Stochastic Inflation revisited: Non-slow roll statistics and DBI inflation

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Aim

- Develop stochastic description of super-Hubble physics
  Generalize previous approaches beyond slow-roll approximation
  Why?
- To give a more complete description of probabilities and initial conditions, can be used to understand evolution on the landscape
- To describe non-slow roll inflationary models such as DBI-inflation and k-inflation, do they undergo eternal inflation?
- *Formulated by Starobinski, Vilenkin, Linde, Linde, Mezhlumian, Salopek and Bond, Rey, Sazaki, Nambu, Kakao, Graziano and others..... more recent work by Winitzki on k-inflation and Chen, Sarangi, Tye and Xu on eternal brane inflation*

Stochastic dynamics: the idea

- Inspiration: Fluctuations of massless scalar field in an approximately inflating spacetime grows at super-Hubble scales as

$$\langle \phi \rangle = 0, \quad \langle \phi^2 \rangle \propto H^3 t$$

- Scalar field undergoes a random walk! just as a particle undergoing Brownian motion
- Sub-Hubble fluctuations provide the stochastic noise that causes the super-Hubble field to jump around

Coarse graining the fields

- More generally we can split the fields ϕ up into super-Hubble modes ϕ_L and sub–Hubble modes ϕ_S

$$\phi = \phi_L + \phi_S = \int \frac{d^3k}{(2\pi)^3} \left(\underbrace{\theta(\epsilon aH - k)\phi_k}_{super-Hubble} + \underbrace{\theta(k - \epsilon aH)\phi_k}_{sub-Hubble} \right)$$

- Short wavelength modes are treated as LINEAR (gaussian) QUANTUM fluctuations
- Long wavelength modes are treated as NONLINEAR CLASSICAL fluctuations

Probabilistic dynamics

- In slow roll approximation, we have a Langevin equation

$$3H\frac{d\phi}{dt} = V_{,\phi} + \eta_\phi$$

- The noise η_ϕ is a gaussian random field with fluctuations

$$\langle \eta_\phi(t)\eta_\phi(t')\rangle = H^3\delta(t-t')$$

- The evolution is probabilistic, and we can write an evolution equation for this probability as a Fokker-Planck equation

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\phi}J_\phi = 0$$

- where the probability current is

$$J_\phi = \frac{V_{,\phi}}{3H}\rho + \underbrace{H^3\frac{\partial}{\partial\phi}\rho}_{\textit{diffusive current}}$$

E-folding time and scaling symmetries

- It is convenient (particularly in inflation) to parameterize time by the number of e-folds $\lambda = \ln a$.
- In a spatially flat universe, the classical action describing a set of fields ϕ_i exhibits a scaling symmetry

$$S = \int d\lambda e^{3\lambda} \left(\pi_i \frac{d\phi_i}{d\lambda} - H_{eq}(\phi_i, \pi_i) \right)$$

- Under $\lambda \to \lambda + c$, $S \to e^{3c} S$, so the equations of motion are e-folding time translation invariant!

Minimally coupled scalar as an example

- To see how this works, start with minisuperspace action for gravity/scalar system

$$S = \int dt N e^{3\lambda} \left(-\frac{3M^2\dot{\lambda}^2}{N^2} + \frac{\dot{\phi}^2}{2N^2} + V \right)$$

- Varying with respect to N gives the Friedman equation

$$\frac{3M^2\dot{\lambda}^2}{N^2} = \frac{\dot{\phi}^2}{2N^2} + V$$

- Solving this for N and substituting back into the action

$$S = -2M \int d\lambda e^{3\lambda} \sqrt{3V \left(1 - \frac{1}{6M^2} (\frac{d\phi}{d\lambda})^2 \right)}$$

Damped Hamiltonian system

- Any system in an FRW spacetime may be written as a damped Hamiltonian system

$$\frac{d\phi}{d\lambda} = \frac{\partial H_{eq}}{\partial \pi} \quad \frac{d\pi}{d\lambda} = -\frac{\partial H_{eq}}{\partial \phi} - \gamma\pi$$

- Where the damping/friction constant $\gamma = 3$. In the absence of damping the energy H_{eq} would be conserved:

$$\frac{dH_{eq}}{d\lambda} = -\gamma\pi\frac{\partial H_{eq}}{\partial \pi}$$

- For an system described by Einstein-Hilbert gravity we this energy is proportional to the Hubble constant

$$H_{eq} = 6M^2 H$$

- E.g. for min. coupled scalar $H_{eq} = 6M^2\sqrt{\pi^2 + V/(3M^2)}$

Detailed balance

- Although energy is lost through damping, energy is input through stochastic noise

$$\frac{d\phi}{d\lambda} = \frac{\partial H_{eq}}{\partial \pi} \qquad \frac{d\pi}{d\lambda} = -\frac{\partial H_{eq}}{\partial \phi} - \gamma\pi + \eta_\pi$$

$$\frac{dH_{eq}}{d\lambda} = -\gamma\pi\frac{\partial H_{eq}}{\partial \pi} + \eta_\pi \frac{\partial H_{eq}}{\partial \pi}$$

- This can lead to the possible existence of an equilibrium state satisfying DETAILED BALANCE. The loss of energy through friction can be balanced by the energy input via noise

$$\langle \frac{dH_{eq}}{d\lambda} \rangle = 0$$

- This describes an eternally inflating state

Fokker-Planck equation

- To see how to implement detailed balance we use the fact that associated with the previous Langevin equations is the dual Fokker-Planck equation

$$\frac{\partial \rho}{\partial \lambda} + \frac{\partial}{\partial \phi} J_\phi + \frac{\partial}{\partial \pi} J_\pi = 0$$

- where the currents split up into a reversible, damping and diffusion parts

$$J_\phi = \underbrace{\frac{\partial H_{eq}}{\partial \pi}}_{\textit{reversible}} \rho$$

$$J_\pi = -\underbrace{\frac{\partial H_{eq}}{\partial \phi}}_{\textit{reversible}} \rho - \underbrace{3\pi\rho}_{\textit{damping}} + \underbrace{D\frac{\partial \rho}{\partial \pi}}_{\textit{diffusion}}$$

Detailed balance in the FP

- The precise statement of detailed balance is that there should exist an equilibrium state in which the irreversible (damping+diffusion) current vanishes identically

$$J_A^{irrev} = 0 \quad \text{i.e.} \quad J_\pi^{irrev} = -3\pi\rho + D\frac{\partial\rho}{\partial\pi} = 0$$

- and the reversible part is conserved

$$\frac{\partial}{\partial\phi}J_\phi^{rev} + \frac{\partial}{\partial\pi}J_\pi^{rev} = [\rho, H_{eq}]_{\text{Poisson}} = 0$$

- Last equation implies $\rho_{eq} = f(H_{eq})$ and so the first determines the diffusion constant

$$D = -3\pi\frac{f(H_{eq})}{f'(H_{eq})}$$

Equilibrium solutions

- One can derive from the Fokker-Planck equation the irreversibility result

$$\frac{P(A \to B, \lambda)}{P(B \to A, \lambda)} = \frac{\rho(H_{eq}^B)}{\rho(H_{eq}^A)}$$

- However the fluctuation theorem tells us that this ratio is given by $\exp(S_B - S_A)$, where S_A is the entropy in the state A.
- Entropy must be a function of H_{eq} and hence H, solution must be entropy of de Sitter!

$$\rho_{eq}(H_{eq}) = \exp(S) = \exp(8\pi^2 M^2 H^{-2}) = \exp(8\pi^2 (6M^2)^2 H_{eq}^{-2})$$

Is detailed balance true? Coarse grained derivation

- Finally predicted Diffusion constant is

$$D = -3\pi \frac{1}{\frac{\partial H_{eq}}{\partial \pi} S'(H_{eq})} \propto H_{eq}^3 \pi \left(\frac{\partial H_{eq}}{\partial \pi} \right)^{-1}$$

- To check if detailed balance is true we can test this against a first principles (coarse grained) calculation of the noise

$$D\delta(\lambda - \lambda') = \frac{1}{2} \langle \eta_\pi(\lambda) \eta_\pi(\lambda') \rangle$$

- Consider arbitrary theories of form

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2M^2} R - P(X, \phi) \right)$$

where $X = -(\partial\phi)^2$. This includes DBI-inflation, k-inflation.

- Results agree to within the ambiguities of coarse graining

Conditions for eternal inflation

- Eternally inflating regime kicks in when stochastic noise fluctuations become comparable to classical evolution
- Can give universal statement from

$$\frac{dH_{eq}}{d\lambda} = -\underbrace{3\pi\frac{\partial H_{eq}}{\partial \pi}}_{\text{classical evolution}} + \underbrace{\eta_\pi \frac{\partial H_{eq}}{\partial \pi}}_{\text{stochastic fluctuations}}$$

  Condition is

$$(3\pi)^2 \geq D = -3\pi\left(\frac{\partial H_{eq}}{\partial \pi} S'(H_{eq})\right)^{-1}$$

  which is equivalent to

$$\frac{dS}{d\lambda} \geq 1$$

- Agrees with Arkani-Hamed et al (arXiv:0704.1814) derived for any theory satisfying null energy condition

DBI inflation

- In DBI-inflation, inflaton is position of D-brane moving in warped throat and 4d effective action takes the form

$$S = \int d^4x\sqrt{-g}\left(\frac{M^2}{2}R + \underbrace{T(\phi)\sqrt{1-(\partial\phi)^2/T(\phi)} - T(\phi)}_{\textit{DBI action}} - V(\phi)\right)$$

- $T(\phi) = T_3 h^4(\phi)$ is the warped brane tension where $h(\phi)$ is warp factor. $\dot{\phi}$ can be large, close to relativistic limit, but geometry can be quasi-de Sitter
- Equilibrium Hamiltonian calculates in the standard way is

$$H_{eq} = 6M^2\sqrt{\frac{2}{3M^2}\sqrt{\pi^2/T + T^2} + \frac{V-T}{3M^2}}$$

Summary

Results

- Derived universal form for stochastic evolution (Langevin or FP) valid without slow-roll assumption
- Equilibrium state with $\rho = \exp(S)$, satisfied detailed balance, where S is the de Sitter entropy
- Condition for eternal inflating regime $\frac{dS}{d\lambda} \approx 1$

Uses

- Probabilities on the Landscape
- Tunneling in the Landscape
- Framing the measure question (e.g. Gibbons-Turok)
- Nongaussianity