Nonequilibrium Dynamics in Particle Physics and Cosmology

Kavli Institute for Theoretical Physics, Santa Barbara

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Organizers: J. Berges (Darmstadt), L. Kofman (CITA), L. Yaffe (U. of Washington)
Information/application at http://www.kitp.ucsb.edu/activities/auto2/?id=834

Nonequilibrium quantum field theory in cosmology

Jürgen Berges

Darmstadt University of Technology

Content

I. Nonequilibrium quantum dynamics

- two-particle irreducible generating functional
- preheating: non-thermal fixed points, (pre-)thermalization

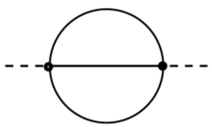
II. Lattice simulations of quantum fields

- real-time stochastic quantization
- optimized updating: non-Abelian gauge theories

Standard QFT techniques fail out of equilibrium

`Secularity´

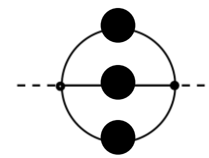
• uniform approximations in time require infinite perturbative orders



$\sim \gamma t$

`Universality´

• nonlinear dynamics necessary for late-time thermalization



$\rightarrow e^{-\gamma t}$

2-particle irreducible generating functionals

$\Rightarrow$ systematic 2PI loop-, coupling- or 1/N-expansions available

$\Rightarrow$ *far-from-equilibrium* dynamics as well as late-time *thermalization* in QFT

Berges, Cox ´01; Aarts, Berges ´01; Berges ´02; Cooper, Dawson, Mihaila ´03; Berges, Serreau ´03; Berges, Borsányi, Serreau ´03; Cassing, Greiner, Juchem ´03; Arrizabalaga, Smit, Tranberg ´04; Berges, Borsányi ´05; Aarts,Tranberg ´06; Rajantie, Tranberg ´06 ...

2PI effective action

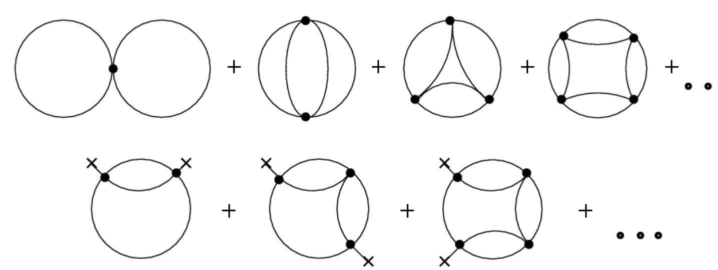
Luttinger, Ward ´60; Baym ´62; Cornwall, Jackiw, Tomboulis ´74

$$\Gamma[\phi, G] = S[\phi] + \frac{i}{2}\mathrm{Tr}\ln G^{-1} + \frac{i}{2}\mathrm{Tr}\, G_0^{-1}(\phi)\, G + \Gamma_2[\phi, G]$$

- Parametrized by macroscopic field: $\phi(x) = \langle \Phi(x) \rangle$ and
- exact connected propagator: $G(x,y) = \langle T\Phi(x)\Phi(y) \rangle - \phi(x)\phi(y)$
- $\Gamma_2[\phi, G]$ contains only *two-particle irreducible* (2PI) diagrams

E.g. scalar *N*-component field theory to NLO in *2PI* 1/*N*-expansion:

$$\Gamma_2[\phi, G] = -\frac{\lambda}{4!N}\int_x G_{aa}(x,x) G_{bb}(x,x) + \frac{i}{2}\,\mathrm{Tr}\ln \mathbf{B}(G)$$
$$+ \frac{i\lambda^2}{(6N)^2}\int_{xyz} \mathbf{B}^{-1}(x,z;G) G^2(z,y)\phi_a(x) G_{ab}(x,y)\phi_b(y)$$



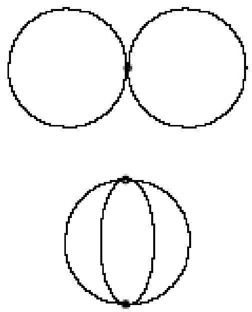
$$\mathbf{B}(x,y;G) \equiv \delta(x-y) + \frac{i\lambda}{6N} G^2(x,y)$$

includes NLO 1PI !

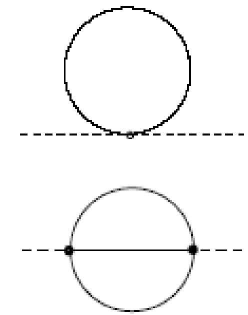
Berges ´02 ; Aarts, Ahrensmeier, Baier, Berges, Serreau ´02

Comparison with standard (1PI) expansion:

E.g. $\Gamma_2[0,G]$ to 3-loop or $\mathcal{O}(\lambda^2)$: Self-energy $\Sigma \sim \delta\Gamma_2/\delta G$:



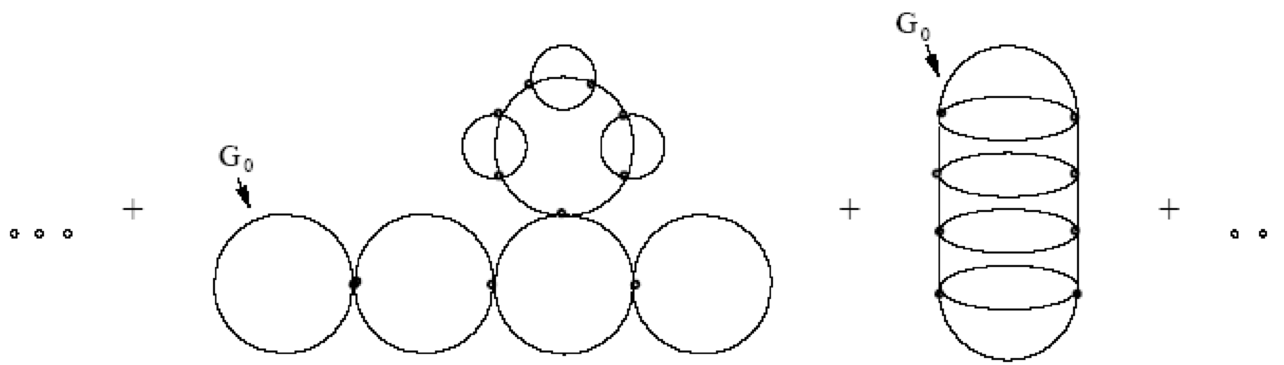
$\rightsquigarrow$



$\delta\Gamma[G]/\delta G = 0 \Rightarrow G^{-1} = G_0^{-1} - \Sigma$:

$$G = G_0 + G_0\,\Sigma\,G_0 + G_0\,\Sigma\,G_0\,\Sigma\,G_0 + \ldots$$

$\rightsquigarrow$ each 2PI diagram correponds to infinite series: daisies, ladders, ...



Lecture notes: *Introduction to nonequilibrium quantum field theory,* Berges, arXiv:hep-ph/0409233

Time evolution equations

Equations of motion: (1) $\frac{\delta\Gamma[\phi,G]}{\delta\phi(x)} = 0$, (2) $\frac{\delta\Gamma[\phi,G]}{\delta G(x,y)} = 0$

spectral function $\sim \langle[\Phi,\Phi]\rangle$

$$\boxed{G(x,y) = F(x,y) - \frac{i}{2}\rho(x,y)\,\mathrm{sign}_{\mathscr{C}}(x^0 - y^0)}$$

statistical propagator $\sim \langle\{\Phi,\Phi\}\rangle$

$$\begin{aligned}
\left[\Box_x\delta_{ac} + M^2_{ac}(x)\right]\rho_{cb}(x,y) &= -\int_{y^0}^{x^0} \mathrm{d}z\, \Sigma^{\rho}_{ac}(x,z)\rho_{cb}(z,y) \\
\left[\Box_x\delta_{ac} + M^2_{ac}(x)\right]F_{cb}(x,y) &= -\int_{0}^{x^0} \mathrm{d}z\, \Sigma^{\rho}_{ac}(x,z)F_{cb}(z,y) \\
&\quad + \int_{0}^{y^0} \mathrm{d}z\, \Sigma^{F}_{ac}(x,z)\rho_{cb}(z,y)
\end{aligned}$$

$$\left(\left[\Box_x + \frac{\lambda}{6N}\phi^2(x)\right]\delta_{ab} + M^2_{ab}(x;\phi=0,F)\right)\phi_b(x) = -\int_0^{x^0} \mathrm{d}y\, \Sigma^{\rho}_{ab}(x,y;\phi=0,F,\rho)\,\phi_b(y)$$

Nonequilibrium:

$F \not\sim \rho$

Equilibrium/Vacuum:
(fluct.-diss. relation)

$F \sim \rho$

Quantum- vs. classical-statistical dynamics

Self-energies from 3-loop 2PI effective action (similarly for NLO $1/N$):

Quantum $\qquad \phi = 0$

$$\Sigma^F(t,t';\mathbf{p}) = -\frac{\lambda^2}{6}\int_{\mathbf{q},\mathbf{k}} F(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k}) \left[F(t,t';\mathbf{q})F(t,t';\mathbf{k}) - \frac{3}{4}\rho(t,t';\mathbf{q})\rho(t,t';\mathbf{k})\right]$$

$$\Sigma^\rho(t,t';\mathbf{p}) = -\frac{\lambda^2}{2}\int_{\mathbf{q},\mathbf{k}} \rho(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k}) \left[F(t,t';\mathbf{q})F(t,t';\mathbf{k}) - \frac{1}{12}\rho(t,t';\mathbf{q})\rho(t,t';\mathbf{k})\right]$$

Classical

$$\Sigma^F_{\mathrm{cl}}(t,t';\mathbf{p}) = -\frac{\lambda^2}{6}\int_{\mathbf{q},\mathbf{k}} F(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k})F(t,t';\mathbf{q})F(t,t';\mathbf{k})$$

$$\Sigma^\rho_{\mathrm{cl}}(t,t';\mathbf{p}) = -\frac{\lambda^2}{2}\int_{\mathbf{q},\mathbf{k}} \rho(t,t';\mathbf{p}-\mathbf{q}-\mathbf{k})F(t,t';\mathbf{q})F(t,t';\mathbf{k})$$

Classicality condition

$\rightsquigarrow$ Analytical form of the *exact* classical evolution equations for $F_{\rm cl}$ and $\rho_{\rm cl}$ is *identical* to the respective quantum ones with replacements

$$\begin{aligned}\Sigma^F &\rightarrow \Sigma^F_{\rm cl} = \Sigma^F(F^2 \gg \rho^2)\\ \Sigma^\rho &\rightarrow \Sigma^\rho_{\rm cl} = \Sigma^\rho(F^2 \gg \rho^2)\end{aligned}$$

(LO large-N/Hartree approximations have $\Sigma^F = \Sigma^\rho \equiv 0$,
i.e. quantum $\equiv$ classical)

$\rightsquigarrow$ *sufficient* condition for classical evolution:

$$\boxed{|F(t,t';\mathbf{q})F(t,t';\mathbf{k})| \gg \frac{3}{4}\,|\rho(t,t';\mathbf{q})\rho(t,t';\mathbf{k})|}$$

i.e. for *all* times and momenta

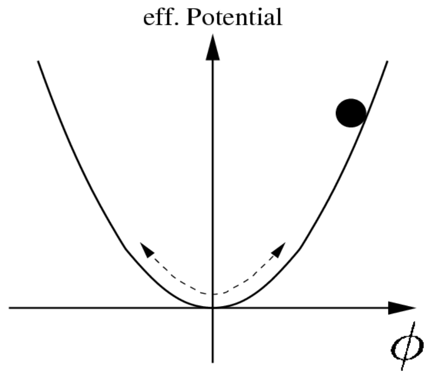
Aarts, Berges, PRL88 (2002) 041603; Berges, Gasenzer, PRA (2007), cond-mat/0703163

Preheating dynamics

Kofman, Linde, Starobinsky, PRL 73 (1994) 3195, …

Berges, Serreau PRL91 (2003) 111601

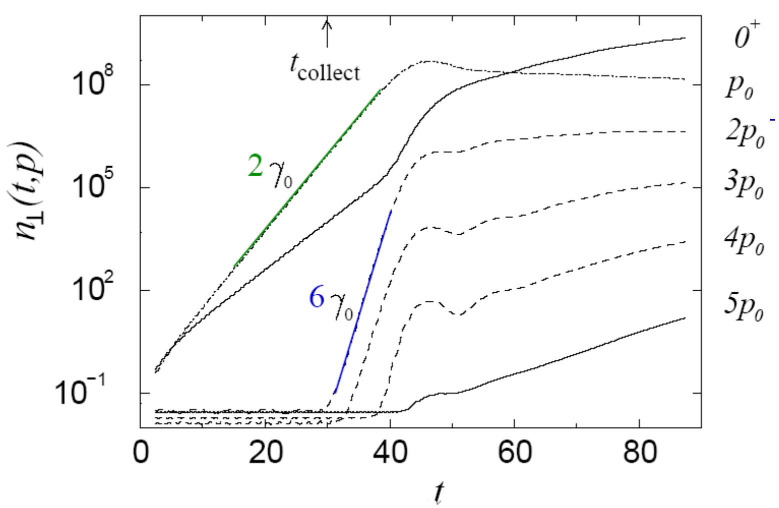
<u>N-component $\lambda\phi^4$, conformal time t</u>

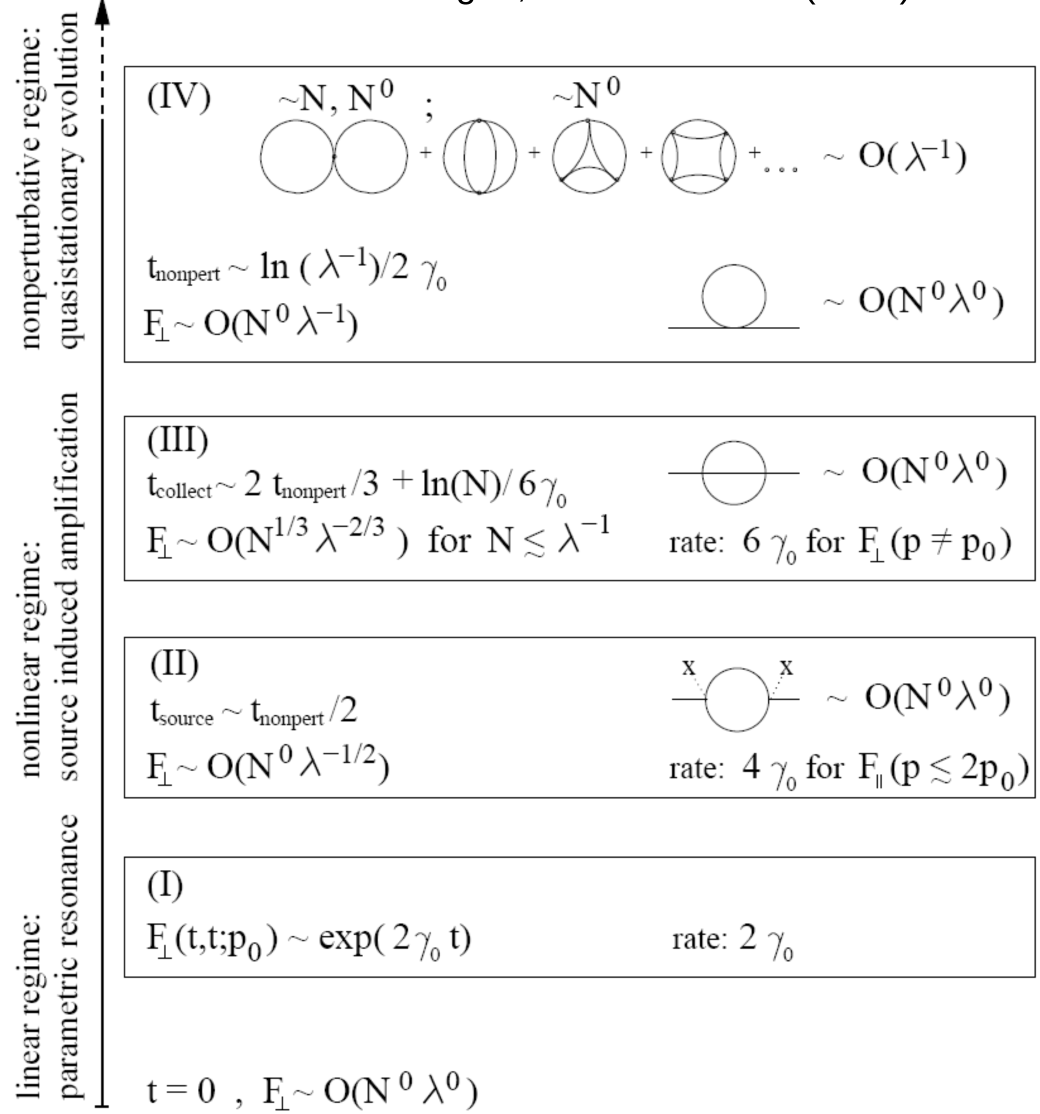


macroscopic field $\phi = \langle \Phi \rangle$

fluctuations $F \sim \langle \Phi\Phi \rangle - \langle \Phi \rangle^2$

Particle production:





Quantum vs. classical preheating dynamics

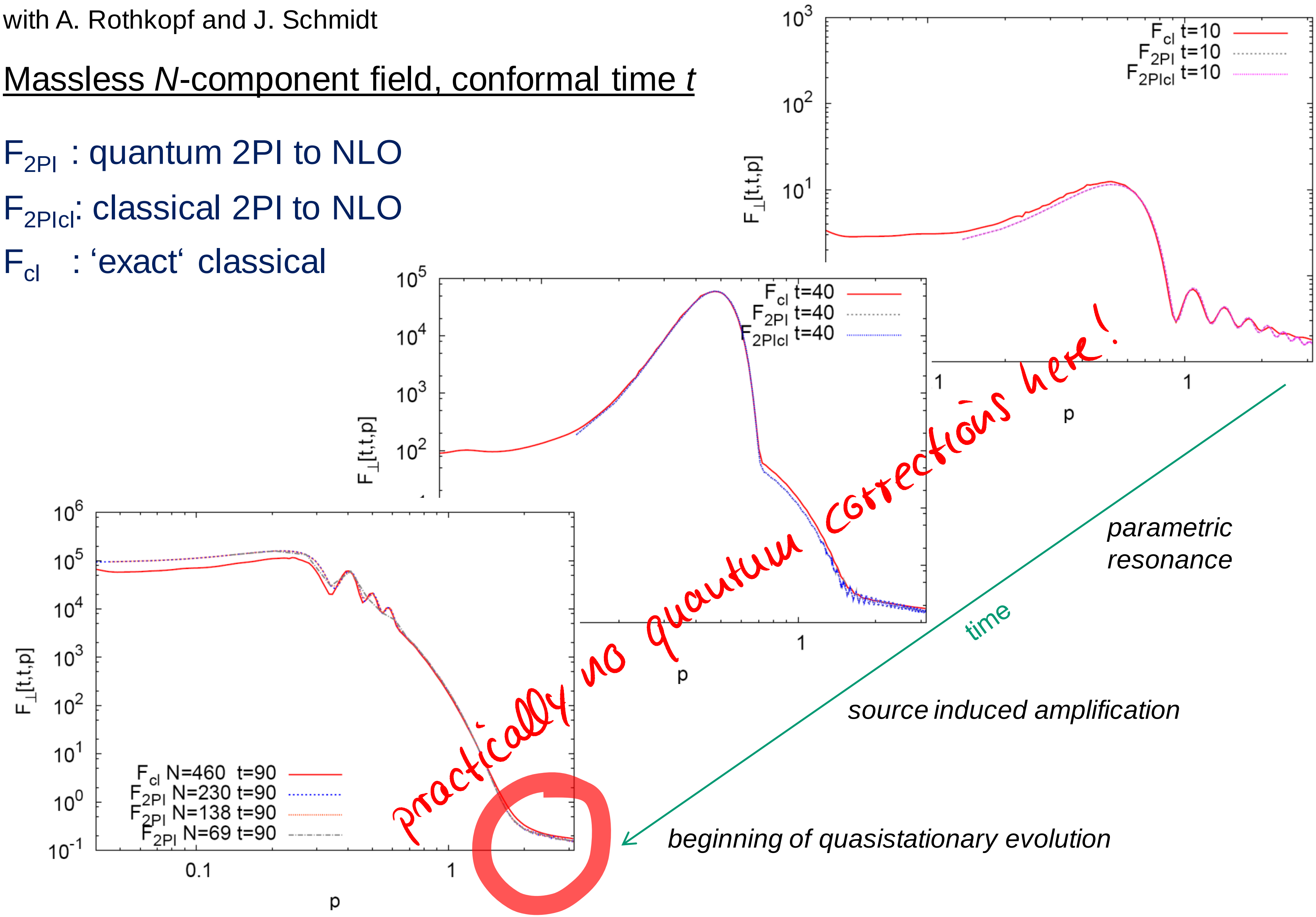
with A. Rothkopf and J. Schmidt

<u>Massless *N*-component field, conformal time *t*</u>

F_{2PI} : quantum 2PI to NLO

F_{2PIcl}: classical 2PI to NLO

F_{cl} : ‘exact‘ classical



Quasistationary regime: Non-thermal fixed point

Fixed point of the time-evolution equation for fluctuations *F*:

$$\left[\Sigma_{\rho}^{(\mathrm{hom})}(\omega,\mathbf{p})F^{(\mathrm{hom})}(\omega,\mathbf{p})-\Sigma_{F}^{(\mathrm{hom})}(\omega,\mathbf{p})\rho^{(\mathrm{hom})}(\omega,\mathbf{p})\right]=0$$

1) Unique solution in quantum theory (H-theorem): Thermal equilibrium

$$F^{(\mathrm{eq})}(\omega,\mathbf{p})=-i\left(\frac{1}{2}+n_{\mathrm{BE}}(\omega)\right)\rho^{(\mathrm{eq})}(\omega,\mathbf{p})\,,\quad \Sigma_F^{(\mathrm{eq})}(\omega,\mathbf{p})=-i\left(\frac{1}{2}+n_{\mathrm{BE}}(\omega)\right)\Sigma_\rho^{(\mathrm{eq})}(\omega,\mathbf{p})$$

fluctuation-dissipation relation

2) Existence of non-thermal stationary (power-law) solutions in classical theory:

$$F^{(\mathrm{hom})}(\omega,\mathrm{p})=\frac{1}{p^{2+\alpha}}\,f\left(\frac{\omega^z}{p}\right)$$

universal ‘critical‘ exponents *universal scaling function*

For $\phi \neq 0$ we find: $\alpha \simeq 3/2$, $z \simeq 0$ *irrespective of the number of field components!*

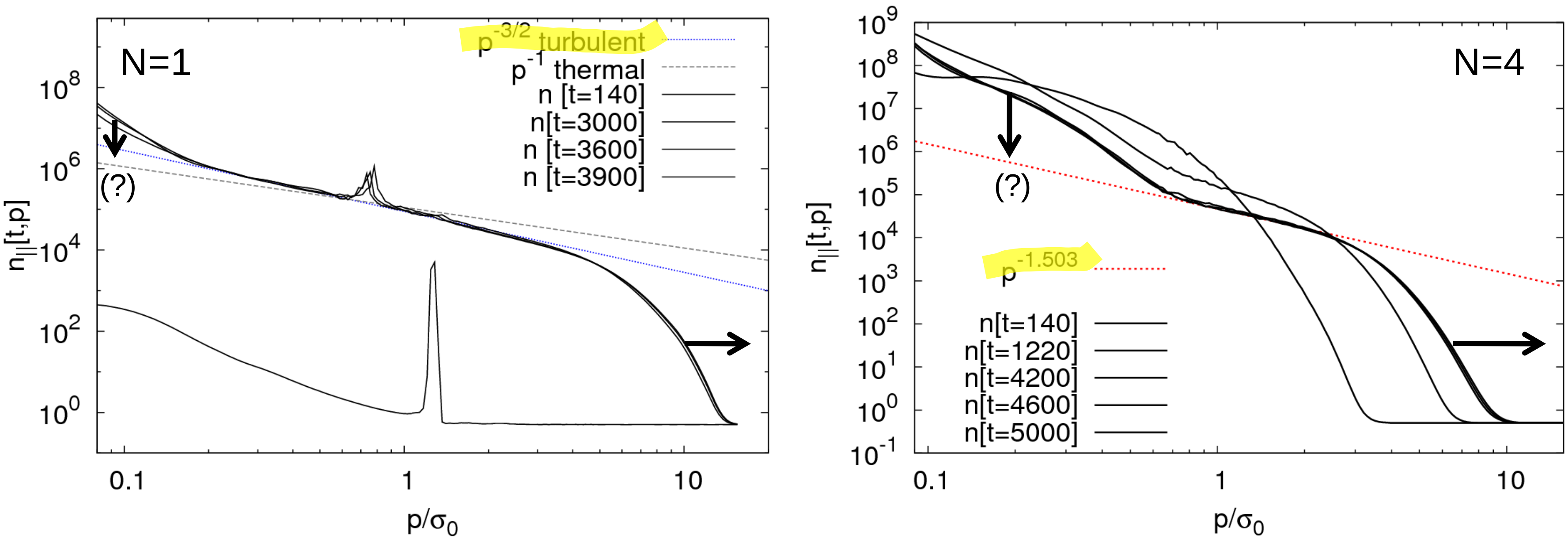
Quasistationary regime: why it is not perturbative (Kolmogorov) wave turbulence

In the context of Kolmogorov wave turbulence power-law solutions are obtained from a classical Boltzmann equation with Micha, Tkachev PRD70 (2004) 043534, …

a) *local four-leg interaction* $\Rightarrow$ $\alpha = 4/3, 5/3$

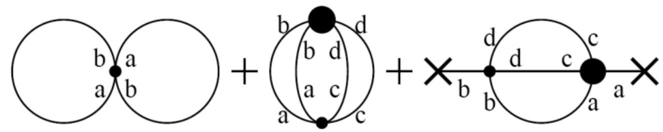
b) *local three-leg interaction* $\Rightarrow$ $\alpha = 3/2$

From comparison with classical-statistical simulations one apparently finds: $\alpha = 3/2$

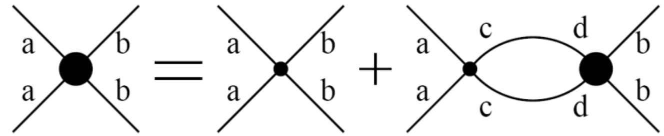


Berges, Rothkopf, Schmidt

In the quasistationary regime all $O(1)$ contributions to Γ_2 can be written as



$\lambda_{\text{eff}}(\omega, p)$:



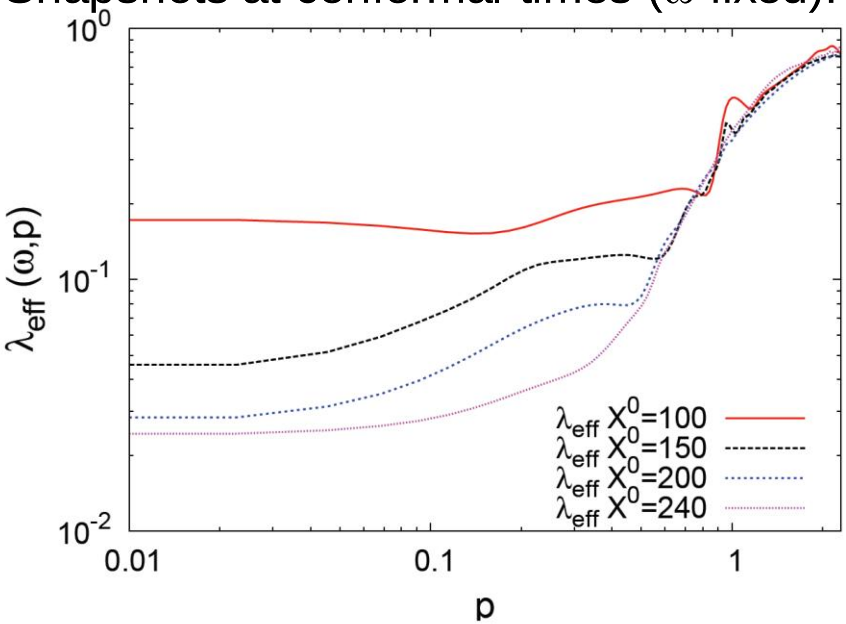
~~Perturbative (Kolmogorov) $\Rightarrow$ $\lambda_{\text{eff}}(\omega,p) = \lambda = \text{const}$~~

perturbative wave turbulence not realized!

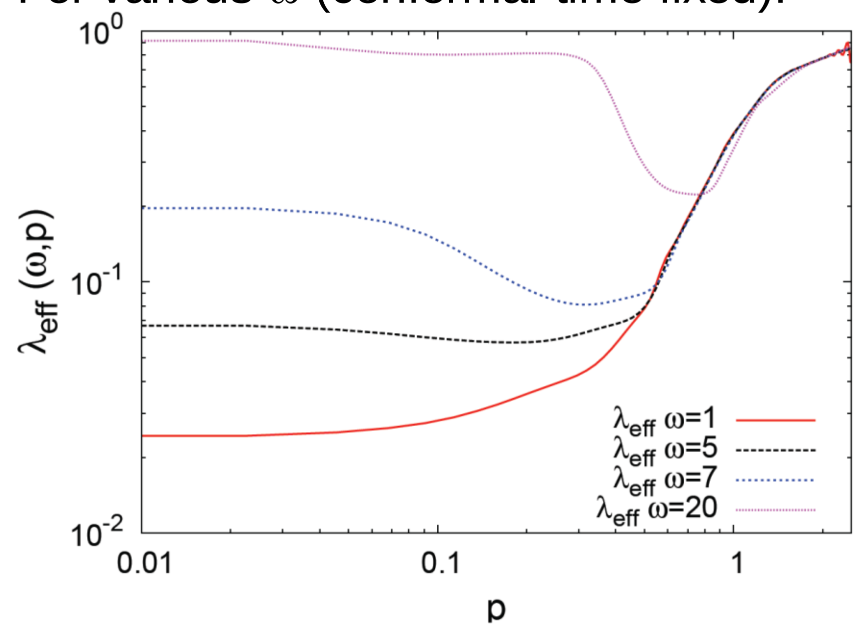
However: *Build-up of power-law for effective coupling!*

$$\lambda_{\text{eff}}(\omega, p)|_{\omega} \sim p^{\alpha}$$

Snapshots at conformal times (ω fixed):



For various ω (conformal time fixed):



Role of quantum fluctuations/fermions

Non-thermal fixed point unstable under inclusion of quantum corrections or fermions

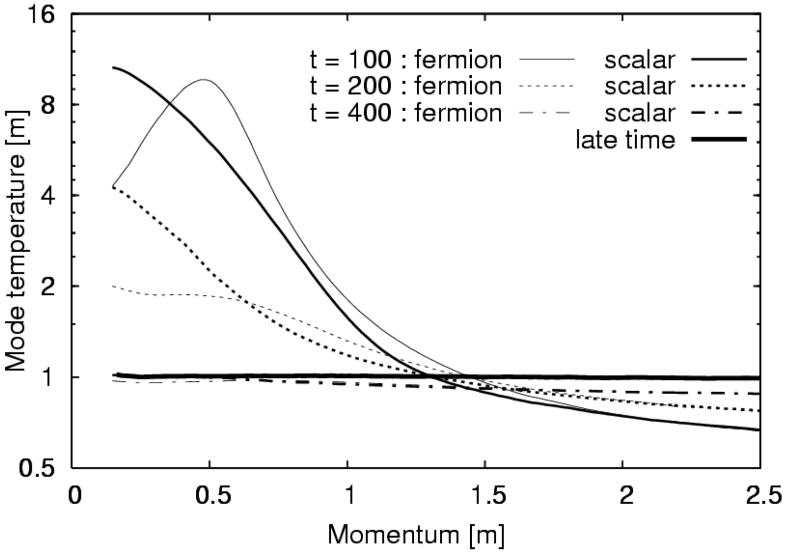
$\Rightarrow$ *late-time thermalization to Bose-Einstein/Fermi-Dirac distributions*

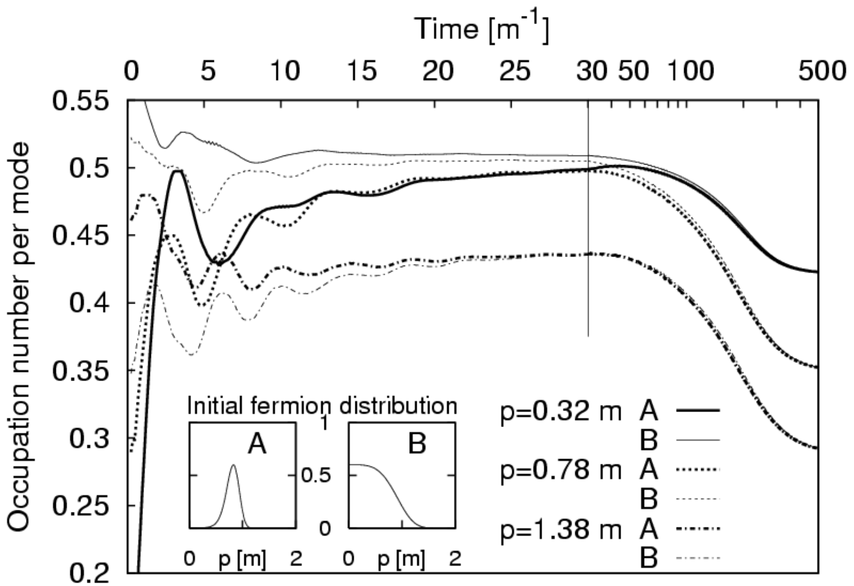
Thermalization example:

$SU(2)_L \times SU(2)_R$ fermionic model coupled to N=4 inflaton

'Mode temperature' T_p: $\left(n_p \sim \mathrm{tr}\, \frac{p^i\gamma^i}{p} \langle[\psi,\bar{\psi}]\rangle_p\right)$

$$n_p(t) \stackrel{!}{=} \frac{1}{\exp\left[\omega_p(t)/T_p(t)\right] \pm 1}$$





Emergence of BE/FD distribution with

$$\boxed{T_p^{(f)}(t) = T_p^{(s)}(t) = T_{\text{eq}}}$$

Berges, Borsányi, Serreau, NPB 660 (2003) 51

Prethermalization

(Approximate) non-thermal fixed points can lead to extremely late thermal equilibration

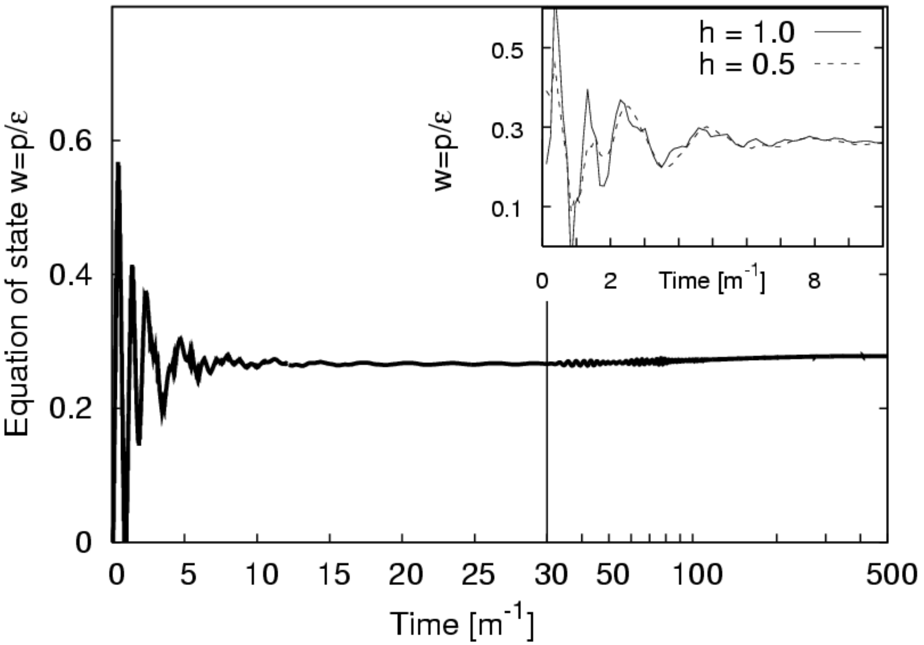
$\Rightarrow$ exceedingly small reheating temperatures T

However, different quantities effectively thermalize on different time scales:

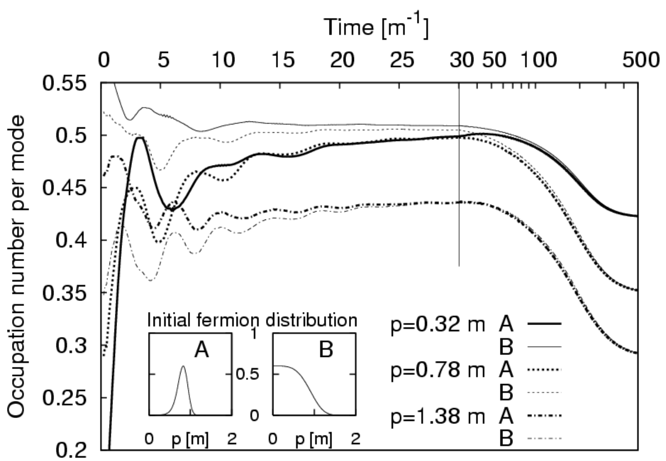
The *prethermalization* of crucial quantities may occur on time scales dramatically shorter than the thermal equilibration time

Berges, Borsányi, Wetterich, PRL93 (2004) 142002; Podolsky, Felder, Kofman, Peloso, PRD 73 (2006) 023501
Dufaux, Felder, Kofman, Peloso, Podolsky, JCAP 0607 (2006) 006

Example:

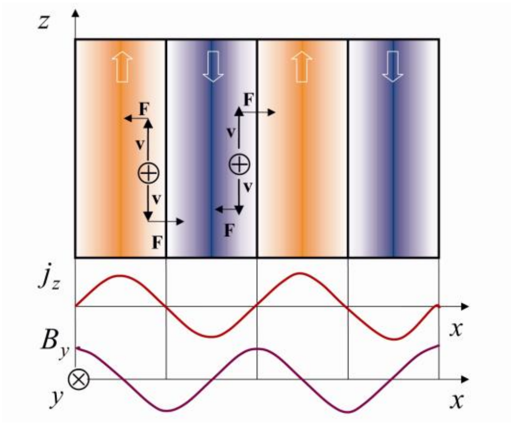


compare:



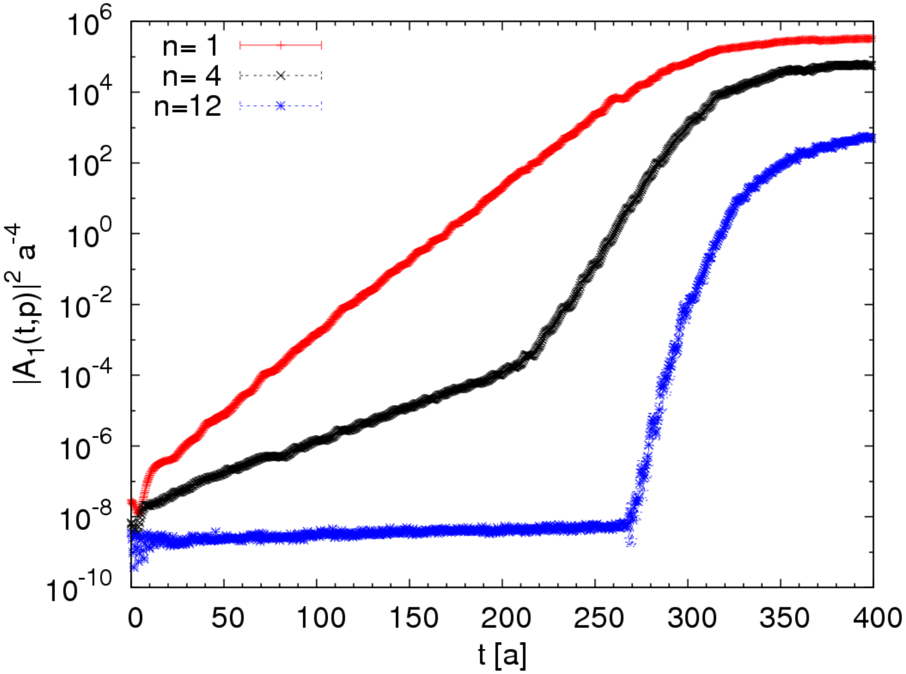
For comparison: Instability-driven particle production in *SU(N)* gauge theory

Anisotropic momentum distribution: plasma instabilities

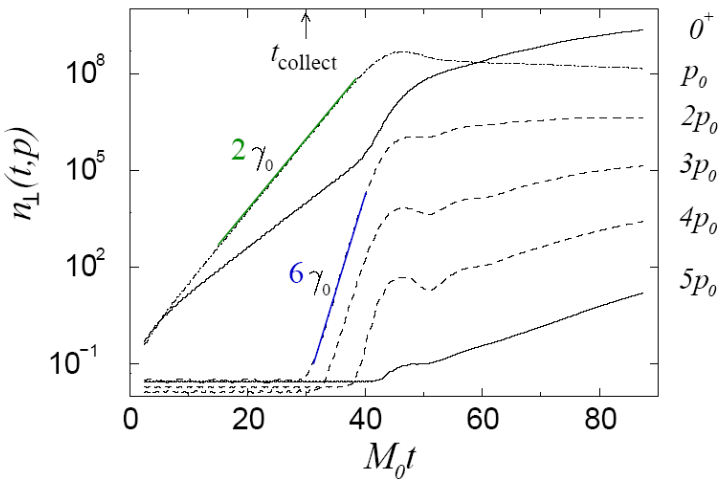


Weibel, PRL 2 (1959) 83;
Mrowczynski, PRC 49 (1994) 2191;
Arnold, Moore, Yaffe, PRL 94 (2005) 072302;
Romatschke, Venugopalan, PRL 96 (2006) 062302; ...

Classical-statistical *SU(2)* gauge theory simulation:



Scalar parametric instability for comparison:



with D. Sexty and S. Scheffler

Lattice simulations of real-time quantum fields

Real time:

$$\sim e^{iS_M}$$

non-positive definite probability measure!

Euclidean stochastic quantization

Parisi, Wu '81; …

- Hamiltonian for *d*-dimensional field theory in *(d+1)*-dimensions

$$H = \int \mathrm{d}^d x \left(\frac{1}{2}\pi^2(x;t_5) + \mathcal{L}_\mathrm{E}(\varphi(x;t_5), \partial_x \varphi(x;t_5)) \right)$$

- expectation values for quantum theory with action $S_\mathrm{E}[\varphi] \equiv \int \mathrm{d}^d x \, \mathcal{L}_\mathrm{E}$

$$\langle F(\varphi) \rangle = Z^{-1} \int \mathcal{D}\pi \mathcal{D}\varphi \, F(\varphi) \, e^{-H[\pi,\varphi]} \quad , \quad Z = \int \mathcal{D}\pi \mathcal{D}\varphi \, e^{-H[\pi,\varphi]}$$

- replace canonical ensemble averages by micro-canonical

$$\frac{\partial \varphi(x;t_5)}{\partial t_5} = \frac{\delta H}{\delta \pi(x;t_5)} = \pi(x;t_5) \, , \quad \frac{\partial \pi(x;t_5)}{\partial t_5} = -\frac{\delta H}{\delta \varphi(x;t_5)} = -\frac{\delta S_\mathrm{E}[\varphi]}{\delta \varphi(x;t_5)}$$

- conjugate momenta have Gaussian distribution; randomly refresh after every single step $\rightarrow$ Langevin dynamics in additional 'time' t_5

Classical dynamics in additional Langevin-time computes quantum averages!

Real-time stochastic quantization

Klauder ’83; Parisi ’83; Hüffel, Rumpf ’84; Okano, Schülke, Zheng ’91 …

- Replace embedded *d*-dimensional Euclidean by Minkowskian action

$\Rightarrow$ Langevin dynamics: $\left(\frac{1}{2}\Delta t_5^2 \to \epsilon \;,\; \sqrt{2}\,\pi(x) \to \eta(x) \right)$

$$\varphi'(t,\mathbf{x}) = \varphi(t,\mathbf{x}) + i\,\epsilon\,\frac{\delta S\,[\varphi]}{\delta\varphi(t,\mathbf{x})} + \sqrt{\epsilon}\,\eta(t,\mathbf{x})\,, \qquad \langle \eta(x)\,\eta(x')\rangle_\eta = 2\,\delta(x-x')$$

$\Rightarrow$ φ, η *in general complex!*

- Lattice gauge theory: $U_{x,\mu\nu} \equiv U_{x,\mu} U_{x+\hat{\mu},\nu} U^{-1}_{x+\hat{\nu},\mu} U^{-1}_{x,\nu}$ gauge invariant plaquette

$$U'_{x,\mu} \;=\; \exp\left\{ i \sum_a \lambda_a \left(\epsilon\, i \underset{\uparrow}{D_{x\mu a}} S[U] + \sqrt{\epsilon}\,\eta_{x\mu a} \right) \right\} U_{x,\mu}\,, \quad \langle \eta_{x\mu a}\,\eta_{y\nu b}\rangle = 2\,\delta_{\mu\nu}\delta_{xy}\delta_{ab}$$

differentiation in group space

e.g. *SU(2)* real-time: $U_{x,\mu} \equiv e^{iA_{x\mu a}\sigma_a/2}$

complex traceless matrix $A_{x\mu a}\sigma_a$ $\Rightarrow$ *extension to SL(2,**C**)*

Berges, Borsanyi, Sexty, Stamatescu ’07

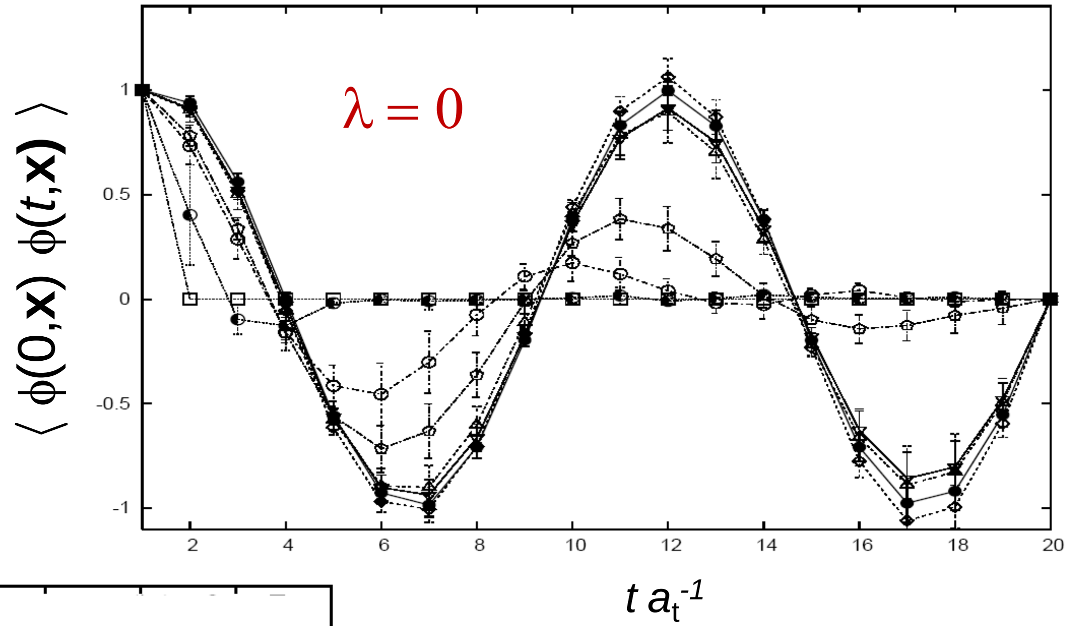
Simulating nonequilibrium quantum fields

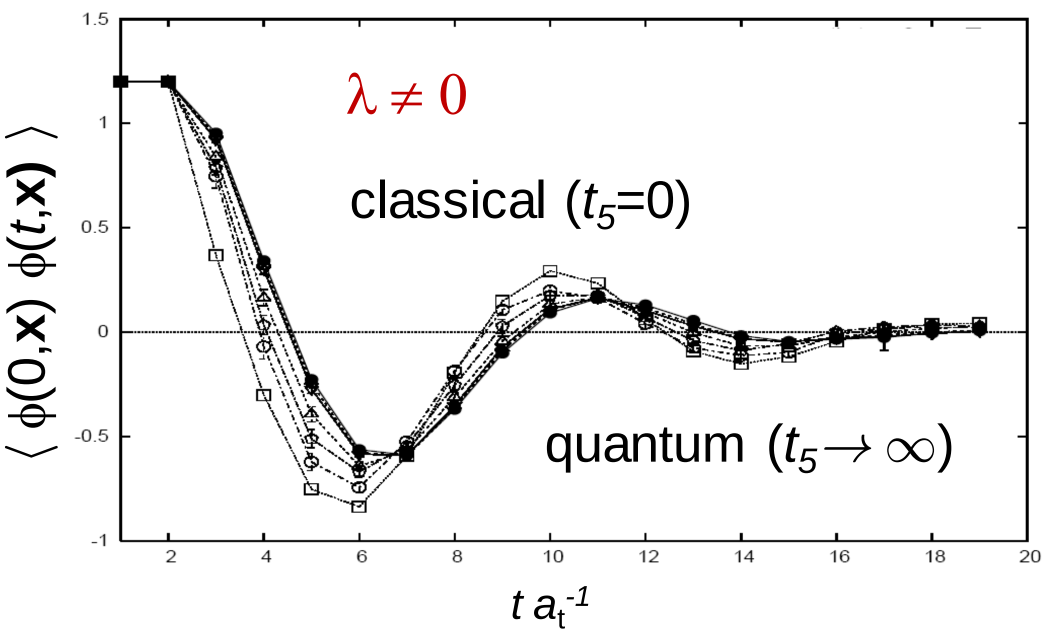
Berges, Stamatescu, PRL (2005) 202003

Scalar λ^4-theory:

null starting configuration

(t_5 = 0), Langevin updating builds up free-field solution



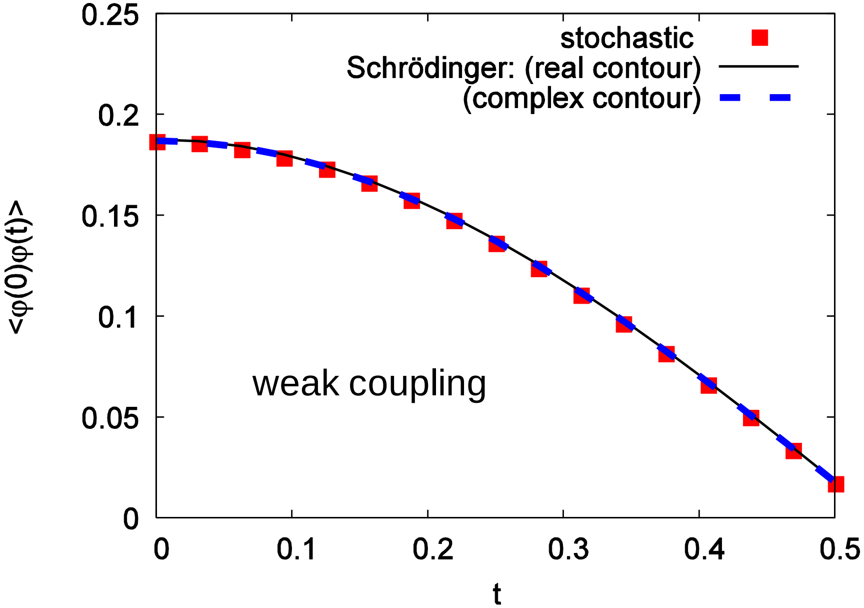


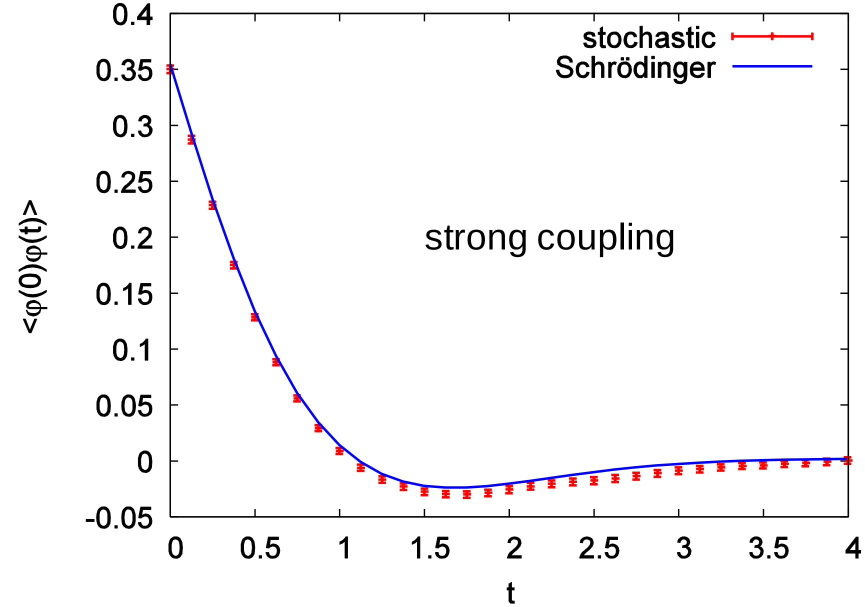
classical starting configuration

(t_5 = 0), Langevin updating takes into account quantum corrections

Precision test

Anharmonic quantum oscillator: comparison with solution of Schrödinger equation





- good agreement with `exact´ results for sufficiently short real-time contours

- breakdown if real-time extent is not small on the scale of the inverse temperature

- gauge theories: physical solutions only approached at intermediate Langevin-times

$\Rightarrow$ *further optimization required*

Berges, Borsanyi, Sexty, Stamatescu, PRD 75 (2007) 045007

Stochastic quantization with optimized updating

Berges, Sexty, arXiv:0708.0779 [hep-lat]

The problem: e.g. for *SU(2)* gauge theory real-time stochastic quantization requires the dynamical variables to become elements of *SL(2;**C**)*. Let $U = a\,\mathbf{1} + i\vec{b}\cdot\vec{\sigma}$

$$\det U = a^2 + \vec{b}^2 = 1$$

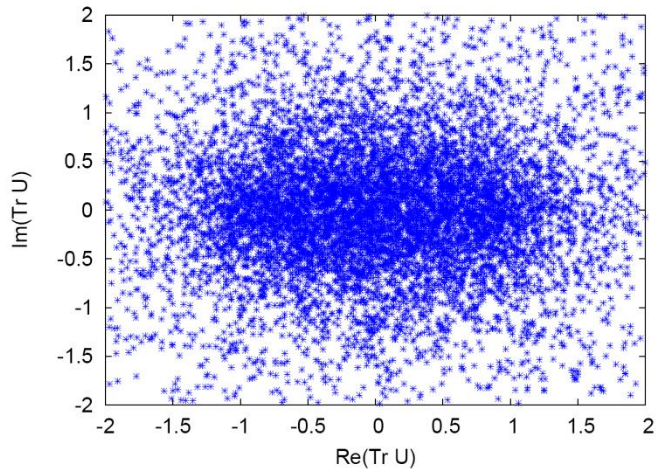
compact SU(2) → *finite* values for *real* *a* and ***b***

non-compact SL(2;C) → *unbounded* values for *complex* *a* and ***b***

Without optimization increasing fluctuations finally lead to a breakdown of the method!

Example: *SU*(2) one-plaquette model ⟨ Tr*U*/2 ⟩ = *i* 0.261… (analytic)

without optimization



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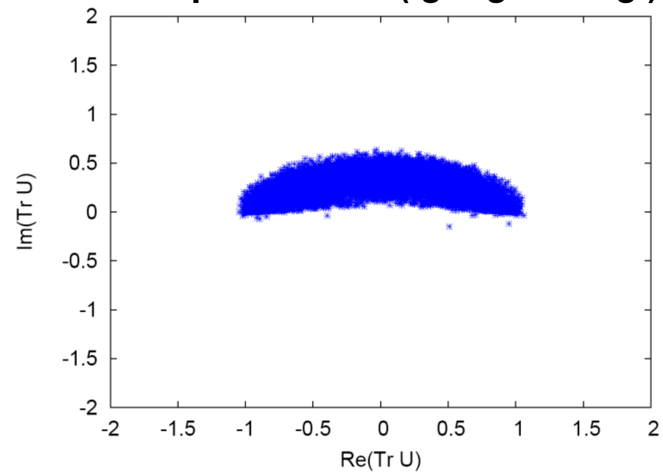
with optimization ('gauge fixing')



✓!

$$\langle \mathrm{Tr}\, U/2\rangle \overset{\text{without optimization}}{=} -0.02(\pm 0.02) - i\,0.01(\pm 0.02)$$

$$\langle \mathrm{Tr}\, U/2\rangle \overset{\text{with optimization}}{=} -0.004(\pm 0.006) + i\,0.260(\pm 0.001)$$

Optimized updating using gauge fixing

Fix by gauge transformations

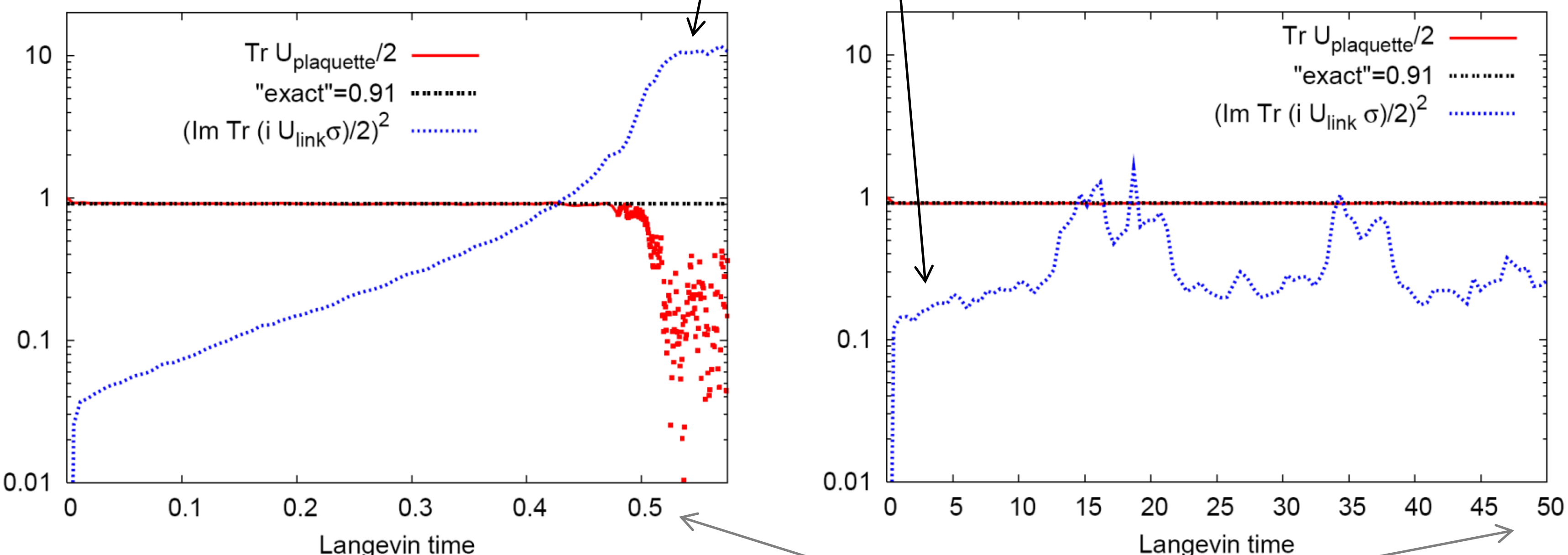
$$U_{x,\mu} \to W^{-1}(x) U_{x,\mu} W(x+\hat{\mu})$$

the following links to one:

$$\mu = 0,\ 0 \le x^0 < N_t - 1,\ 0 \le x^1, x^2, x^3 < N_s\,,$$
$$\mu = 1,\ 0 \le x^1 < N_s - 1,\ 0 \le x^2, x^3 < N_s,\ x^0 = 0\,,$$
$$\mu = 2,\ 0 \le x^2 < N_s - 1,\ 0 \le x^3 < N_s,\ x^0 = x^1 = 0\,,$$
$$\mu = 3,\ 0 \le x^3 < N_s - 1,\ x^0 = x^1 = x^2 = 0\,,$$

'maximal axial gauge'

"Distance from *SU*(2)"



Factor $\sim$ *100*

Conclusions

- Classical-statistical simulations extremely valuable tool for bosons

  Preheating: quantum corrections for most part negligible even for highest momenta,
  relevant only for late-time thermalization

- Non-thermal fixed points of the classical-statistical theory

  Universal attractor solutions after an instability (tachyonic, parametric, …)
  $\Rightarrow$ exceedingly small heating temperatures (once quantum corrections included)
  $\Rightarrow$ prethermalization (early equation of state, …)

  Non-thermal fixed points unstable under inclusion of quantum corr./fermions

- No rigorous classical-statistical simulations for fermions (“sign problem”)

- 2PI quantum effective action techniques (“analytic & approximative”)

  $\Rightarrow$ resolve secularity $\Rightarrow$ far-from-equilibrium as well as thermalization
  $\Rightarrow$ nonperturbative 2PI *1/N*-expansion for scalars ✓, fermions ✓,
  2PI loop-expansions for gauge fields (1/*N* too complex: summing all 2PI planar diagrams!)

- Stochastic quantization techniques (“numerical & exact” - poorly developed so far)

  $\Rightarrow$ optimized updating using stochastic reweighting and/or gauge fixing
  $\Rightarrow$ inclusion of quantum corrections to classical results by Langevin updating