Spherically symmetric spacetimes in $f(R)$ gravity

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Introduction

- Solar System constraints / Post-Newtonian parameter

$$\mathrm{d}s^2 \equiv g_{\mu\nu}x^\mu x^\nu = -e^{A(r)}\mathrm{d}t^2 + e^{B(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

$$\mathrm{d}s^2 = -(1 - \frac{2GM}{r})\mathrm{d}t^2 + (1 + \frac{2\gamma_{\mathrm{PPN}}GM}{r})\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

$$\gamma_{\mathrm{PPN}} - 1 \approx -\frac{B}{A} - 1 = (2.1 \pm 2.3) \times 10^{-5}$$

Cassini time delay
Bertotti et al
Nature 425 (2003) 374

- Debate

$$\gamma_{\mathrm{PPN}} = ?$$

1/2 Chiba, Phys Lett B575 (2003) 1
1 Faraoni, Phys Rev D74 (2006) 023529 …
1/2 Erickcek et al, Phys Rev D74 (2006) 121501 …
1 Zhang, Phys. Rev D76 (2007) 024007 …

- Goal: Compute the continuous metric by solving the full field equations

Metric vs Palatini

- Beyond Einstein-Hilbert action: Same action yields different EOM:s depending on choice of free variables

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4x\sqrt{-g}f(R) + S_{\mathrm{m}}(g,\psi)\ , \qquad R \equiv g^{\mu\nu}R_{\mu\nu}(\Gamma)$$

- Metric formalism

$g_{\mu\nu}$ independent

$$\Gamma^{\rho}{}_{\mu\nu} \equiv \left\{{\rho \atop \mu\nu}\right\}_g$$

$$FR_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - \nabla_\mu\nabla_\nu F + g_{\mu\nu}\Box F = 8\pi G T_{\mu\nu}$$

$$F \equiv \frac{\partial f}{\partial R}$$

Metric vs Palatini

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- Metric formalism

$g_{\mu\nu}$ independent

$$\Gamma^{\rho}{}_{\mu\nu} \equiv \left\{ {}^{\;\rho}_{\mu\nu} \right\}_g$$

- Palatini formalism

$g_{\mu\nu}\,,\ \Gamma^{\rho}{}_{\mu\nu}$ independent

Larger space of allowed spacetime manifolds
Min may be found outside Levi-Civita subspace

$$FR_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - \nabla_\mu\nabla_\nu F + g_{\mu\nu}\Box F = 8\pi G T_{\mu\nu}$$

$$F \equiv \frac{\partial f}{\partial R}$$

$$\nabla_\rho(\sqrt{-g}F(R)g^{\mu\nu}) = 0 \quad \Rightarrow \quad \Gamma^{\rho}{}_{\mu\nu} \equiv \left\{ {}^{\;\rho}_{\mu\nu} \right\}_{Fg}$$

Metric case

- Trace equation

$$\Box F + \frac{1}{3}(FR - 2f) = \frac{8\pi G}{3}T$$

- For the static and spherically symmetric case

$$\begin{aligned} A' &= \frac{-1}{1+\gamma}\left(\frac{1-e^B}{r} - \frac{re^B}{F}8\pi G p + \frac{re^B}{2}(R - \frac{f}{F}) + \frac{4\gamma}{r}\right) \\ B' &= \frac{1-e^B}{r} + \frac{re^B}{F}\frac{8\pi G}{3}(2\rho + 3p) + \frac{re^B}{6}(R + \frac{f}{F}) - \gamma A' \end{aligned} \qquad \gamma \equiv \frac{rF'}{2F}$$

$$F'' + \frac{2}{r}F' + \frac{A' - B'}{2}F' + \frac{e^B}{3}(FR - 2f) = \frac{e^B}{3}8\pi G T$$

Kainulainen, Piilonen, Reijonen, and Sunhede, Phys Rev D76 (2007) 024020

- Together with the conservation law and a given eqn of state, these form a complete generalization of the Tolman-Oppenheimer-Volkov eqns

$$p' = -\frac{A'}{2}(\rho + p)\,, \quad p = p(\rho)$$

Metric case

- Weak field limit $A, B \ll 1$, Newtonian approx $p \ll \rho$
  Example: $f(R) = R - \mu^4/R$, $\mu^2 = 4\Lambda/\sqrt{3}$

$$d \equiv F - 1$$

$$d'' + \frac{2}{r}d' - \underbrace{\frac{\mu^2(1-3d)}{3\sqrt{d}}}_{\text{Neglect}} = -\frac{8\pi G}{3}\rho \quad \Rightarrow \quad d = \frac{2GM}{3r} + \underbrace{d_0}_{\text{Boundary condition}}$$

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Neglect

Boundary condition

$$A' = \frac{B}{r} - 2d' + \frac{\mu^2 r\sqrt{d}}{1+d} \quad \Rightarrow \quad A = -\frac{8GM}{3r} - \frac{(rB)_0}{3r} + A_0$$

$$(rB)' = \frac{16\pi G}{3}r^2\rho + \frac{(\mu r)^2}{3\sqrt{d}(1+d)} \quad \Rightarrow \quad B = \frac{4GM}{3r} + \frac{(rB)_0}{3r}$$

Metric case

- Weak field limit $A, B \ll 1$, Newtonian approx $p \ll \rho$
  Example: $f(R) = R - \mu^4/R\,,\;\; \mu^2 = 4\Lambda/\sqrt{3}$

$$d \equiv F - 1$$

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$$(rB)' = \frac{16\pi G}{3}r^2\rho + \frac{(\mu r)^2}{3\sqrt{d}(1+d)} \quad \Rightarrow \quad B = \frac{4GM}{3r} + \frac{(rB)_0}{3r}$$

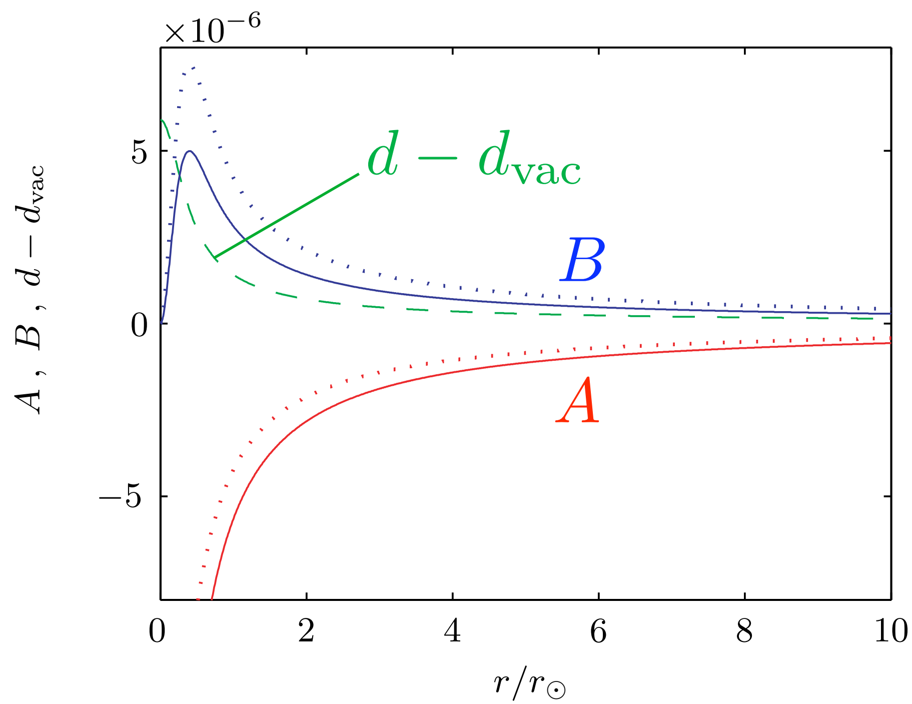
$$\gamma_{\mathrm{PPN}} = -\frac{B}{A} = \frac{4GM + 3(rB)_0}{8GM + 3(rB)_0} = \frac{1}{2}$$

For *any* finite value of B_0

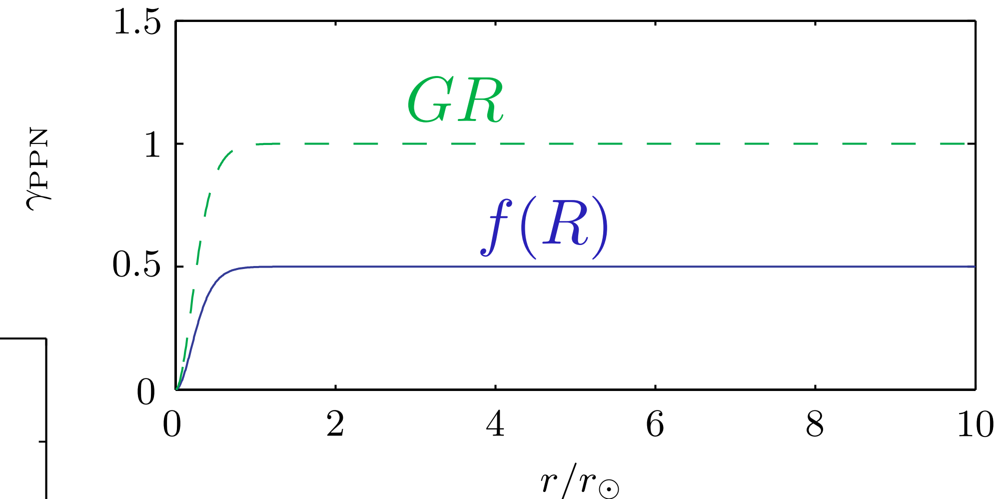
Metric case

- CASE I: $d_0 > 10^{-6}$, non-linear term is neglible and expected results follow

Metric components and $d - d_{\mathrm{vac}}$



PPN parameter



Results independent of boundary conditions (set at the center of the star)

Possible background such as DM and solar wind has *no* effect on the solution

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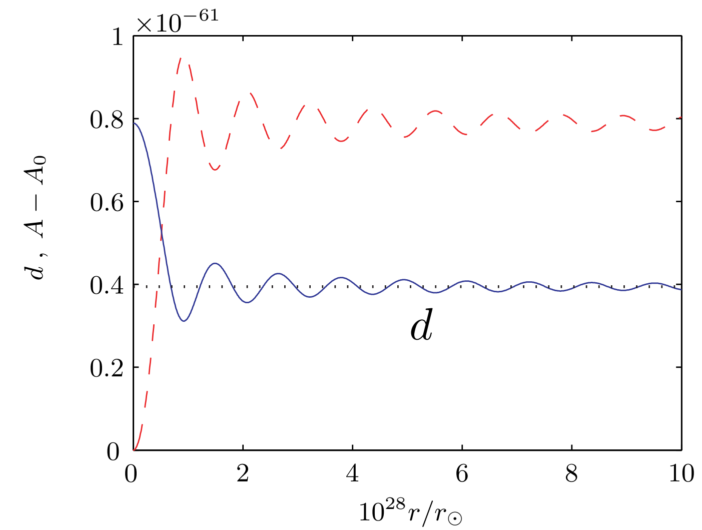
Metric case

- CASE II: $d_0 < 10^{-6}$, non-linear term dominant and solution gets trapped in Palatini track

$$d'' + \frac{2}{r}d' - \frac{\mu^2(1-3d)}{3\sqrt{d}} = -\frac{8\pi G}{3}\rho$$

Neglible

$$FR - 2f = 8\pi GT$$



- If in this limit also

  $F \approx 1 \quad \Rightarrow \quad R \approx 8\pi G\rho$

  GR-like limit exists! However…

  $f(R) = R - \mu^4/R$ solution unstable in time

  Stabilized $+\alpha R^2$ solution not compatible with obs

Dolgov & Kawasaki,
Phys Lett B573 (2003) 1

Nojiri & Odintsov,
Phys Rev D68 (2003) 123512

Kainulainen, Reijonen, and Sunhede
Phys Rev D76 (2007) 043503

Palatini case

- For constant density, R is also constant

$$F(R)R - 2f(R) = 8\pi GT \quad \Rightarrow \quad R = \text{const.} \quad \Rightarrow$$

$$G_{\mu\nu} + \Lambda_\rho g_{\mu\nu} = \frac{8\pi G}{F_\rho} T_{\mu\nu}\,, \qquad \Lambda_\rho \equiv \frac{1}{2}\left(R_\rho - \frac{f_\rho}{F_\rho}\right)$$

Schwarzschild-dS exterior, $\gamma_{\mathrm{PPN}} = 1$. However…

$$M = \int_0^{r_\odot} \mathrm{d}r \frac{4\pi r^2 \rho}{F_\rho} + \frac{\Lambda_\rho - \Lambda_0}{6G} r_\odot^3$$

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- For $f(R) = R - \mu^4/R$

$$R_\rho = \frac{1}{2}\left(8\pi G\rho \pm \sqrt{(8\pi G\rho)^2 + 12\mu^4}\right) \approx 8\pi G\rho$$

$$F = 1 + \frac{\mu^4}{R^2} \approx 1$$

OK!

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Palatini case

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- For $f(R) = R - \mu^4/R + \alpha R^2$

$$R_\rho = \frac{1}{2}\left(8\pi G\rho \pm \sqrt{(8\pi G\rho)^2 + 12\mu^4}\right) \approx 8\pi G\rho$$

$$F = 1 + \frac{\mu^4}{R^2} + 2\alpha R \gg 1 \qquad \text{NOT OK!}$$

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Summary

METRIC

- Non-linear terms negligible for major part of parameter space and $\gamma_{\rm PPN} = \frac{1}{2}$ follows
- Small region exists for which the solution follows a Palatini track. It appears that, as long as $\delta f(R) \equiv f(R) - R$ is *not* dominated by a cosmological constant, this solution is unstable or incompatible with obs.

Kainulainen & Sunhede, in progress

PALATINI

- Exterior solution always Schwarschild-dS, $\gamma_{\rm PPN} = 1$ Relation between density and apparent mass non-standard, only functions $\delta f(R)$ increasing slower than R will yield GR gravitational field