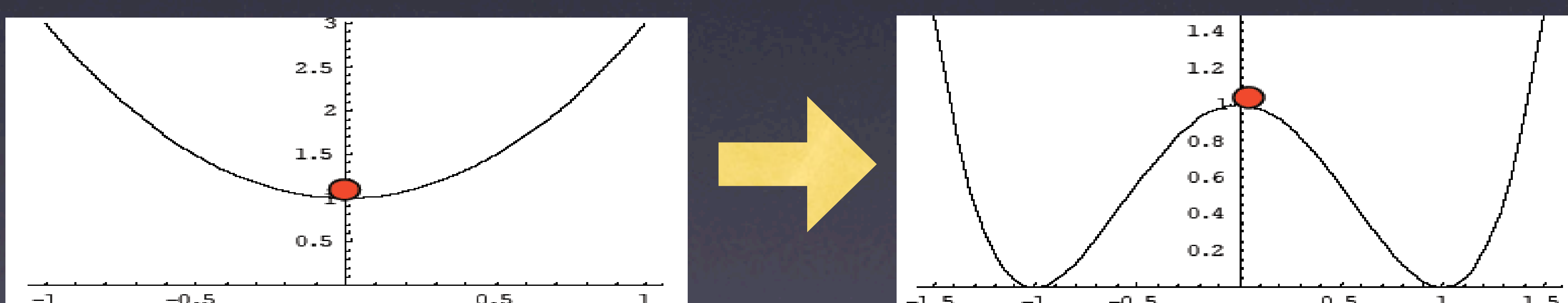
James Fergusson and Paul Shellard (DAMTP, Cambridge)
[astro-ph/0612713] (also Michele Liguori (DAMTP))
with Gerasimos Rigopoulos (Utrecht/Helsinki)
and Bartjan van Tent (Paris-Orsay) [astro-ph/0511041 etc.]

Brane Inflation

Dvali & Tye, 2000
Burgess, Quevedo *et al* 01
Jones, Stoica & Tye, 2002
KKLMMT, 2003



- *Interbrane interaction creates inflationary potential*



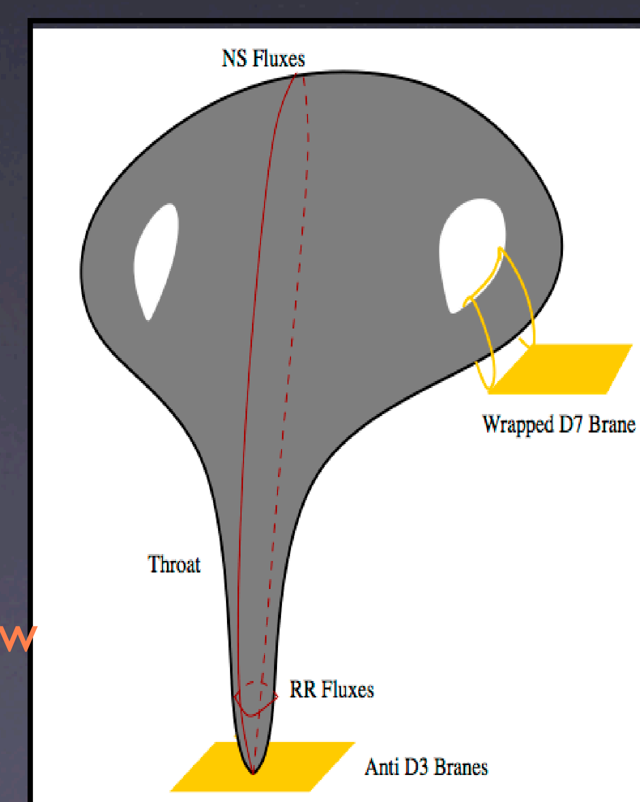
- *Brane collision = hybrid inflation reheating*

  'Generic' formation of cosmic strings

  Extra fields and nonGaussianity

  Observable signatures of extra dimensions?

Sarangi & Tye, 2002
See Majumdar review
hep-th/0512062



Multifield inflation

Gravity is inherently nonlinear

Interacting inflationary potentials

→ *NonGaussianity!*

CMB observations → *discriminating inflation models*

- ***Gaussian***

$$\hat{\Phi}_{\rm lin} = \Phi_{\rm lin}\, \hat{a}^\dagger + \Phi^*_{\rm lin}\, \hat{a} \quad \Rightarrow \quad \text{Gaussian with } \langle \hat{\Phi}\hat{\Phi}\hat{\Phi} \rangle = 0$$

- ***Non-Gaussian***

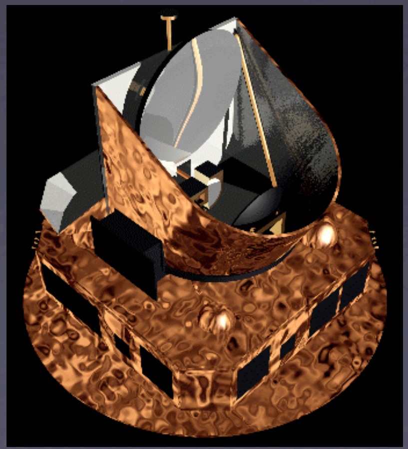
$$\hat{\Phi} = \hat{\Phi}_{\rm lin} + \hat{\Phi}_{\rm NL} \text{ where } \hat{\Phi}_{\rm NL} = f_{\rm NL}\, \hat{\Phi}^2_{\rm lin}$$

$$\Rightarrow \quad \text{nonGaussian with } \langle \hat{\Phi}\hat{\Phi}\hat{\Phi} \rangle \ \sim \ f_{\rm NL}\, \langle \hat{\Phi}^2_{\rm lin} \rangle^2$$

- ***Observational prospects*** *(Komatsu astro-ph/0206039)*

WMAP will observe $|f_{NL}| \geq 20$, Planck $|f_{NL}| > 5$

Current WMAP bound: $-58 < f_{NL} < 134$ (95%)



Superhorizon non-Gaussianity

- *'Evolution' equations (multifield inflation)*

$$\frac{dH}{dt} = -\frac{\kappa^2}{2} N \Pi_B \Pi^B \,, \quad (1)$$

$$\mathcal{D}_t \Pi^A = -3NH\Pi^A - NG^{AB} V_B \,, \quad (2)$$

where $V_B \equiv \partial_B V \equiv \partial V / \partial \phi^B$ and $\kappa^2 \equiv 8\pi G = 8\pi / m_{\rm pl}^2$

- *'Constraint' equations*

$$H^2 = \frac{\kappa^2}{3} \left(\tfrac{1}{2} \Pi_B \Pi^B + V \right) \,, \quad (3)$$

$$\partial_i H = -\frac{\kappa^2}{2} \Pi_B \partial_i \phi^B \,, \quad (4)$$

- *Separate Universe approach* — *Salopek & Bond, 1990*

  initial data must respect energy and momentum constraints

  evolving collection of indpt universes preserve constraints

- *But how to self-consistently generate fluctuations?*

General semi-analytic solution

- <u>Recast master equation and perturbatively expand</u>

Defining $v_{i\,a} \equiv (\zeta_i^1, \theta_i^1, \zeta_i^2, \theta_i^2, \ldots)^T$,

implies $$\dot{v}_{i\,a}(t, \boldsymbol{x}) + A_{ab}(t, \boldsymbol{x}) v_{i\,b}(t, \boldsymbol{x}) = 0,$$

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 3 & -6\tilde{\eta}^{\perp} & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 3\chi & 3 \end{pmatrix}$$

with $\chi(t, \boldsymbol{x}) \equiv \frac{e_2^A V_{;AB}\, e_2^B}{3H^2} + \tilde{\epsilon} + \tilde{\eta}^{\|}$

two-field case

slow-roll example (exact case used)

Perturbative expansion:

$$\begin{aligned} \dot{v}_{i\,a}^{(1)} + A_{ab}^{(0)}(t) v_{i\,b}^{(1)} &= b_{i\,a}^{(1)}(t, \boldsymbol{x}), \\ \dot{v}_{i\,a}^{(2)} + A_{ab}^{(0)}(t) v_{i\,b}^{(2)} &= -A_{ab}^{(1)}(t, \boldsymbol{x}) v_{i\,b}^{(1)} , \end{aligned}$$

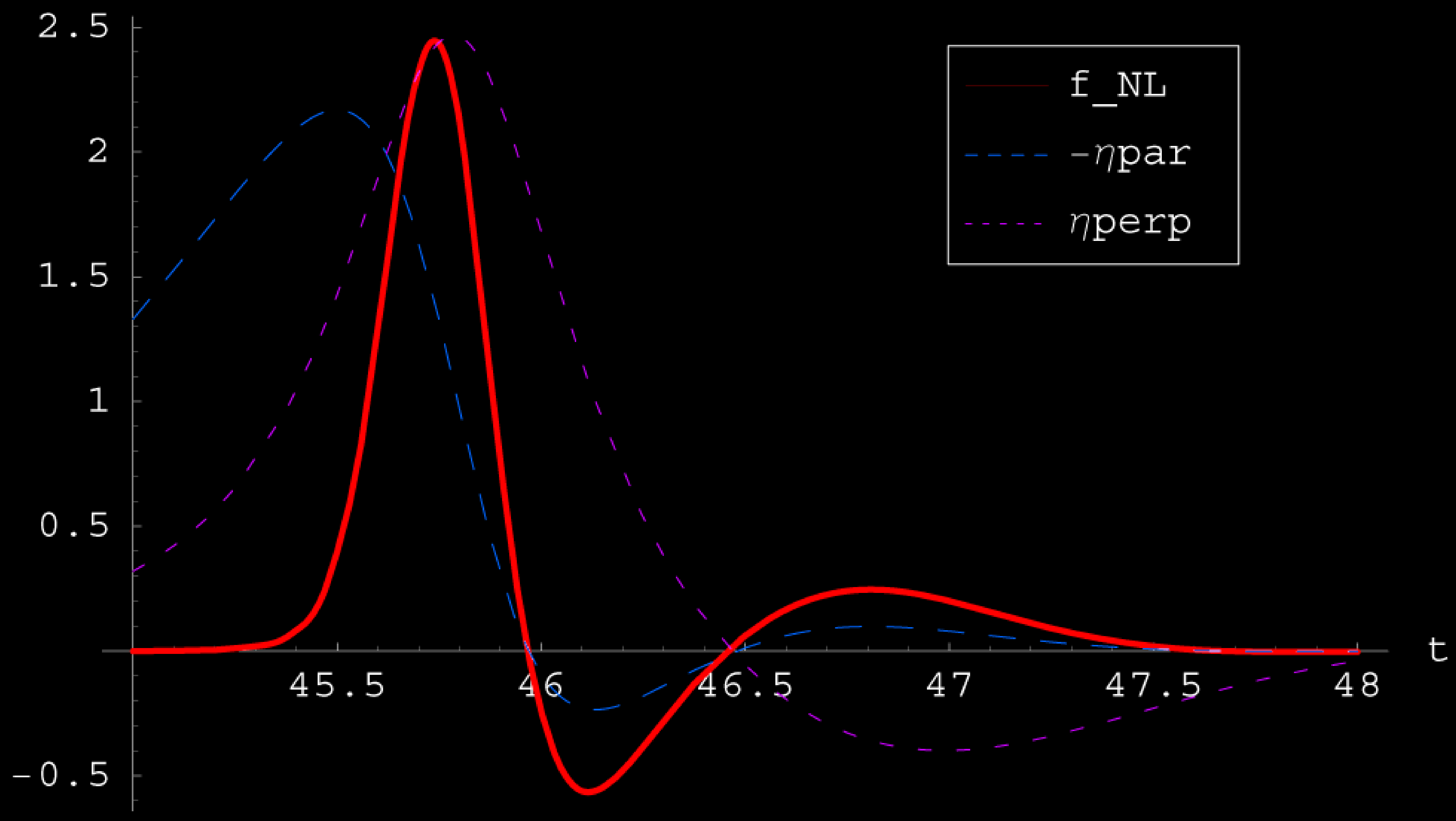
where $v_{i\,a} = v_{i\,a}^{(1)} + v_{i\,a}^{(2)}$ and
$$\begin{aligned} A_{ab}(t, \boldsymbol{x}) &= A_{ab}^{(0)} + A_{ab}^{(1)} = A_{ab}^{(0)} + \partial^{-2}\partial^i(\partial_i A_{ab})^{(1)} \\ &\equiv A_{ab}^{(0)}(t) + \bar{A}_{abc}^{(0)}(t) v_c^{(1)}(t, \boldsymbol{x}). \end{aligned}$$

First order solution:

$$v_{a\,m}^{(1)}(k, t) \equiv \int_{-\infty}^{t} \mathrm{d}t'\, G_{ab}(t, t') \dot{\mathcal{W}}(k, t') X_{bm}^{(1)}(k, t').$$

Green's function *horizon-crossing linear soln*

Bispectrum expression

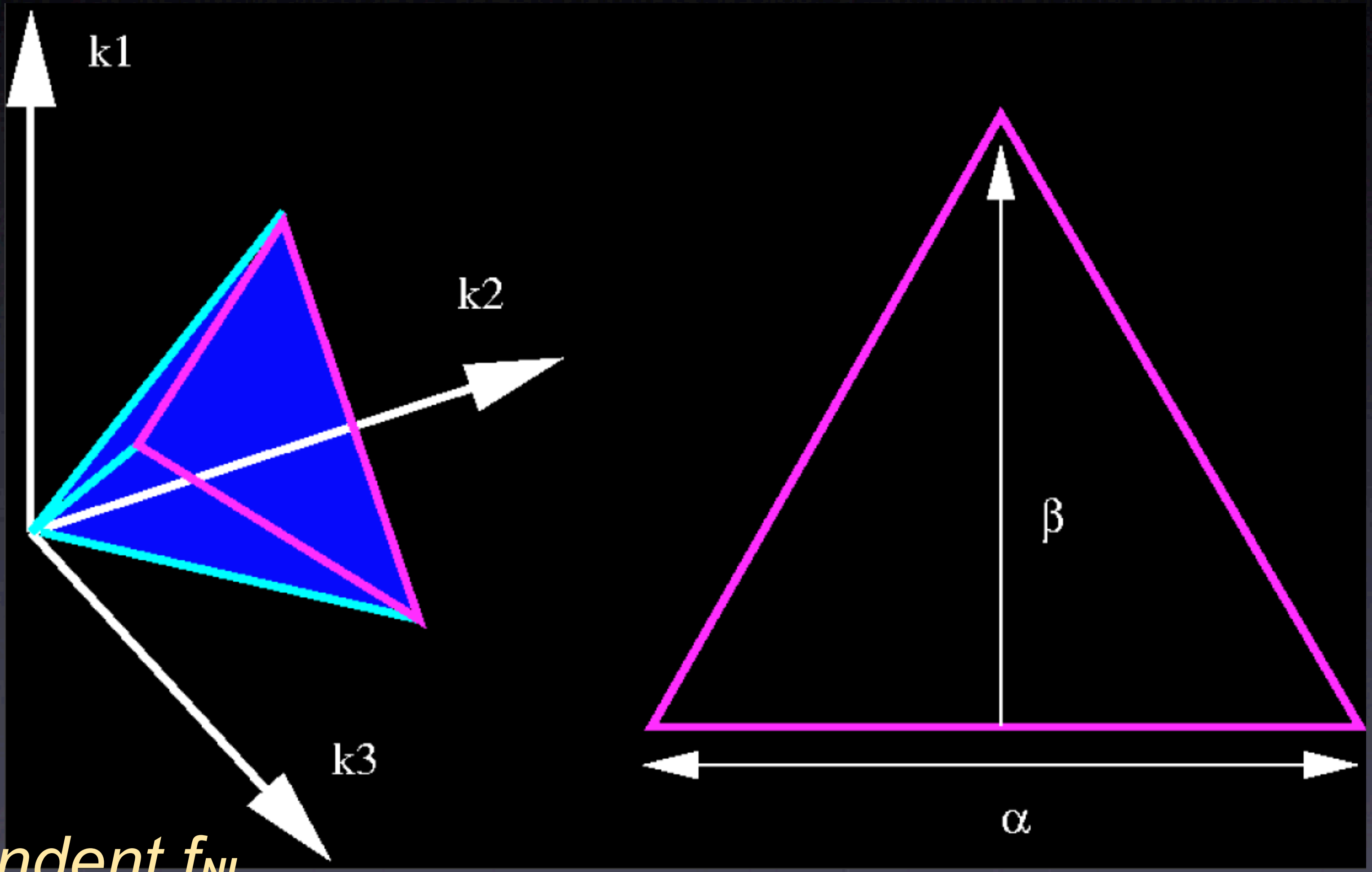


Momentum dependence

Approach suited to calculating <ζ(k_1)ζ(k_2)ζ(k_3)> ➜ 'shape' information

- *Triangular parametrisation appropriate (scale out $k = k_1+k_2+k_3$)*

$$
\begin{aligned}
k_1 &= ka &= k\,(1-\beta) \\
k_2 &= kb &= \frac{1}{2}k\,(1+\alpha+\beta) \\
k_3 &= kc &= \frac{1}{2}k\,(1-\alpha+\beta)\,,
\end{aligned}
$$



- *General momentum dependent f_{NL}*

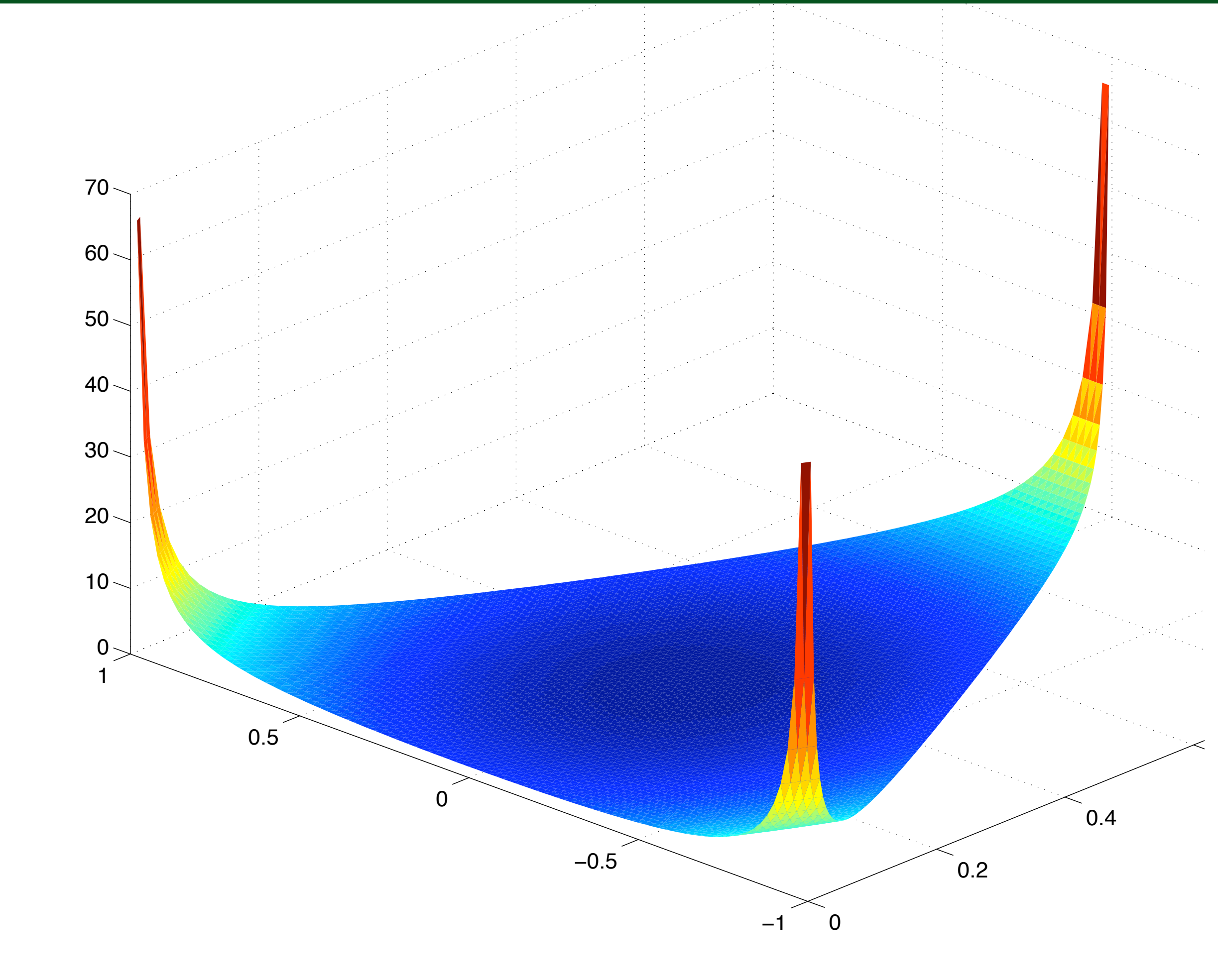
$$
2f_{NL}(k_1,k_2,k_3) = \frac{B^{\Psi}(k_1,k_2,k_3)}{P^{\Psi}(k_1)P^{\Psi}(k_2)+P^{\Psi}(k_2)P^{\Psi}(k_3)+P^{\Psi}(k_3)P^{\Psi}(k_1)}\,.
$$

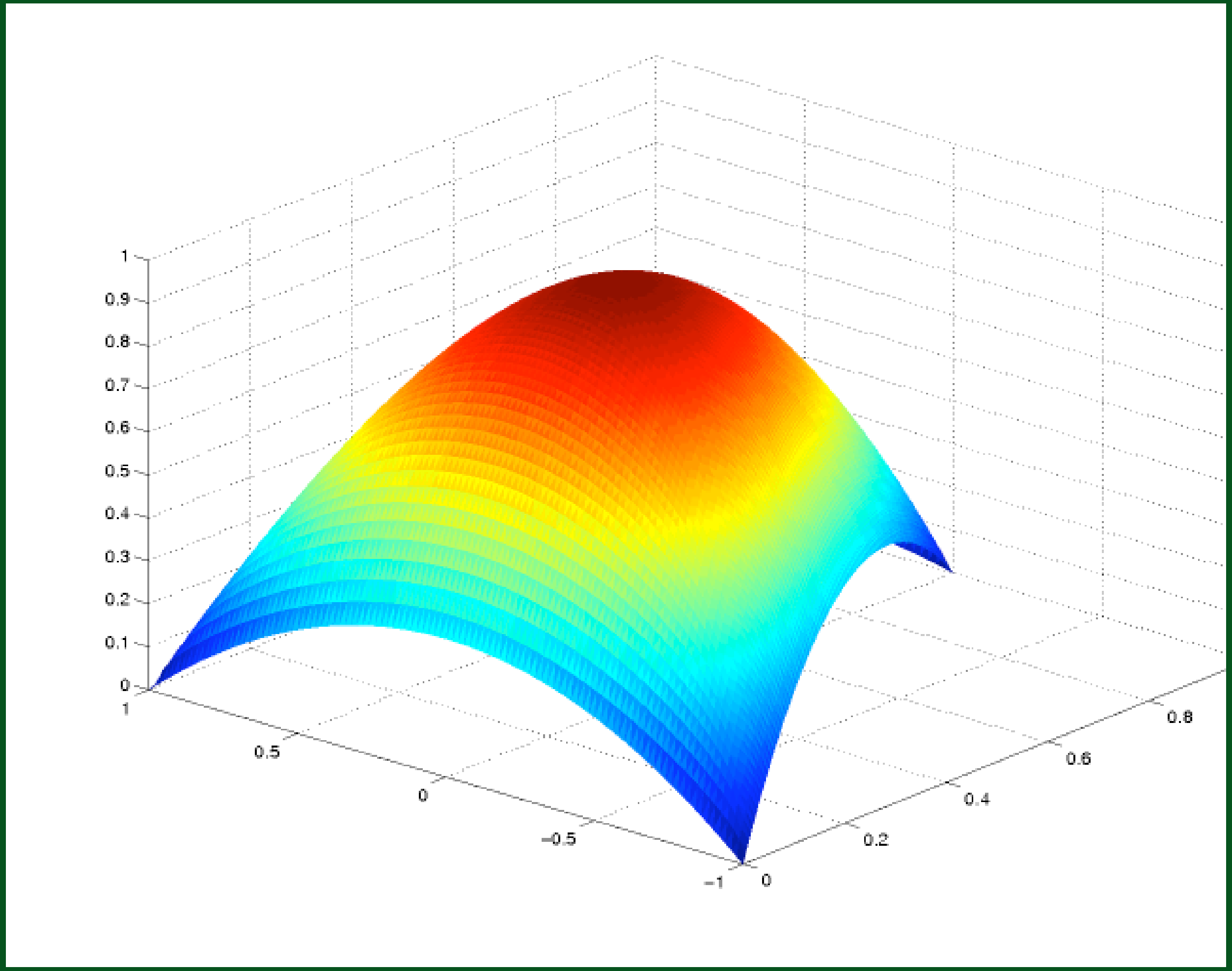
'Local' vs 'Equilateral'

- *In the new parametrisation local and approx. equilateral are:*

$$B^{SI}_{local}(a,b,c) \;=\; \frac{a^3+b^3+c^3}{a\,b\,c}$$

$$B^{SI}_{equilateral}(a,b,c) \;=\; \frac{(1-a)(1-b)(1-c)}{a\,b\,c}\,.$$





Primordial and CMB bispectra

- The angle-averaged bispectrum

$$
\begin{aligned}
B_{l_1 l_2 l_3} &= (8\pi)^3 \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \\
&\times \int dx \int dk_1 \int dk_2 \int dk_3 \, (xk_1k_2k_3)^2 \, B^\Psi(k_1, k_2, k_3) \\
&\times \Delta_{l_1}(k_1)\Delta_{l_2}(k_2)\Delta_{l_3}(k_3) \\
&\times j_{l_1}(k_1 x) j_{l_2}(k_2 x) j_{l_3}(k_3 x).
\end{aligned}
$$

Wigner 3j symbol

Primordial bispectrum

Transfer functions

More problems

- If the primordial bispectrum is separable this simplifies

$$B^\Psi(k_1, k_2, k_3) = \sum_i^N X_i(k_1) Y_i(k_2) Z_i(k_3) \,,$$

- Example: the local approximation

$$B^\Psi(k_1, k_2, k_3) = 2\left(P^\Psi(k_1)P^\Psi(k_2) + P^\Psi(k_2)P^\Psi(k_3) + P^\Psi(k_3)P^\Psi(k_1)\right).$$

The integral reduces to products of 1D integrals

$$\int x^2 dx \; b^L_{l_1}(x) b^L_{l_2}(x) b^{NL}_{l_3}(x) + perms$$

where

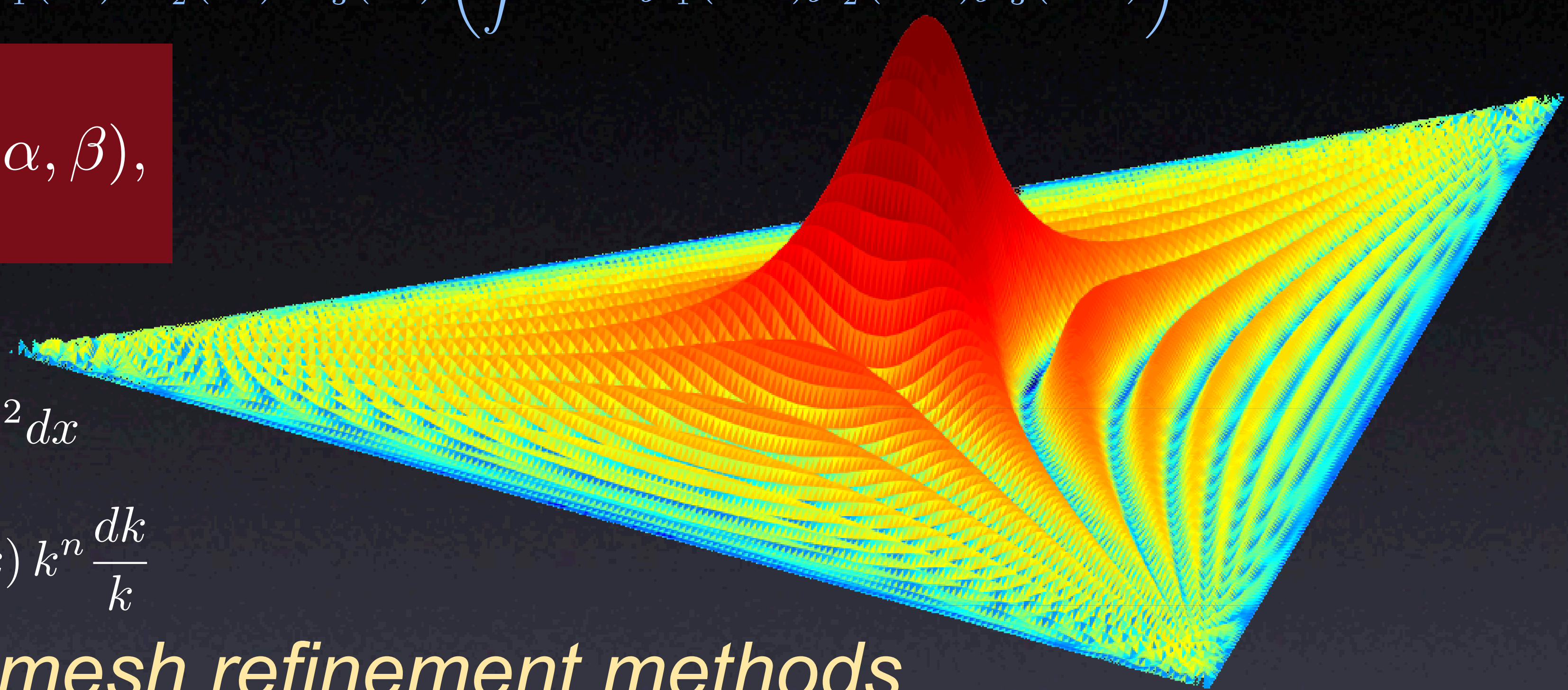
$$
\begin{aligned}
b^L_l(x) &= \int k^2 dk P^\Psi(k) \Delta_l(k) j_l(kx) \\
b^{NL}_{l_3}(x) &= f_{NL} \int k^2 dk \Delta_l(k) j_l(kx) \,,
\end{aligned}
$$

Adaptive integration

- *Assuming an overall scale-dependence f(k)*

$$\int dk_1 dk_2 dk_3 (k_1k_2k_3)^2 B^{\Psi}(k_1,k_2,k_3)\Delta_{l_1}(k_1)\Delta_{l_2}(k_2)\Delta_{l_3}(k_3)\left(\int x^2 dx j_{l_1}(k_1x)j_{l_2}(k_2x)j_{l_3}(k_3x)\right).$$

$$\int d\alpha d\beta\, B^{SI}(\alpha,\beta)\, I^T(\alpha,\beta)\, I^G(\alpha,\beta),$$

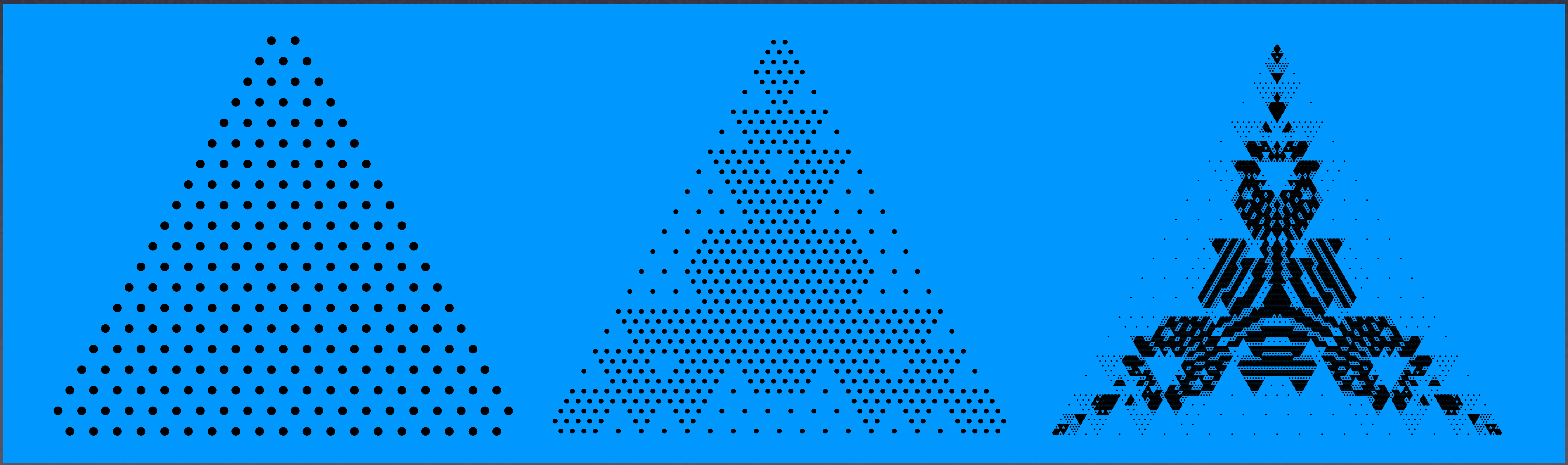


$$B^{SI}(\alpha,\beta) \equiv (abc)^2\, B^{\Psi}(\alpha,\beta),$$

$$I^G(\alpha,\beta) \equiv \int j_{l_1}(ax)\, j_{l_2}(bx)\, j_{l_3}(cx)\, x^2 dx$$

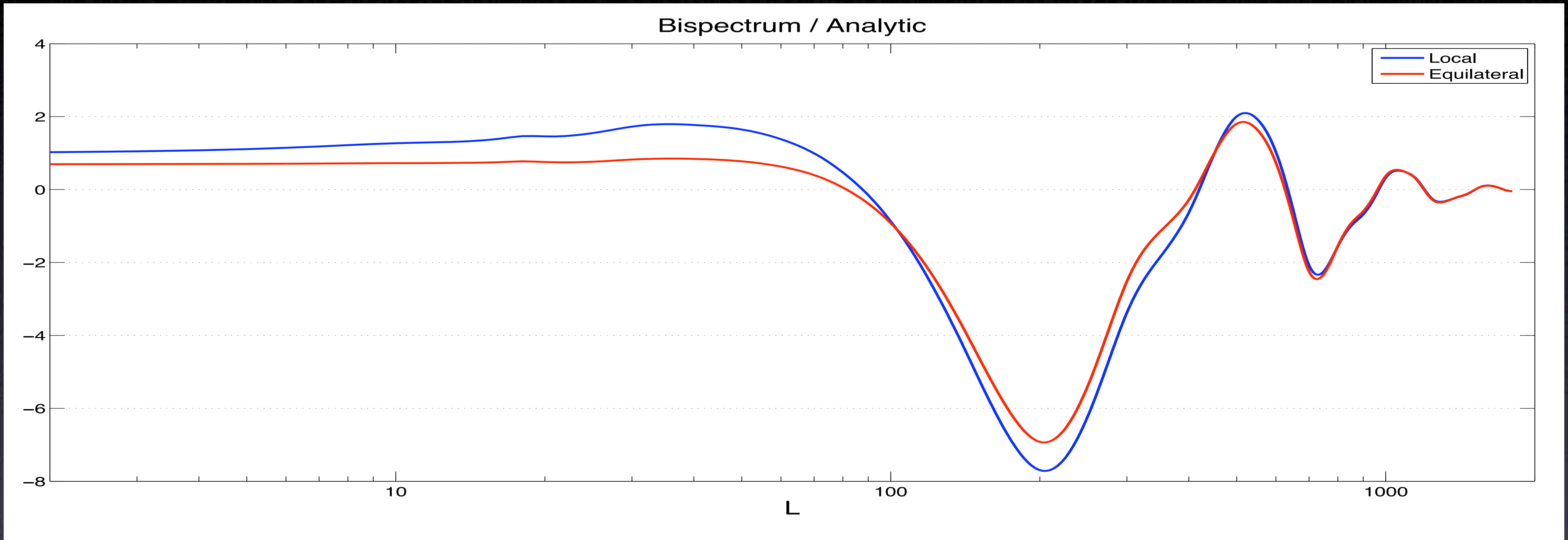
$$I^T(\alpha,\beta) \equiv \int \Delta_{l_1}(ak)\,\Delta_{l_2}(bk)\,\Delta_{l_3}(ck)\, k^n \frac{dk}{k}$$

- *Hierarchical adaptive mesh refinement methods*

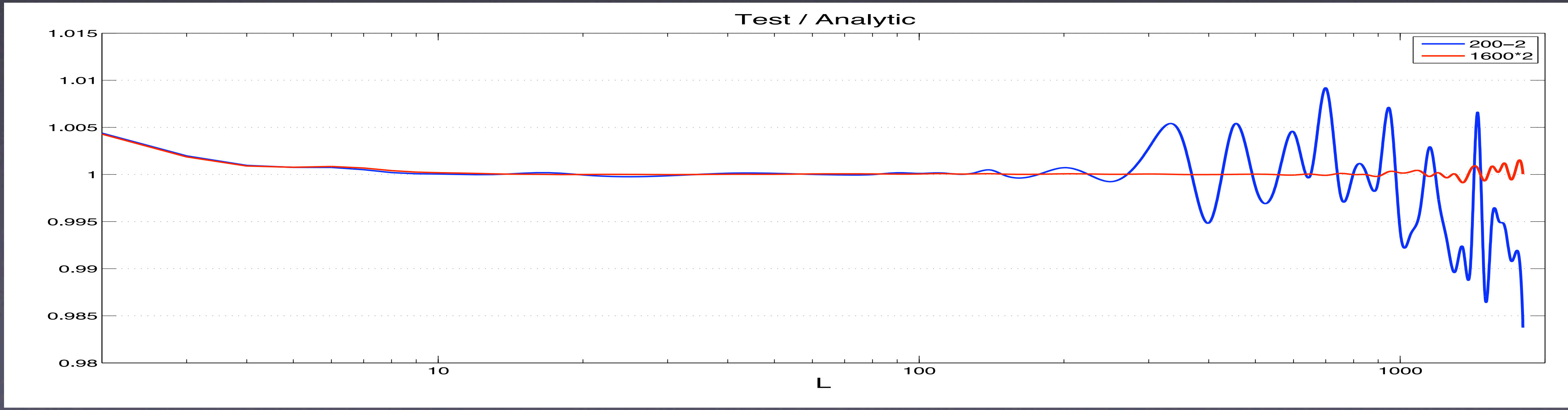


Equal multipole bispectra

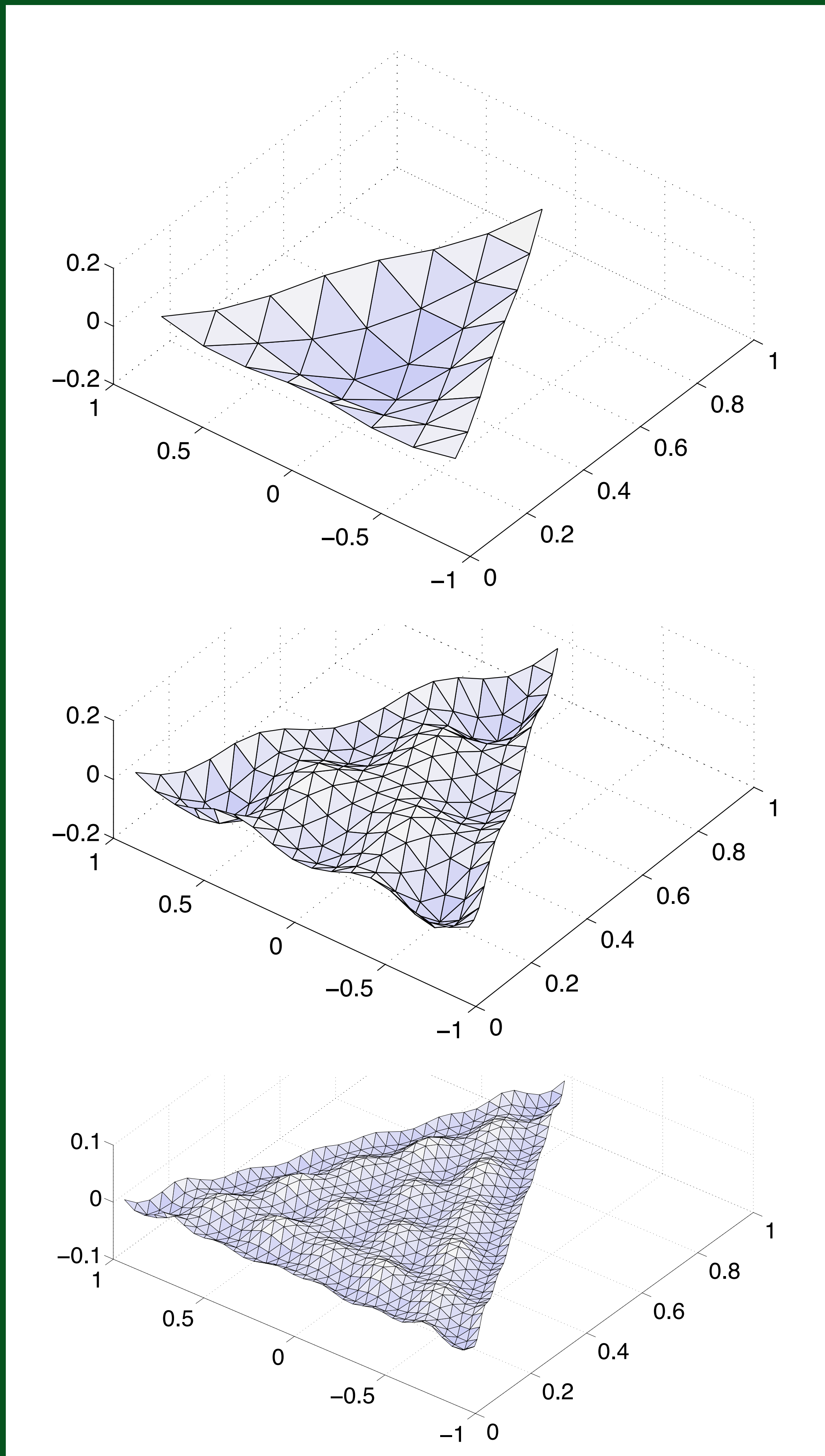
- *Local vs equilateral bispectra with full radiation transfer fns*

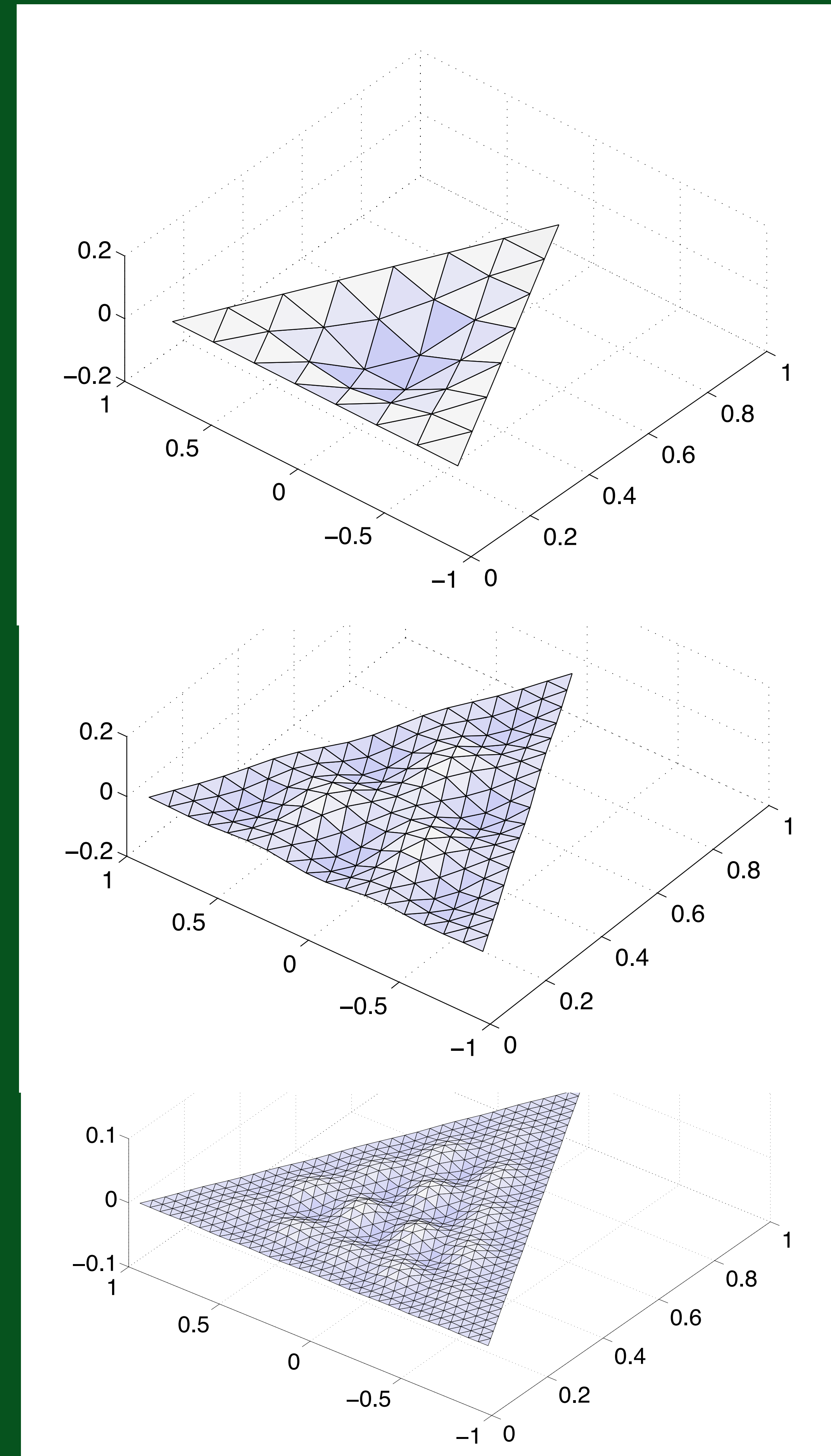


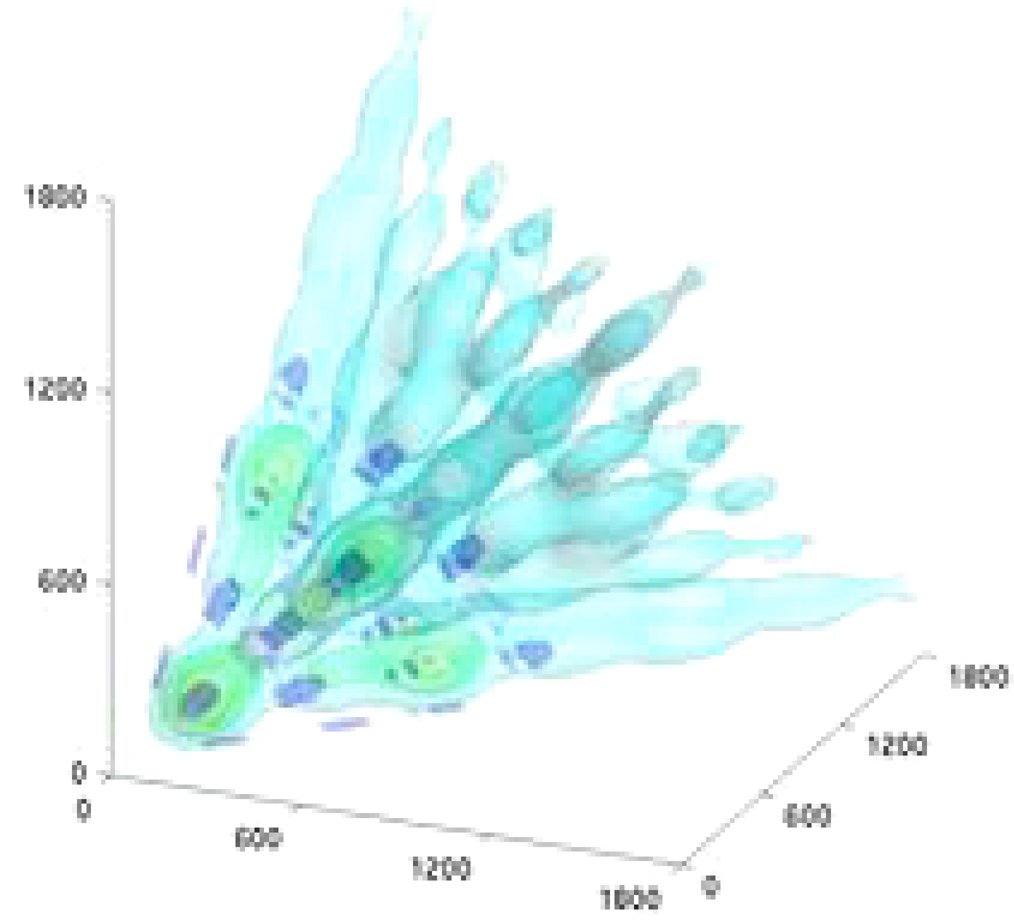
- *Equilateral errors for the large angle approx. (stringent)*



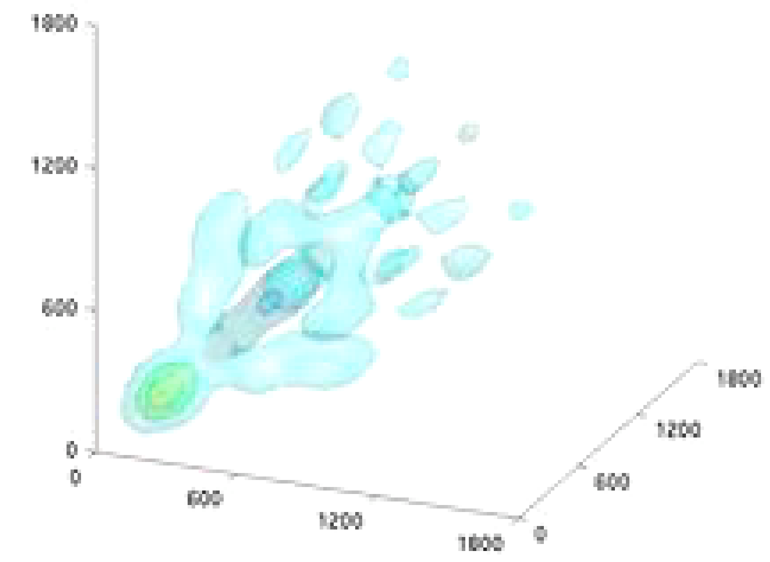
Local vs equilateral bispectra

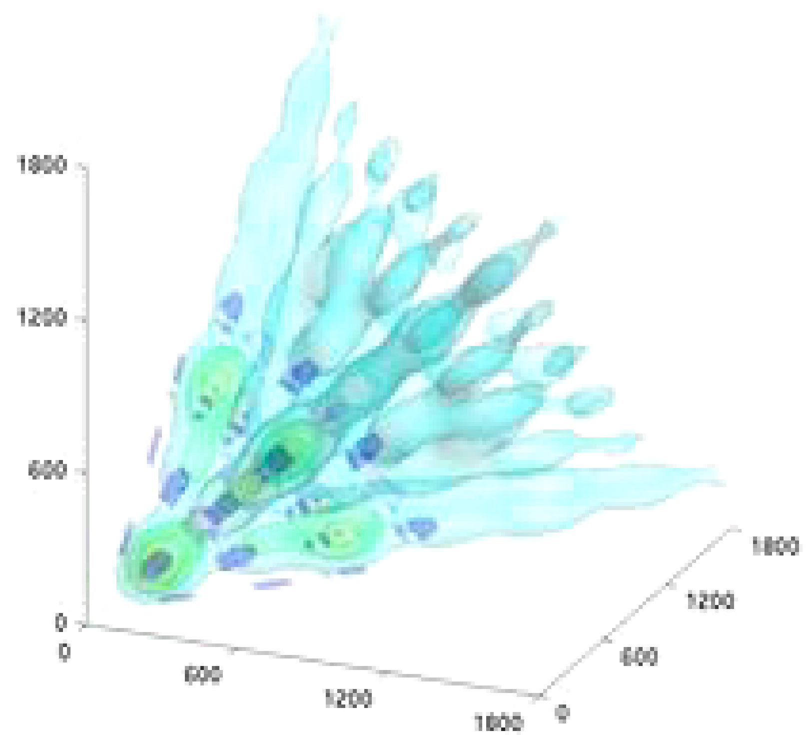




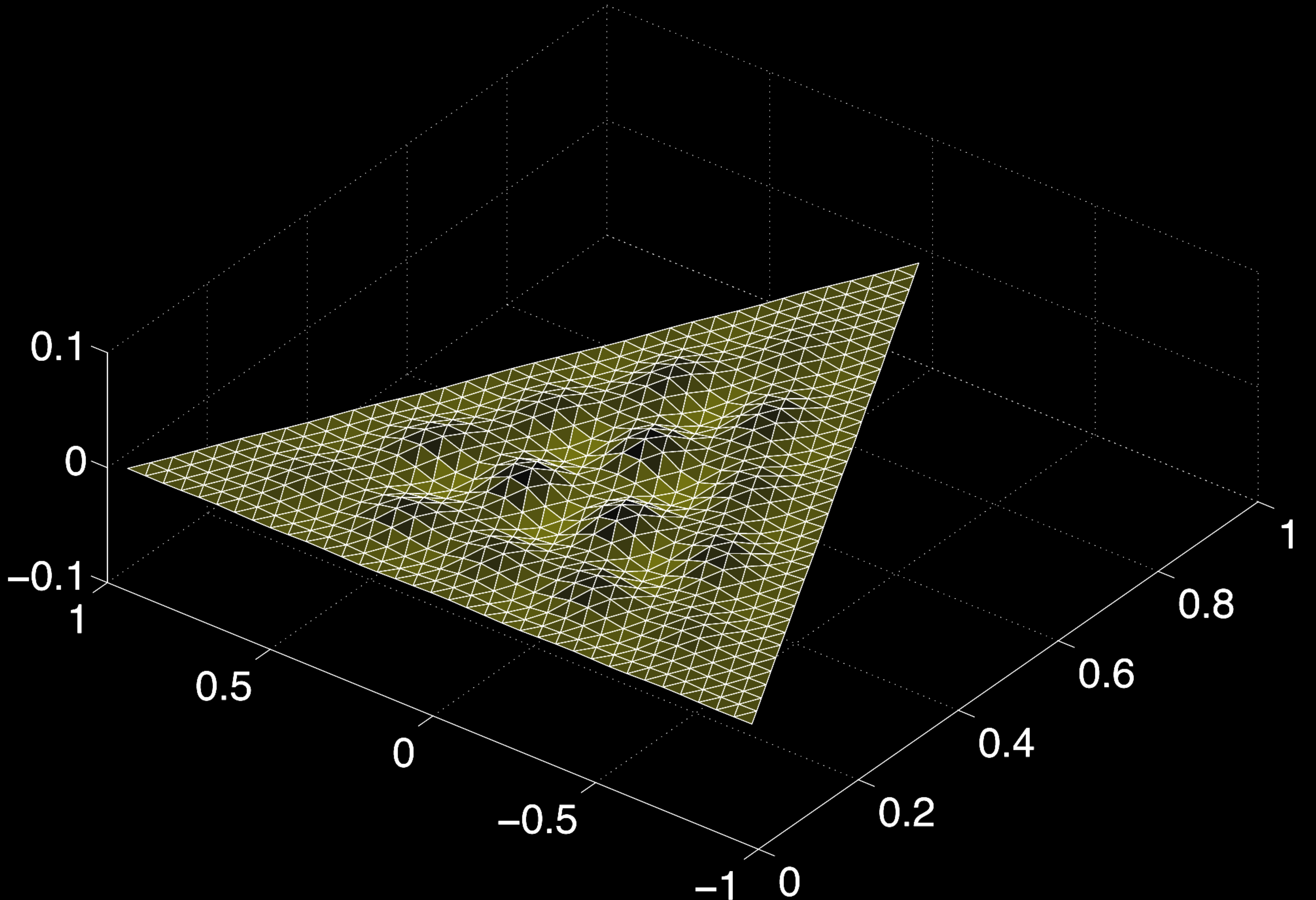

1800
1200
600
0
0
600
1200
1800
0
600
1200
1800

DBI Inflation »»»»

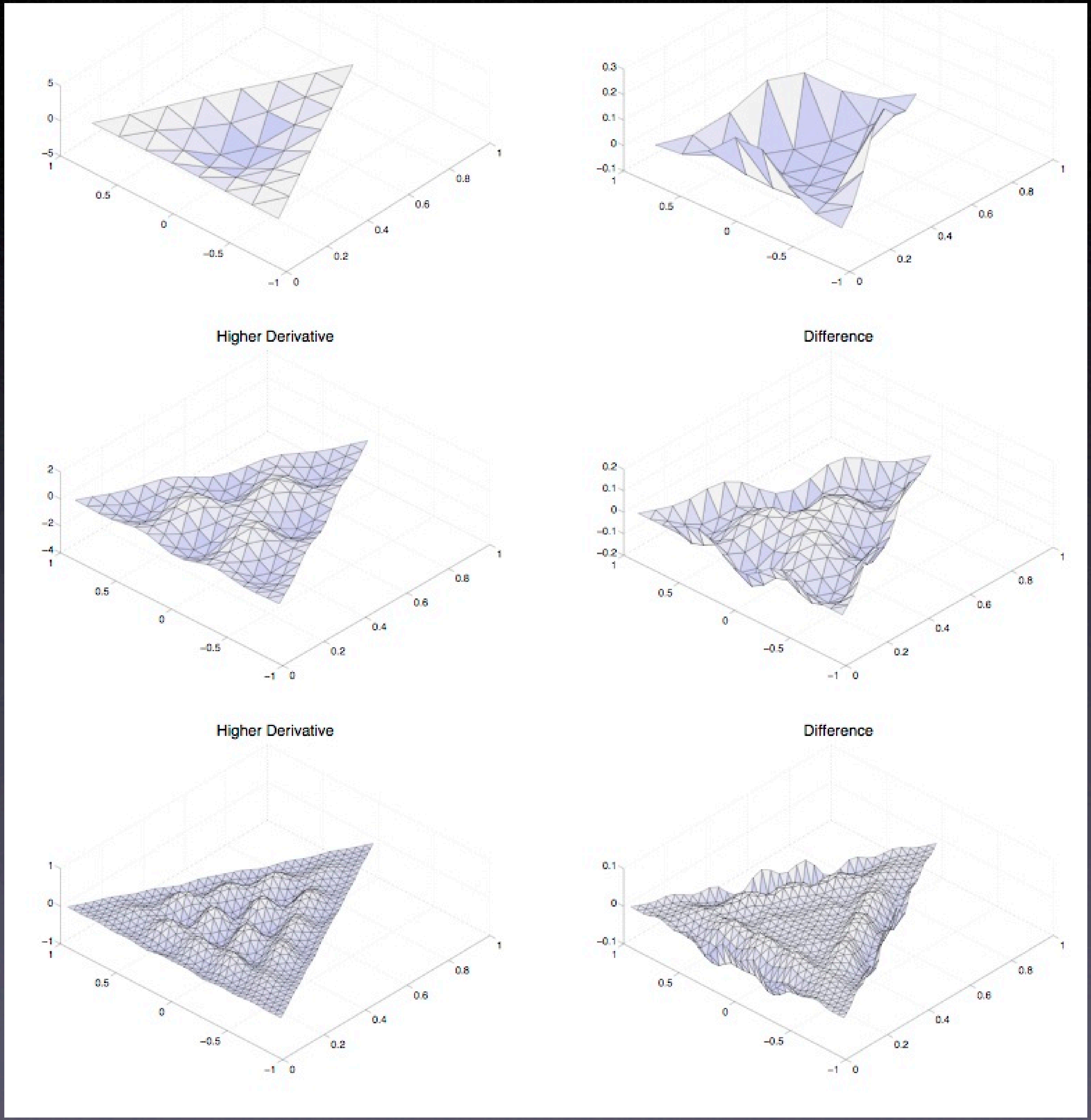




«««« Multifield Inflation


0.1
0
−0.1
1
0.5
0
−0.5
−1
0
0.2
0.4
0.6
0.8
1

DBI vs equilateral bispectra



Non-separable DBI bispectrum

Difference with equilateral approx.

Likelihood analysis

- *Minimising 'least squares' for general primordial bispectra*

$$\mathcal{E} = \frac{1}{N} \sum_{l_i m_i} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \frac{B_{l_1 l_2 l_3}}{C_{l_1} C_{l_2} C_{l_3}} a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3}$$

Wigner 3j symbol

Theoretical model

Planck full sky map

where $N = \sum_{l_1 l_2 l_3} \frac{(B_{l_1 l_2 l_3})^2}{C_{l_1} C_{l_2} C_{l_3}}$

- *Estimator with bispectrum in separable form*

$$b_{l_1 l_2 l_3} = \frac{1}{6} \sum_{i=1}^{N_{fact}} \left(X_{l_1}^{(i)} Y_{l_2}^{(i)} Z_{l_3}^{(i)} + 5 \text{ perms} \right), \qquad X_a^{(i)}(\hat{\mathbf{n}}) = \sum_{lm} X_l^{(i)} \frac{a_{lm}}{C_l} Y_{lm}(\hat{\mathbf{n}}),$$

$$S = \frac{1}{N} \sum_{i=1}^{N_{fact}} \int d\hat{\mathbf{n}} X_a^{(i)}(\hat{\mathbf{n}}) Y_a^{(i)}(\hat{\mathbf{n}}) Z_a^{(i)}(\hat{\mathbf{n}}).$$

Separable expansion

- Smooth bispectrum implies accurate sum with basis functions

$$b_{l_1 l_2 l_3} = \frac{1}{3} \sum_{\alpha\beta\gamma} a_{\alpha\beta\gamma} \left(X'_\alpha(l_1) X'_\beta(l_2) X_\gamma(l_3) + 2 \text{ permutations} \right) ,$$

-

$$X_\alpha(l) = P_\alpha(\frac{2l - l_{max}}{l_{max}}), \qquad X'_\alpha(l) = \frac{X_\alpha(l)}{l(l+1)}.$$

- With expansion coefficients given by ...

$$\left(\frac{l_1(l_1+1) l_2(l_2+1) l_3(l_3+1)}{l_1(l_1+1) + l_2(l_2+1) + l_3(l_3+1)} \right) b_{l_1 l_2 l_3} = \sum_{\alpha\beta\gamma} a_{\alpha\beta\gamma} X_\alpha(l_1) X_\beta(l_2) X_\gamma(l_3)$$

$$a_{\alpha\beta\gamma} = (2\alpha+1)(2\beta+1)(2\gamma+1) \int \frac{dl_1 dl_2 dl_3}{l_{max}^3} \left(\frac{l_1(l_1+1) l_2(l_2+1) l_3(l_3+1)}{l_1(l_1+1) + l_2(l_2+1) + l_3(l_3+1)} \right) b_{l_1 l_2 l_3} X_\alpha(l_1) X_\beta(l_2) X_\gamma(l_3$$

- So the estimator becomes ...

$$\bar{X}_\alpha(\hat{\mathbf{n}}) = \sum_{lm} X_\alpha(l) \frac{a_{lm}}{C_l} Y_{lm}(\hat{\mathbf{n}}), \qquad \bar{X}'_\alpha(\hat{\mathbf{n}}) = \sum_{lm} \frac{X_\alpha(l)}{l(l+1)} \frac{a_{lm}}{C_l} Y_{lm}(\hat{\mathbf{n}}),$$

$$S = \frac{1}{N} \sum_{\alpha\beta\gamma} a_{\alpha\beta\gamma} M_{\alpha\beta\gamma} \qquad \text{where} \qquad M_{\alpha\beta\gamma} = \frac{1}{3} \int d\hat{\mathbf{n}} \left(\bar{X}'_\alpha(\hat{\mathbf{n}}) \bar{X}'_\beta(\hat{\mathbf{n}}) \bar{X}_\gamma(\hat{\mathbf{n}}) + 2 \text{ perms} \right) .$$

Conclusions

- *Quantitative calculations of primordial non-Gaussianity*
  - tractable with full momentum dependence
- *Quantitative calculation of resulting CMB non-Gaussianity*
  - without simplifying assumptions of separability
- *Separable expansion for CMB bispectrum estimators*
  - Smooth primordial models well-approximated (Chebyshev)
- *Aim is seamless confrontation between early universe bispectrum predictions and CMB observations* ***[see astro-ph/0612713]***

Generalised stochastic approach

- Nonlinear spatial gradients *(time-slice invariant)*:

Rigopoulos & EPS (astro-ph/0306620)
see also Langlois & Vernizzi (0503416)

$$\zeta_i^A = \frac{\Pi^A}{\Pi}\partial_i \ln a - \frac{H}{\Pi}\partial_i\phi\,, \qquad \text{with} \qquad H(t,\mathbf{x}) = \frac{1}{N}\frac{\dot a}{a}\,, \quad \Pi^A = \frac{\dot\phi^A}{N}$$

- Master equation *(direct from long-wavelength Einstein eqns)*:

Rigopoulos, EPS, van Tent (astro-ph/0504508)

$$\begin{cases} \mathcal{D}_\tau \zeta_i^A - \theta_i^A = \mathcal{S}_i^A \\ \mathcal{D}_\tau \theta_i^A + \left(\frac{3-2\tilde\epsilon+2\tilde\eta^{\parallel}-3\tilde\epsilon^2-4\tilde\epsilon\tilde\eta^{\parallel}}{(1-\tilde\epsilon)^2}\delta_{AB} + \frac{2}{1-\tilde\epsilon}Z_{AB}\right)\theta_i^B + \frac{1}{(1-\tilde\epsilon)^2}\,\Xi^A{}_B\,\zeta_i^B = \mathcal{J}_i^A \end{cases}$$

- Stochastic source terms:

(RSvT-1 following Starobinsky)

The source terms $\mathcal{S}_i^A$ and $\mathcal{J}_i^A$ emulate small-scale quantum effects:

$$\mathcal{S}_i^A \equiv \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\,\dot{\mathcal{W}}(k)\,\zeta^A_{\mathrm{lin}}(\boldsymbol{k},\boldsymbol{x})\,\mathrm{i}k_i\,\mathrm{e}^{\mathrm{i}\boldsymbol{kx}} + \mathrm{c.c.}\,,$$

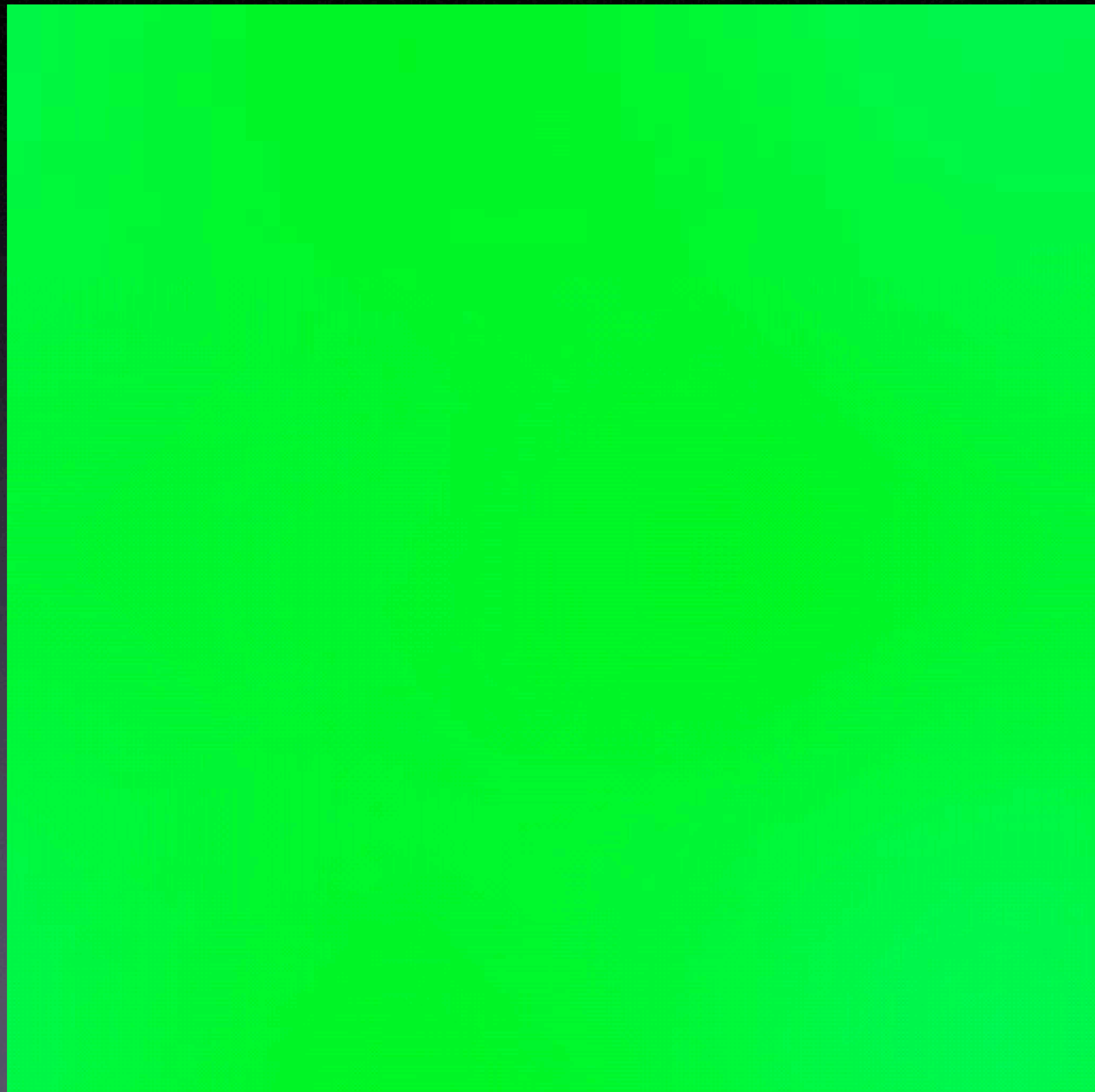
$$\mathcal{J}_i^A \equiv \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^{3/2}}\,\dot{\mathcal{W}}(k)\,\theta^A_{\mathrm{lin}}(\boldsymbol{k},\boldsymbol{x})\,\mathrm{i}k_i\,\mathrm{e}^{\mathrm{i}\boldsymbol{kx}} + \mathrm{c.c.}\,,$$

with linear solns $\zeta^A_{\mathrm{lin}} = \frac{-\kappa}{a\sqrt{2\tilde\epsilon}}\,q^A_{\mathrm{lin}}\,, \qquad \theta^A_{\mathrm{lin}} = \mathcal{D}_\tau\zeta^A_{\mathrm{lin}}\,, \qquad q^A_{\mathrm{lin}} = Q^A_{\mathrm{lin}\,B}(k)\alpha^B(\mathbf{k})$

where the $\alpha(\mathrm{k})$ are Gaussian complex random numbers satisfying

$$\langle\alpha^A(\boldsymbol{k})\alpha^*_B(\boldsymbol{k}')\rangle = \delta^3(\boldsymbol{k}-\boldsymbol{k}')\delta^A_B, \qquad \langle\alpha^A(\boldsymbol{k})\alpha_B(\boldsymbol{k}')\rangle = 0.$$

Nonlinear stochastic evolution



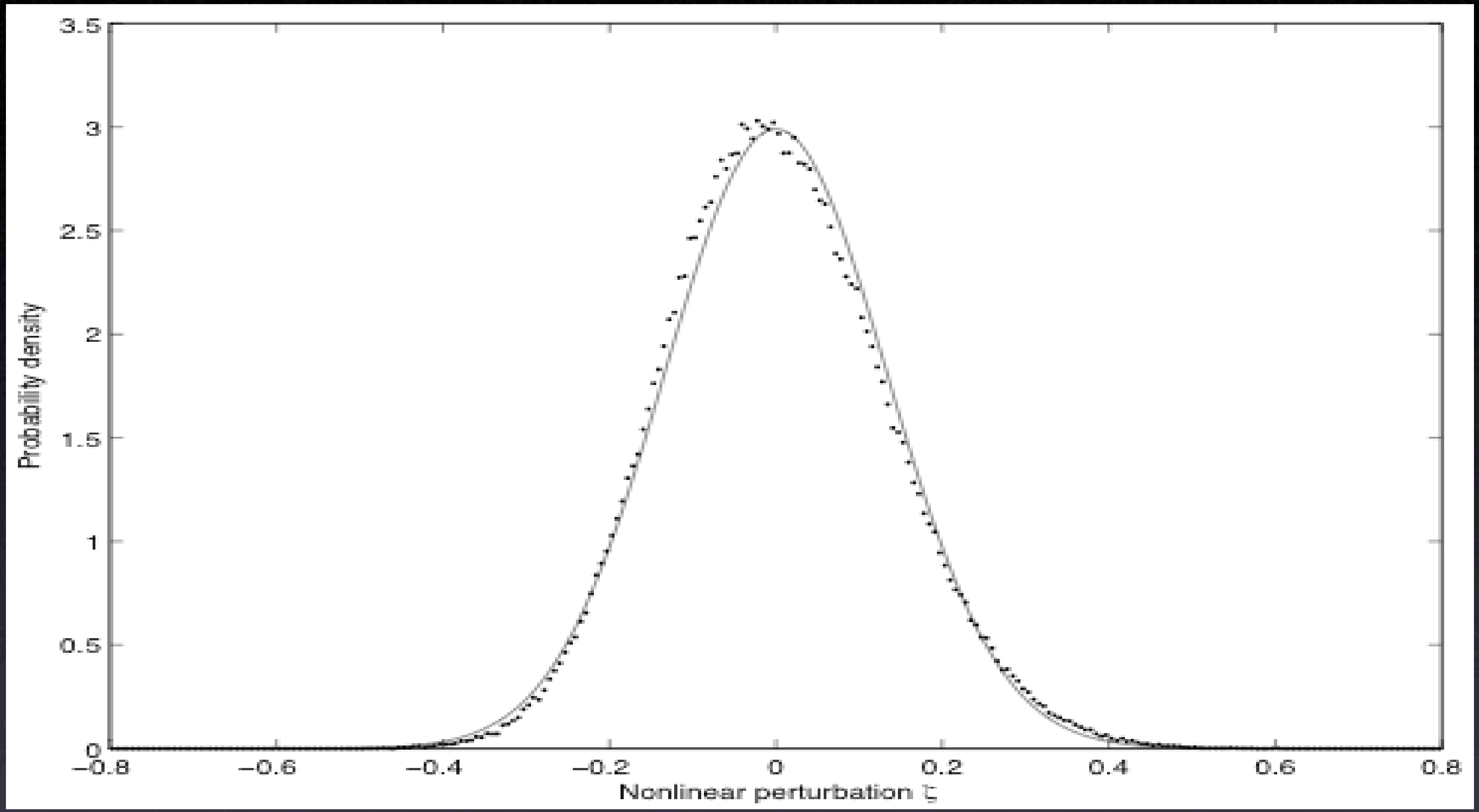
Probability density function

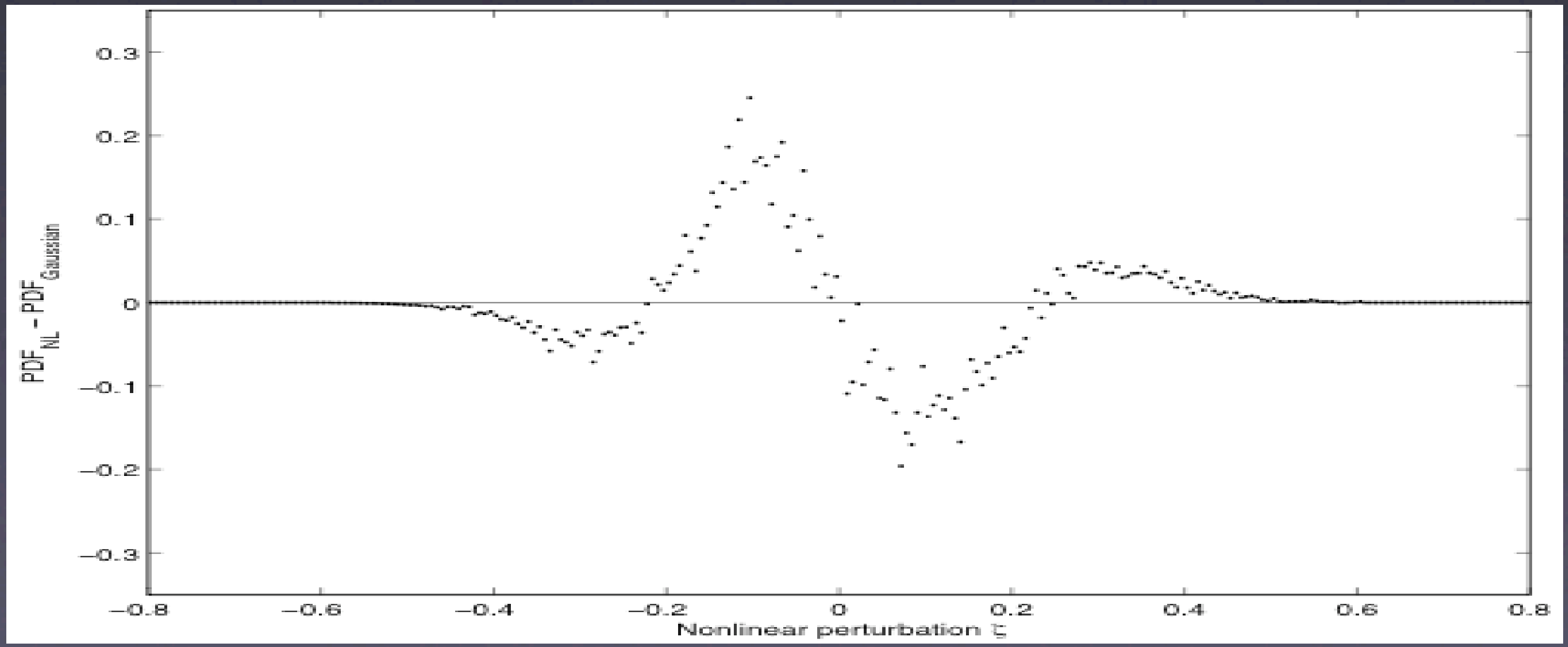
Preliminary results:

Single-field inflation generically very Gaussian

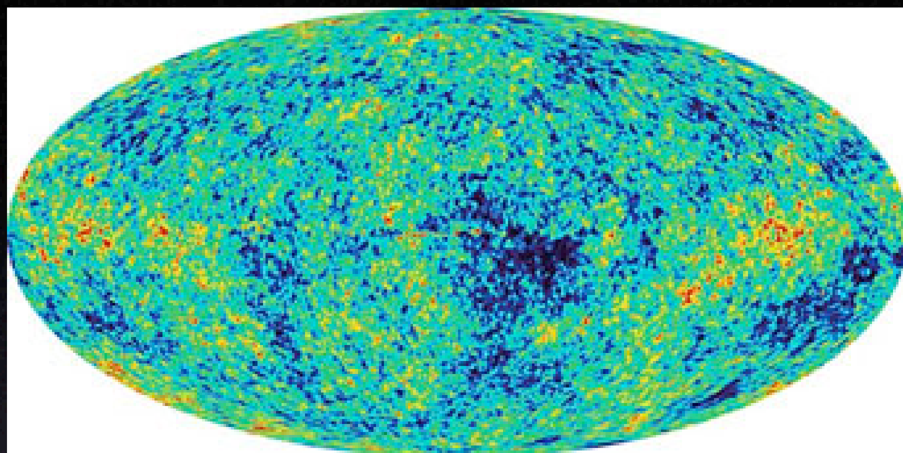
Significant skewness (right) only with extreme params (eternal inflation?)

Focus on multifield simulations …



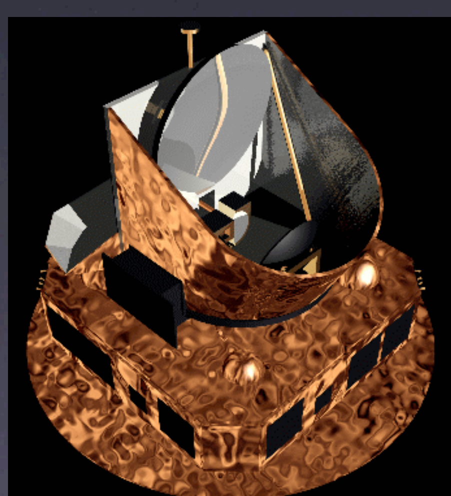


CMB future prospects



- ***WMAP*** Komatsu *et al*, '03; Wright *et al*, '05
  Non-Gaussianity limits $-58 < f_{NL} < 134$
  Direct searches for strings inconclusive
  Planck satellite limit $|f_{NL}| < 5$

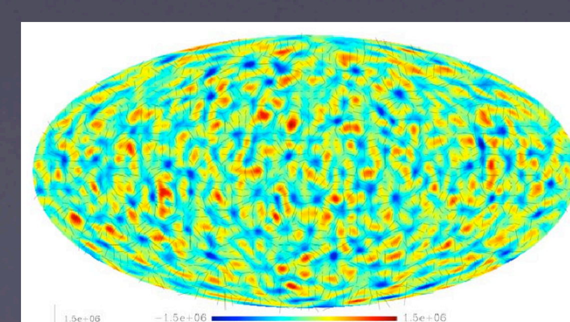
- ***High resolution CMB experiments***
  Interferometers (e.g. AMI, ACT) with arcminute resolution may detect line-like discontinuities



- ***B-mode polarization***
  Vector/tensor modes seed B-mode polarization (unlike dominant inflation scalar mode)
  e.g. Planck, CLOVER & CMBPOL

Pen et al, '97; Benabed & Bernardeau, '03; Seljak & Slosar, '06

- ***Other observational tests***
  Weak lensing, grav. waves etc.



Landriau, Komatsu, EPS, in prepn