Non-Gaussianity of the primordial perturbation in the curvaton model

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A: a fully non-linear calculation with one curvaton; comparison of an analytical sudden-decay approximation to numerical results
M. Sasaki (YITP, Kyoto), JV, & D. Wands (ICG, Portsmouth)
Phys. Rev. D **74**, 103003 (2006) [astro-ph/0607627].

B: non-Gaussianity in two-curvaton model in the sudden-decay approximation
H. Assadullahi, JV, & D. Wands (ICG, Portsmouth)
arXiv:0708.0223v1 [hep-ph].

Motivation to study non-Gaussianity in the curvaton model:
The curvaton model is one of the very few examples where it is possible to produce an observable amount of non-Gaussianity relatively easily.

Curvaton model in its simplest form

- During inflation two light ($m^2 \ll H^2$) scalar fields: inflaton ϕ and curvaton χ. The potential could be, e.g., $V = \frac{1}{2}M^2\phi^2 + \frac{1}{2}m^2\chi^2$.
- **Inflaton drives the expansion, curvaton is subdominant:** $\rho_\phi \gg \rho_\chi$.
- At horizon exit during inflation both fields acquire classical **Gaussian** perturbations that freeze in:

  in slow-roll $P_{\delta\phi} \sim P_{\delta\chi} \sim \left(\frac{H_*}{2\pi}\right)^2$, but $\frac{P_{\zeta_\phi}}{P_{\zeta_\chi}} \sim \left(\frac{\dot{\chi}}{\dot{\phi}}\right)^2$, which can be $\ll 1$.
- Since the observed CMB and LSS perturbations are thought to result from curvaton, **the curvature perturbation induced by the inflaton perturbation, ζ_ϕ, can be much smaller** than the usually demanded 10^{-5}.
- **After inflation**, in reheating, the energy of the inflaton field is converted into ultra-relativistic particles (radiation, $r0$); $\rho_{r0} \gg \rho_\chi$.

... Curvaton model in its simplest

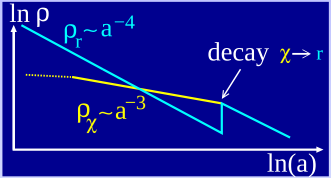
- H is a decreasing function of time $\Rightarrow$ finally $m_\chi^2 \gtrsim H^2$.

$\Rightarrow$ **Curvaton starts to oscillate** at the bottom of its quadratic potential. In what follows I call this time as $t_{\rm in}$.

- Often, it is assumed that still at this time $\rho_{r0} \gg \rho_\chi$.
- Due to the harmonic oscillation curvaton behaves (on super-Hubble scales) like pressureless dust $\Rightarrow \rho_\chi \propto \mathbf{a^{-3}} \Rightarrow \rho_\chi/\rho_{r0}$ grows as a.
- Finally (when $H \leq \Gamma_\chi$) the curvaton decays into radiation $\Rightarrow$ immediately after the decay $\rho_r = \rho_{r0} + \rho_\chi$.
- Only "radiation" left, and the standard adiabatic "initial conditions" follow.
- To ensure the standard later evolution the curvaton must decay before nucleosynthesis etc.

Curvaton model schematically

- "Curvaton" as an origin of perturbations [Lyth&Wands 2001, Enqvist&Sloth 2001]
- In addition to inflaton, ϕ, another light scalar field, **curvaton** χ



↑ Re-heating $\rho_\phi \rightarrow \rho_{r0}$

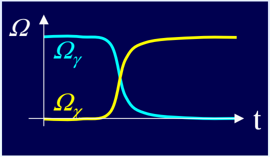
↑ Start of curvaton oscillation $t_{\rm in}$

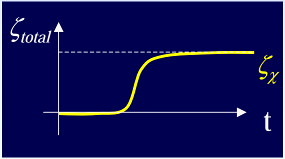
↑ Primordial nucleosynthesis

Before the decay:

the density parameters

the curvature perturbation $\zeta = \frac{\dot\rho_{r0}}{\dot\rho}\zeta_{r0} + \frac{\dot{\boldsymbol\rho}_{\boldsymbol\chi}}{\dot{\boldsymbol\rho}}\boldsymbol\zeta_{\boldsymbol\chi,\mathbf{in}}$





Curvature perturbation

- Gauge invariant **first order definition** of perturbation; $\zeta_i = -\mathcal{H}\frac{\delta\rho_i}{\dot\rho_i} - \psi\,.$
- The total perturbation can be composed as

$$\begin{aligned}\zeta_{\rm tot} &= \frac{\dot\rho_{r0}}{\dot\rho}\zeta_{r0} + \frac{\dot\rho_\chi}{\dot\rho}\zeta_\chi \\ \text{(for pure curvaton model } \zeta_{r0} \ll \zeta_{\chi,\rm in}) &\approx \frac{3\rho_\chi}{4\rho_{r0}+3\rho_\chi}\zeta_\chi \\ \text{(NOTE: during osc. } \zeta_\chi = \zeta_{\chi,\rm in} = \text{const.)} &= \frac{3\Omega_\chi}{4\Omega_{r0}+3\Omega_\chi}\zeta_{\chi,\rm in}\,.\end{aligned}$$

- In the **sudden-decay** approximation: $\zeta_{\rm tot} \xrightarrow{\text{decay at } t_{\rm dec}} \zeta_{\rm r,out}$.
- Hence $\zeta_{\rm r,out} = r\zeta_{\chi,\rm in}$, where $r = 3\Omega_\chi/(4\Omega_{r0}+3\Omega_\chi)|_{\rm dec} \sim \Omega_{\chi,\rm dec}$.
- If the decay happens **non-instantaneously** we can still write

$$\zeta_{\rm r,out} = \mathbf{r}\zeta_{\chi,\rm in}\,,$$

but now the perturbation transfer efficiency **r** needs to be calculated numerically.

The non-linearity parameters $f_{\rm NL}$ and $g_{\rm NL}$

- In the simplest case the (possible) **non-gaussianity** is caused by the square of the first order perturbation, so that $\zeta_{\rm 2nd\ ord} \propto \zeta_{\rm 1st\ ord}^2$.
- The constant of proportionality is $f_{\rm NL}$:

$$\zeta_{\rm non-lin} = \zeta_1 + \frac{3}{5} f_{\rm NL} \zeta_1^2 + \frac{9}{25} g_{\rm NL} \zeta_1^3 + \mathcal{O}(\zeta_1^4).$$

- **Connection to the bi- and tri-spectrum**

If we write the primordial power spectrum as

$$\langle \zeta(\mathbf{k_1}) \zeta(\mathbf{k_2}) \rangle = (2\pi)^3 P(k_1) \delta^3(\mathbf{k_1} + \mathbf{k_2}) \,,$$

then the leading order contributions to the bispectrum and (connected part of the) trispectrum are given by

$$\begin{aligned} \langle \zeta(\mathbf{k_1}) \zeta(\mathbf{k_2}) \zeta(\mathbf{k_3}) \rangle &= (2\pi)^3 B(\mathbf{k_1}, \mathbf{k_2}) \delta^3(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3}) \,, \\ \langle \zeta(\mathbf{k_1}) \zeta(\mathbf{k_2}) \zeta(\mathbf{k_3}) \zeta(\mathbf{k_4}) \rangle &= (2\pi)^3 T(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}) \delta^3(\mathbf{k_1} + \mathbf{k_2} + \mathbf{k_3} + \mathbf{k_4}), \quad \text{where} \end{aligned}$$

$$B(\mathbf{k_1}, \mathbf{k_2}) = \mathbf{(6/5) f}_{\rm NL} \left[P(k_1) P(k_2) + 2 \,\mathrm{perms} \right],$$

$$\begin{aligned} T(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}) &= \mathbf{(36/25) f}_{\rm NL}^{\mathbf{2}} \left[P(k_1) P(k_2) P(|\mathbf{k_1} + \mathbf{k_3}|) + 11 \,\mathrm{perms} \right] \\ &\quad + \mathbf{(54/25) g}_{\rm NL} \left[P(k_1) P(k_2) P(k_3) + 3 \,\mathrm{perms} \right] . \end{aligned}$$

Here $P(k) = \frac{2\pi^2}{k^3} \mathcal{P}_\zeta(k) = \frac{2\pi^2}{k^3} A^2_{\rm pivot} \left(\frac{k}{k_{\rm pivot}} \right)^{n-1}$, $A^2_{\rm pivot} \sim 6.25 \times 10^{-10}$.

The main objectives of our work [Phys. Rev. D 74, 103003 (2006)]

1. Calculate $f_{\rm NL}$ and $g_{\rm NL}$ analytically in the sudden-decay approximation, and compare to the numerical results in the non-instantaneous decay case.
2. Go beyond any perturbative expansion and calculate $\zeta(\chi_*)$ fully non-linearly in the long-wavelength limit (δN-formalism, separate universes assumption).
3. Find the **p**robability **d**ensity **f**unction pdf(ζ) of the full non-linear primordial perturbation ζ.
   - The full pdf describes the non-Gaussianity at all orders, not just up to the 3rd order like $f_{\rm NL}$ and $g_{\rm NL}$.

... The non-linearity parameters $f_{\rm NL}$ and $g_{\rm NL}$

- In the **sudden decay approx.** the second order calculation leads to

$$f_{\rm NL}(r) = +\frac{5}{4}\frac{1}{r}\left(1+\frac{gg''}{g'^2}\right) - \frac{5}{3} - \frac{5}{6}r\,,$$

where $\chi_{\rm in} \equiv g(\chi_*)$, $' \equiv \partial/\partial\chi_*$ and $\chi_* =$ the curvaton field value at horizon exit. This $f_{\rm NL}$ found, e.g., in [Bartolo, Matarrese & Riotto, Lyth & Rodriguez, JV, Sasaki & Wands].

- At third order we find in astro-ph/0607627

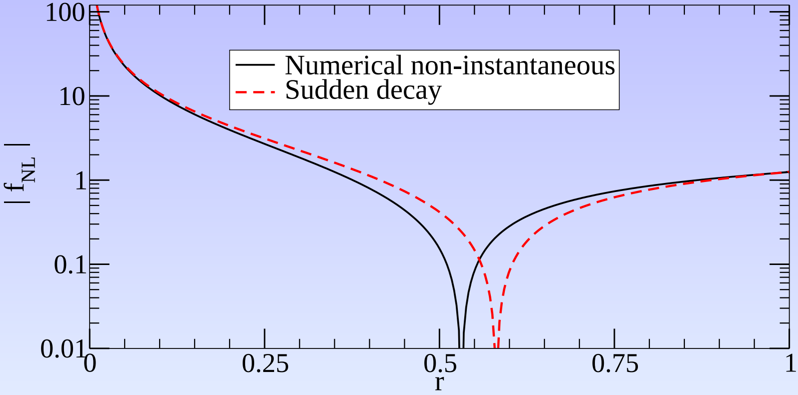
$$g_{\rm NL}(r) = \frac{25}{54}\left[\frac{9}{4r^2}\left(\frac{g^2g'''}{g'^3}+3\frac{g\,g''}{g'^2}\right) - \frac{9}{r}\left(1+\frac{g\,g''}{g'^2}\right) + \frac{1}{2}\left(1-9\frac{g\,g''}{g'^2}\right) + 10r + 3r^2\right].$$

- From now on, I assume linear evolution of χ, so $g'' = g''' = 0$. For the possibility of a non-linear g, see [Enqvist & Nurmi].

- **If $\Omega_{\chi,\rm dec} \ll 1$, then $r \approx 3\Omega_{\chi,\rm dec}/4 \ll 1$, and we have**

$$f_{\rm NL} \approx \frac{5}{4r} \approx \frac{5}{3\Omega_{\chi,\rm dec}} \qquad \textbf{and} \qquad g_{\rm NL} \approx -\frac{9}{r} \approx -\frac{12}{\Omega_{\chi,\rm dec}}.$$

Comparison of an exact calculation and the sudden-decay approximation of $f_{\rm NL}$ versus r $(\sim \Omega_{\chi,\rm dec})$



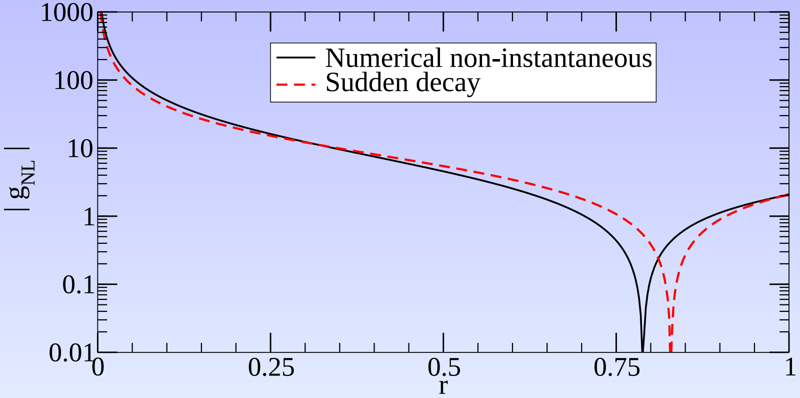
$f_{\rm NL} > 0$ $\quad$ $f_{\rm NL} < 0$

[astro-ph/0607627]

Figure assuming linear evolution of the field perturbation, i.e., $g'' = g''' = 0$.

Numerical comparisons of $f_{\rm NL}$ based on 2nd ord pert. eqns. also done in [Malik & Lyth].

Comparison of an exact calculation and the sudden-decay approximation of $g_{\rm NL}$ versus r $(\sim \Omega_{\chi,\rm dec})$



$g_{\rm NL} < 0$ $g_{\rm NL} > 0$

[astro-ph/0607627]

Figure assuming linear evolution of the field perturbation, i.e., $g'' = g''' = 0$.

Fully non-linear calculation

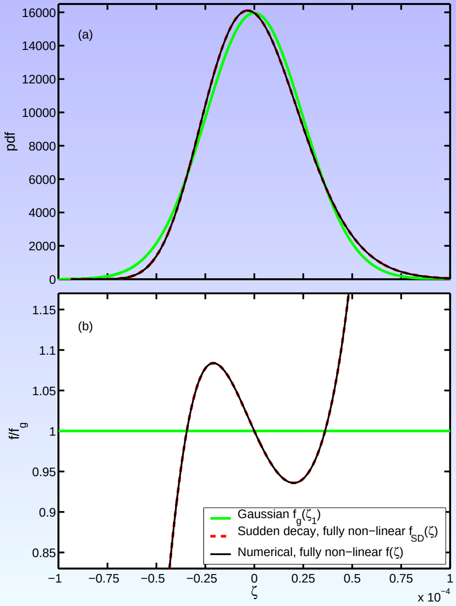
Hubble exit during inflation	(Possibly non-lin) evolution	Start of curvaton oscillation	Curvaton decay	Primordial adiabatic curvature perturbation
χ_* Gaussian	$\rightarrow$	$\chi = g(\chi_*)$, ζ_χ non-Gaussian	$\rightarrow$	ζ Highly non-Gaussian, if $\Omega_{\chi,\mathrm{dec}} \ll 1$ (i.e. $r \ll 1$)

- We have **derived and solved** a fully **non-linear equation** that relates ζ to the field value at Hubble exit, χ_*, in the sudden-decay approximation:

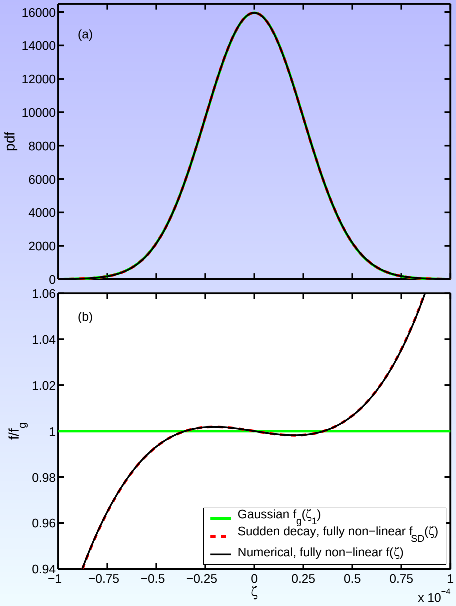
$$e^{4\zeta} - \left[\Omega_{\chi,\mathrm{dec}} e^{3\zeta_\chi(\chi_*)}\right] e^{\zeta} + \left[\Omega_{\chi,\mathrm{dec}} - 1\right] = 0\,.$$

- In the non-instantaneous decay case we have found the fully non-linear mapping **numerically using δN-formalism**!

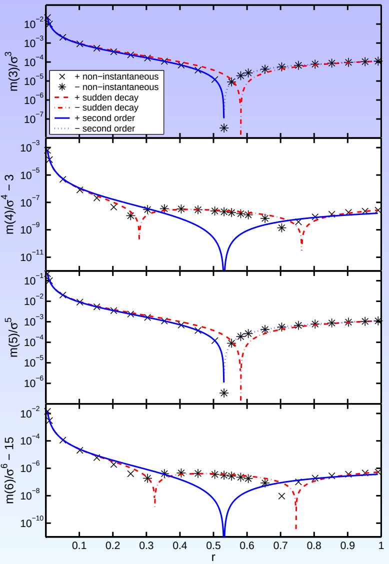
PDF in a model with $r = 0.00028$ ($\Rightarrow f_{NL} = 4432$)



PDF in a model with $r = 0.0108$ ($\Rightarrow f_{NL} = 114$)



Moments of the probability density function



The $i^{\rm th}$ moment $m_\zeta(i)$ is defined as

$$m_\zeta(i) = \int (\zeta - \mu_\zeta)^i \, \mathrm{pdf}(\zeta) d\zeta \,,$$

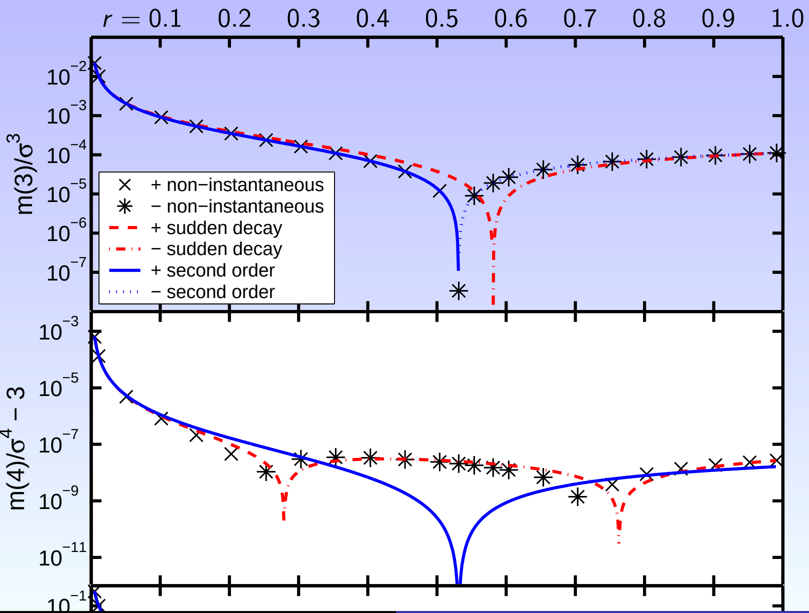
where

$$\mu_\zeta = \int \zeta \; \mathrm{pdf}(\zeta) d\zeta$$

is the mean.

[astro-ph/0607627]

... Moments of the probability density function



Conclusions and questions (for single-curvaton model)

- In δN formalism it is easy to calculate fully non-linearly the primordial curvature perturbation ζ, and find **the full probability density function of ζ.**
- The analytic **sudden-decay approx. is very accurate** if $|f_{\rm NL}| \& |g_{\rm NL}| \gtrsim 1$.
- If one is interested in perturbative expansions, they follow directly.
- The best observational strategy to constrain non-Gaussianity (and curvaton model)?
  - Extract $f_{\rm NL}$, $g_{\rm NL}$, etc, from the data? Or somehow directly the full pdf?
  - As the curvaton model predicts certain ratios of the successive moments, could this be probed somehow?
  - If Planck would see a non-Gaussian bi-spectrum **we have a prediction for the tri-spectrum and its scale dependence** in the curvaton model!
  - If $f_{\rm NL} = 0$, then $\mathbf{g}_{\rm NL} \approx \mathbf{-6}$, and the non-Gaussianity at the leading order is described by the trispectrum

$$
\begin{aligned}
T(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}) &= \mathbf{(54/25)g}_{\rm NL} \left[P(k_1)P(k_2)P(k_3) + 3\,\mathrm{perms} \right] \\
&\approx -13 \times 8\pi^6 \times 10^{-28} \times \left[\frac{1}{k_1^3 k_2^3 k_3^3} + \frac{1}{k_1^3 k_2^3 k_4^3} + \frac{1}{k_1^3 k_4^3 k_3^3} + \frac{1}{k_4^3 k_2^3 k_3^3} \right].
\end{aligned}
$$

Two curvatons a and b; arXiv:0708.0223v1 [hep-ph]

- a decays first, b decays later.
- Like in the single-curvaton model, we can relate the curvaton perturbations ζ_a and ζ_b to the resulting primordial perturbation as

(at first order) $\zeta_{\mathrm{r,out}} = r_a \zeta_{a,\mathrm{in}} + r_b \zeta_{b,\mathrm{in}} = (r_a + \beta r_b)\zeta_{a,\mathrm{in}} = (r_a/\beta + r_b)\zeta_{b,\mathrm{in}}.$

- In the sudden-decay approximation we find for the perturbation transfer efficiencies

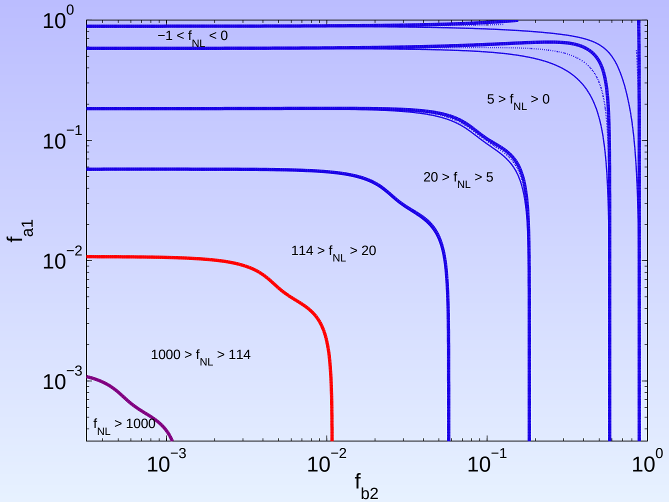
$$\begin{aligned} r_a &= [(1-f_{b2})(3+f_{a1})f_{a1}] \,/\, [3(1-f_{b1})+f_{a1}] \,, \\ r_b &= [(1-f_{b1})f_{b2}(3+f_{a1})+f_{b1}f_{a1}] \,/\, [3(1-f_{b1})+f_{a1}] \,, \quad \text{where} \\ f_{a1} &= 3\Omega_a / (4\Omega_{r0}+3\Omega_a+3\Omega_b)\,|_{\mathrm{at\ 1st\ dec}}, \\ f_{b1} &= 3\Omega_b / (4\Omega_{r0}+3\Omega_a+3\Omega_b)\,|_{\mathrm{at\ 1st\ dec}}, \\ f_{b2} &= 3\Omega_b / (4\Omega_r+3\Omega_b)\,|_{\mathrm{at\ 2nd\ dec}}. \end{aligned}$$

- In addition to these 3 energy-density ratios (f-parameters), we'll need

$$\beta^2 \stackrel{\mathrm{def}}{=} P_{\zeta_{b,\mathrm{in}}} / P_{\zeta_{a,\mathrm{in}}} = (a_*/b_*)^2.$$

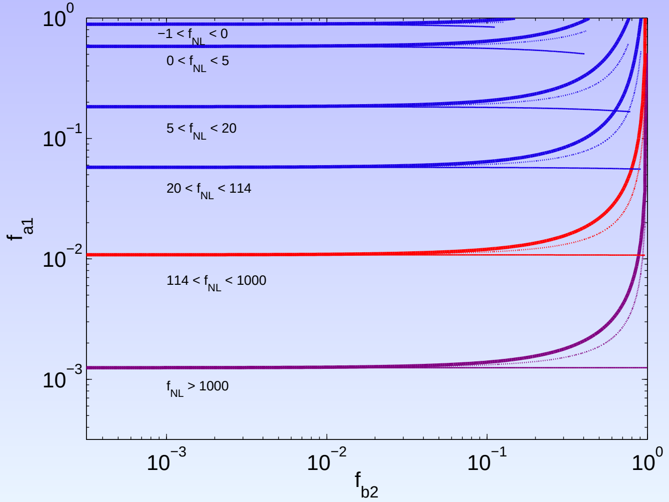
From slow-roll $P_{a*} = P_{b*}$, but then $P_{\zeta_{b*}}/P_{\zeta_{a*}} = a_*^2/b_*^2$.

$f_{\rm NL}$ for a model with $P_{\zeta_{b,{\rm in}}} = P_{\zeta_{a,{\rm in}}}$, i.e., $\beta^2 = 1$.



Recall: $f_{a1} \sim \Omega_a|_{\rm at\ 1st\ dec}$ and $f_{b2} \sim \Omega_b|_{\rm at\ 2nd\ dec}$. [arXiv:0708.0223v1]

$f_{\rm NL}$ for a model with $P_{\zeta_{b,\rm in}} \ll P_{\zeta_{a,\rm in}}$, i.e., $\beta^2 = 0$, i.e., the second curvaton b is almost homogeneous.



Recall: $f_{a1} \sim \Omega_a|_{\rm at\ 1st\ dec}$ and $f_{b2} \sim \Omega_b|_{\rm at\ 2nd\ dec}$.

[arXiv:0708.0223v1]

Conclusions (for two-curvaton model; arXiv:0708.0223)

- If the last decaying curvaton has significant perturbations, then **large non-Gaussianity** of the primordial perturbation follows **only when all the curvatons are highly subdominant at their decay time**; in the two-curvaton example $f_{a1} \ll 1$, $f_{b1} \ll 1$, and $f_{b2} \ll 1$.
- But if the last decaying curvaton is almost homogeneous, it dilutes the first order perturbation, and **large non-Gaussianity can follow even when all the curvatons dominate the energy density at their decay time**;
  in the two-curvaton example $f_{a1} \sim 1$, $f_{b1} \ll 1$, and $f_{b2} \sim 1$.
- Two-curvaton example: If the energy density of both curvatons is small when they decay, then $f_{a1} \ll 1$ and $f_{b2} \ll 1$ [& $f_{b1} \ll 1$], and

(at leading order in f-parameters)

$$
\begin{aligned}
r_a &= f_{a1} = \tfrac{3}{4}\Omega_a|_{\text{at 1st dec}}, \\
r_b &= f_{b2} = \tfrac{3}{4}\Omega_b|_{\text{at 2nd dec}}, \\
f_{\mathrm{NL}} &= \frac{5}{4}\frac{r_a^3 + \beta^4 r_b^3}{\left(r_a^2 + \beta^2 r_b^2\right)^2}\,.
\end{aligned}
$$