Solutions of field equations in f(R) theories of gravity

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Outline of Topics

Accelerating expansion?!

There is strong evidence that the observed expansion history of the universe is not consistent with a *homogeneous* Einstein-de Sitter -model. Possible explanations include:

- Mysterious new form of energy
  - Λ, quintessence, Chaplygin gas, ...
- Inhomogeneous universe
  - Lemaitre-Tolman-Bondi model
- Modified gravity
  - Extra dimensions (DGP model), scalar-tensor -theories, f(R) theories, ...

f(R) theories: Basic equations

Action:

$$S = \int d^4x \sqrt{-g}\Big(f(R) + \mathcal{L}_m\Big)$$

The field equations in the metric approach ($8\pi G = 1$):

$$\boxed{F(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu F(R) + g_{\mu\nu}\Box F(R) = T^m_{\mu\nu}},$$

$T^m_{\mu\nu}$ is the standard minimally coupled stress-energy tensor and $F(R) \equiv df/dR$.

Contracting and assuming a perfect fluid gives

$$F(R)R - 2f(R) + 3\Box F(R) = \rho - 3p.$$

Cosmological solutions

In a Friedmann-Robertson-Walker universe:

$$H^2+\frac{k}{a^2}=\frac{1}{3}\Big(\rho_c+\frac{\rho_m}{f'(R)}\Big)$$
$$2\dot{H}+3H^2+\frac{k}{a^2}=-\left(p_c+\frac{p_m}{f'(R)}\right),$$

where the energy density and pressure of the curvature fluid are

$$\rho_c=\frac{1}{f'(R)}\Big\{\frac{1}{2}\Big(f(R)-Rf'(R)\Big)-3H\dot{R}f''(R)\Big\}$$
$$p_c=\frac{1}{f'(R)}\Big\{\dot{R}^2f'''(R)+2H\dot{R}f''(R)+\ddot{R}f''(R)-\frac{1}{2}\Big(f(R)-Rf'(R)\Big)\Big\}.$$

- Fourth order equations, the expansion history does not uniquely determine the form of $f(R)$
- Eg. assuming EdS-type expansion, $H^2\sim a^{-3}$, one can show that

$$f(R)=R+c_+(-R)^{(7+\sqrt{73})/12}+c_-(-R)^{(7-\sqrt{73})/12},$$

  where $c_\pm$ are arbitrary constants, is a solution

Solutions in empty space

Consider a static spherically symmetric metric:

$$ds^2 = A(r)^2 dt^2 - B(r)^2 dr^2 - r^2(d\theta^2 + \sin^2\varphi^2).$$

The field equations can be written as ($' \equiv d/dr$)

$$F\,R_{\mu\nu} - \nabla_\mu\nabla_\nu F = \frac{1}{4}g_{\mu\nu}(F\,R - \Box F)$$
$$RF' - R'F + 3(\Box F)' = 0,$$

so that given a metric $g_{\mu\nu}$, one can solve for $F(r)$ and consequently for $f(R)$.
Eg.

$A(r)$	$A(r)B(r)$	$f(R)$
$1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2$	2	$R \pm \frac{2}{3M}\sqrt{-R-2\Lambda} + \Lambda$
$s_0 r^m$	$F_0 r^n,\ n = 2\frac{m(m-1)}{m-2}$	$\frac{2F_0}{2-n}\left(\frac{3n(2-n)}{n^2-2n-2}\right)^{n/2} R^{1-n}.$

Solutions in empty space: constant scalar curvature case

Looking for constant curvature solutions, $R = R_0$, the field equations reduce to

$$AB' + AB' = 0,\ 1 - B + \frac{r}{2}(\frac{B'}{B} + \frac{A'}{A})(\frac{r}{2}\frac{A'}{A} - 1) - \frac{r^2}{2}\frac{A''}{A} = 0$$

which are straightforwardly solvable:

$$B(r) = \frac{c_0}{A(r)}, A(r) = c_0 + \frac{c_1}{r} + c_2 r^2$$

which after rescaling of the time coordinate is equivalent to the Scwarzschild-de Sitter (SdS) -metric (the Schwarzschild solution in the presence of a cosmological constant Λ).

Hence any $f(R)$ theory that satisfies $R_0 f'(R_0) - 2f(R_0) = 0$ has the SdS solution as an exact vacuum solution with $R_0 = -4\Lambda$.

Static fluid sphere solutions

Consider static, spherically symmetric perfect fluid solutions (SSSPF-solutions). The surface of the fluid sphere is defined by the surface of zero pressure where the interior solution is matched to the outside metric.

Field equations and continuity equation:

$$FR_{\mu\nu} - \frac{1}{4}(FR - \Box F)g_{\mu\nu} - \nabla_\mu \nabla_\nu F = T^m_{\mu\nu} - \frac{1}{4}(\rho - 3p)g_{\mu\nu}$$

$$\frac{p'(r)}{\rho(r) + p(r)} = -\frac{1}{2}\frac{A'(r)}{A(r)}.$$

This set can be reduced to a single non-linear equation, the modified Tolman-Oppenheimer-Volkov (mTOV) -equation:

$$\boxed{mTOV(F, \rho, \rho', \rho'', p, p', p'', p''') = S_f},$$

where $S_f = S_f(F', F'', F''', \rho, \rho', \rho'', p, p', p'', p''')$ represents a source term.

Static fluid sphere solutions: Examples

- In the GR limit, i.e. $F = 1$, $S_f = 0$, mTOV is satisfied, whenever ρ and p satisfy the usual Tolman-Oppenheimer-Volkov (TOV) -equation.
- Given $\rho(r)$ and $p(r)$, one can solve for $f(R)$
- mTOV is again a higher-order equation: number of solutions exist
  - as a result, the matter distribution alone does not uniquely determine the form of $f(R)$
  - eg. if the ordinary TOV-equation satisfied, non-trivial solutions for $F(r)$ exist
  - eg. for a given standard fluid sphere -solution of general relativity, $F(r) = 1$ is not the only possible solution.
  - eg. $p = 0$, $\rho = 3r$, $f(R) = -2\sqrt{-6 - R}$ or $f(R) = 6\sqrt{3}/\sqrt{-R}$ (non-trivial $B(r)$, $F(r)$ and $R(r)$)

Static fluid sphere solutions: Boundary conditions

Boundary conditions are crucial in determining which solution is relevant in each case:

- Requiring Schwarzschild-de Sitter -space time (SdS) as the outside solution of a fluid sphere constrains the star more than in GR
- $\rho(r_0) = \rho'(r_0) = p(r_0) = p'(r_0) = p''(r_0) = 0$ on the surface for a general $f(R)$ theory
- The Scwarzschild -fluid sphere with a Schwarzschild solution outside is not a valid solution unless $F(r) = 1$ i.e. in the GR limit

Polytropic stars, $p = \kappa\rho^\gamma$:

- in general, $1 < \gamma < 2$
- Requiring that the SdS metric is the outside solution on the boundary of the star leads to a *singular metric* at the center of the star in the $R - \mu^4/R$ model
- Conversely, a regular polytropic solution sets the outside solution to be inconsistent with observations ($\gamma_{PPN} = 1/2$) in the same model

Conclusions

- Modified gravity is an interesting possibility to explain the accelerating expansion of the universe
- The higher order nature of the field equations leads to degeneracies in trying to determine the form of $f(R)$
- Exact static spherically symmetric solutions can be constructed both in the vacuum and in the presence of a perfect fluid
- Boundary conditions are crucial in determining the physical relevance of each solution