

**New Methods for Feynman Integrals.
Evaluating Multiloop Feynman Integrals
by Mellin–Barnes Representation**

- Historiographical remarks

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- Simple one-loop examples

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Slides and `Mathematica` files are available at

<http://science.sander.su>

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Systematic evaluation of dimensionally regularized Feynman integrals (in particular, systematic resolution of the singularities in ϵ) [V.A. Smirnov'99, J.B. Tausk'99]

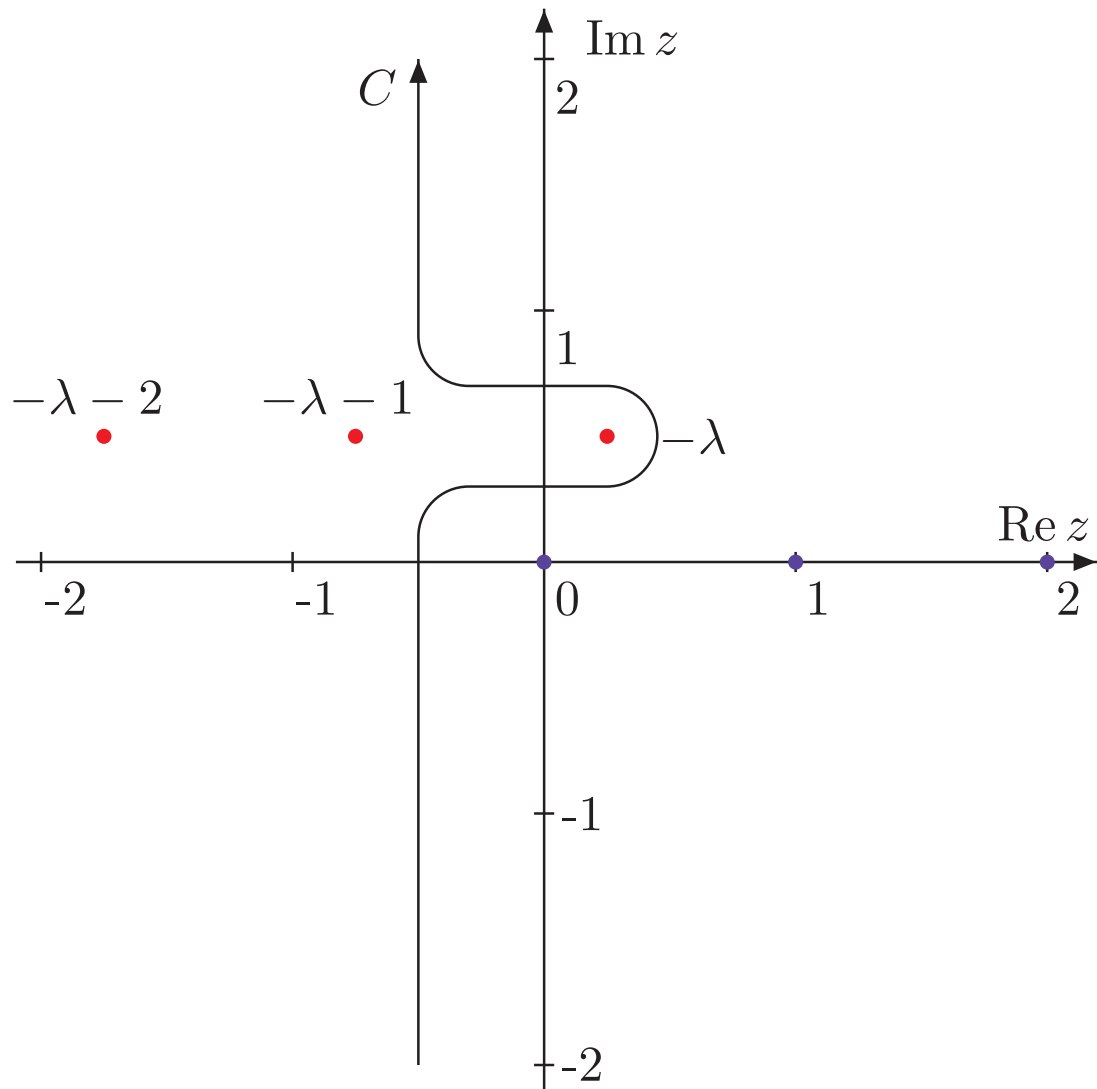
The basic formula:

$$\frac{1}{(X + Y)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{Y^z}{X^{\lambda+z}} \Gamma(\lambda+z) \Gamma(-z) .$$

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The poles with a $\Gamma(\dots +z)$ dependence are to the left of the contour and the poles with a $\Gamma(\dots -z)$ dependence are to the right



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- Expand in a Laurent series in ϵ
- Evaluate expanded MB integrals

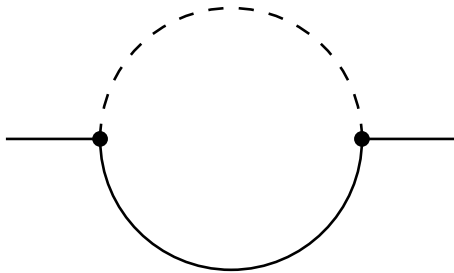
The simplest possibility:

$$\frac{1}{(m^2 - k^2)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{(m^2)^z}{(-k^2)^{\lambda+z}} \Gamma(\lambda + z) \Gamma(-z)$$

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Example 1



$$F_\Gamma(q^2, m^2; a_1, a_2, d) = \int \frac{\mathbf{d}^d k}{(m^2 - k^2)^{a_1} (-(q - k)^2)^{a_2}}$$

$$F_{\Gamma} = \frac{1}{\Gamma(a_1)} \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathbf{d}z (m^2)^z \Gamma(a_1 + z) \Gamma(-z) \\ \times \int \frac{\mathbf{d}^d k}{(-k^2)^{a_1+z} (-(q-k)^2)^{a_2}}$$

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$$\int \frac{\mathbf{d}^d k}{(-k^2)^{a_1+z} [-(q-k)^2]^{a_2}} = i\pi^{d/2} \frac{G(a_1 + z, a_2)}{(-q^2)^{a_1+a_2+\epsilon-2+z}},$$

$$G(a_1, a_2) = \frac{\Gamma(a_1 + a_2 + \epsilon - 2) \Gamma(2 - \epsilon - a_1) \Gamma(2 - \epsilon - a_2)}{\Gamma(a_1) \Gamma(a_2) \Gamma(4 - a_1 - a_2 - 2\epsilon)}$$

$$\begin{aligned}
F_{\Gamma}(q^2, m^2; a_1, a_2, d) &= \frac{i\pi^{d/2}\Gamma(2 - \epsilon - a_2)}{\Gamma(a_1)\Gamma(a_2)(-q^2)^{a_1+a_2+\epsilon-2}} \\
&\times \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathbf{d}z \left(\frac{m^2}{-q^2}\right)^z \Gamma(a_1 + a_2 + \epsilon - 2 + z) \\
&\times \frac{\Gamma(2 - \epsilon - a_1 - z)\Gamma(-z)}{\Gamma(4 - 2\epsilon - a_1 - a_2 - z)}
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&\times \frac{\Gamma(2 - \epsilon - a_1 - z)\Gamma(-z)}{\Gamma(4 - 2\epsilon - a_1 - a_2 - z)}
\end{aligned}$$

Unambiguous prescriptions for contours:

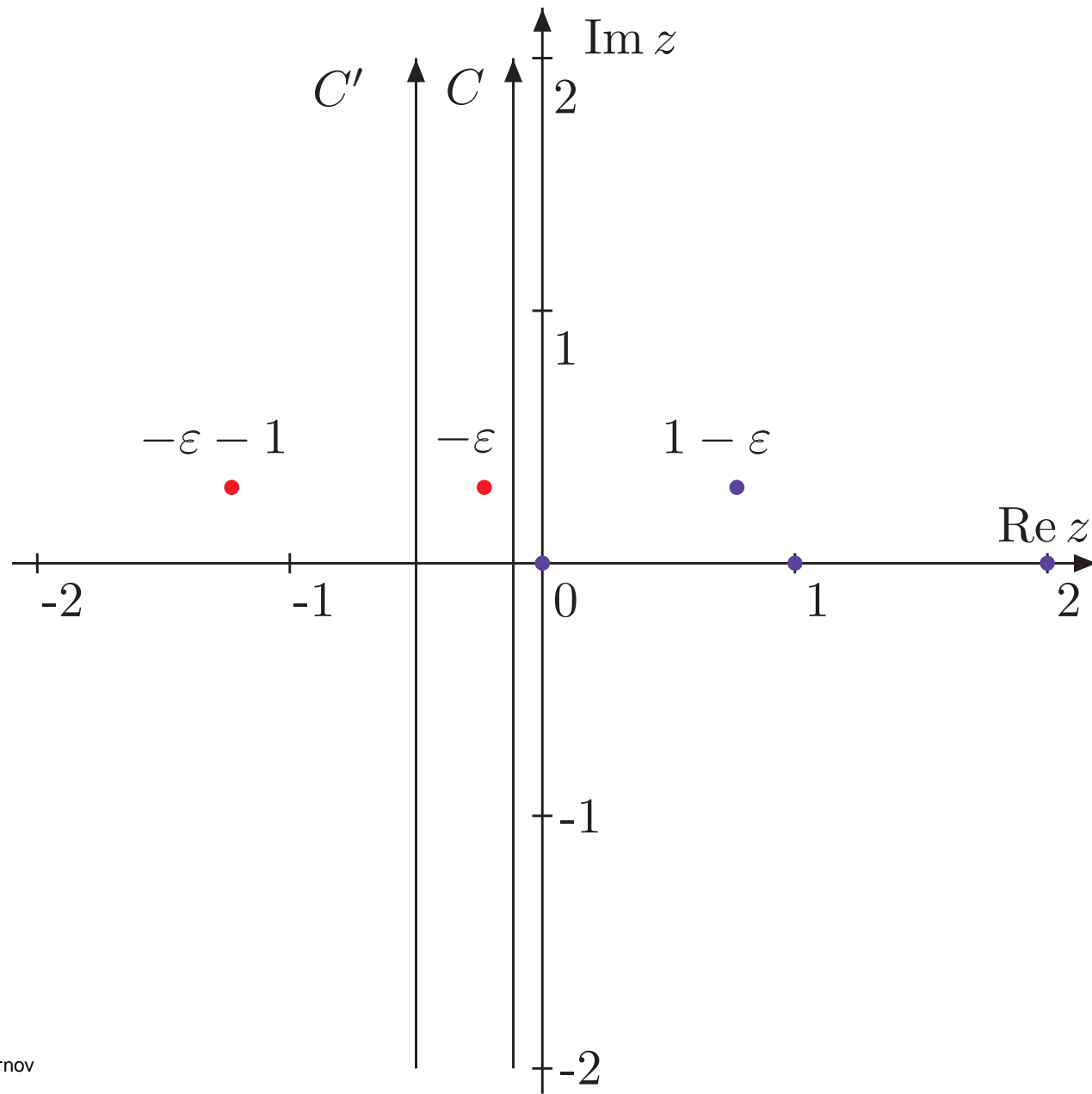
the poles with a $\Gamma(\dots +z)$ dependence are to the left and

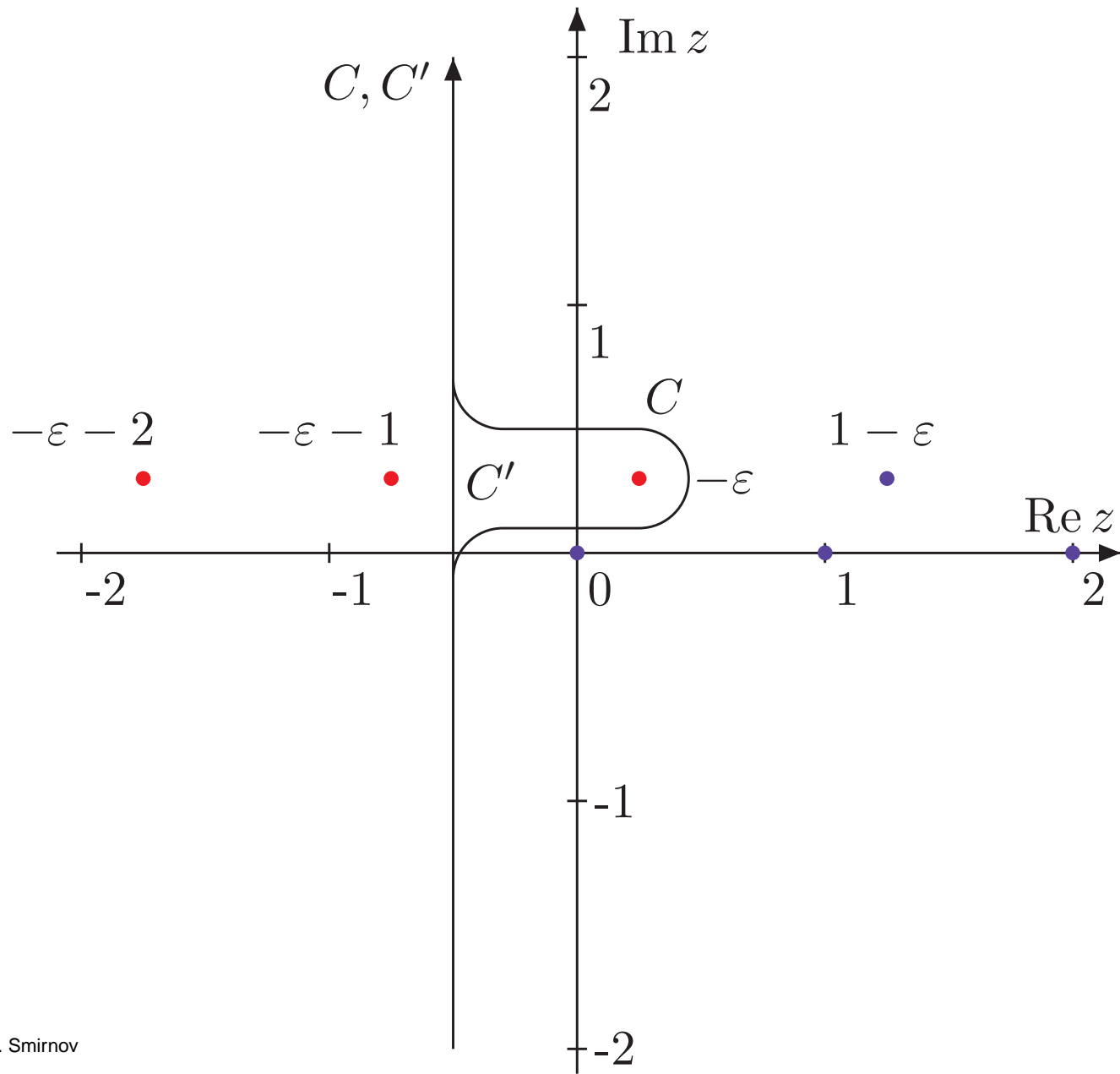
the poles with a $\Gamma(\dots -z)$ dependence are to the right of a contour

$$\begin{aligned}
F_{\Gamma}(q^2, m^2; 1, 1, d) &= \frac{i\pi^{d/2}\Gamma(1 - \epsilon)}{(-q^2)^{\epsilon}} \\
&\times \frac{1}{2\pi i} \int_C \mathbf{d}z \left(\frac{m^2}{-q^2} \right)^z \frac{\Gamma(\epsilon + z)\Gamma(-z)\Gamma(1 - \epsilon - z)}{\Gamma(2 - 2\epsilon - z)}
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$\Gamma(\epsilon + z)\Gamma(-z) \rightarrow$ a singularity in ϵ





Take a residue at $z = -\epsilon$:

$$i\pi^2 \frac{\Gamma(\epsilon)}{(m^2)^\epsilon (1 - \epsilon)}$$

and shift the contour:

$$\frac{i\pi^{d/2}\Gamma(1 - \epsilon)}{(-q^2)^\epsilon} \frac{1}{2\pi i} \int_{C'} \mathbf{d}z \left(\frac{m^2}{-q^2} \right)^z \frac{\Gamma(\epsilon + z)\Gamma(-z)\Gamma(1 - \epsilon - z)}{\Gamma(2 - 2\epsilon - z)}$$

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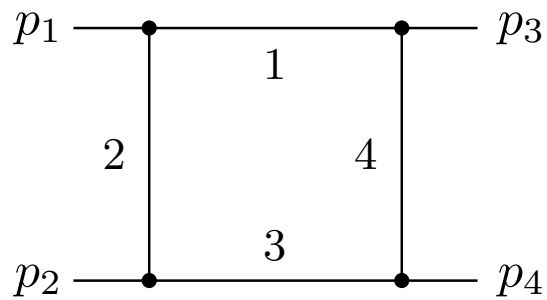
$$\frac{i\pi^{d/2}\Gamma(1 - \epsilon)}{(-q^2)^\epsilon} \frac{1}{2\pi i} \int_{C'} \mathbf{d}z \left(\frac{m^2}{-q^2}\right)^z \frac{\Gamma(\epsilon + z)\Gamma(-z)\Gamma(1 - \epsilon - z)}{\Gamma(2 - 2\epsilon - z)}$$

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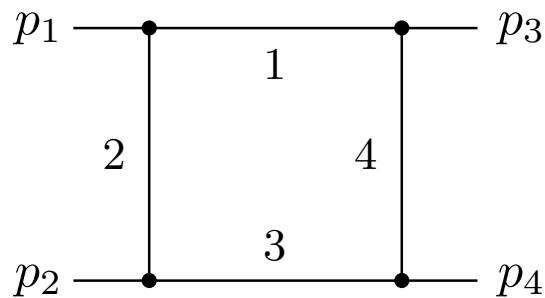
NB:

$$\Gamma(\epsilon + z)\Gamma(1 - \epsilon - z) = -\Gamma(1 + \epsilon + z)\Gamma(-\epsilon - z)$$

Example 2. The massless on-shell box diagram, i.e. with $p_i^2 = 0, i = 1, 2, 3, 4$



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$$F_{\Gamma}(s, t; a_1, a_2, a_3, a_4, d) = \int \frac{d^d k}{(-k^2)^{a_1} [-(k + p_1)^2]^{a_2} [-(k + p_1 + p_2)^2]^{a_3} [-(k - p_3)^2]^{a_4}},$$

where $s = (p_1 + p_2)^2$ and $t = (p_1 + p_3)^2$

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$$a = a_1 + \dots + a_4.$$

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \quad \mathcal{V} = t\alpha_1\alpha_3 + s\alpha_2\alpha_4.$$

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$$a = a_1 + \dots + a_4.$$

Introduce new variables by $\alpha_1 = \eta_1 \xi_1$, $\alpha_2 = \eta_1(1 - \xi_1)$, $\alpha_3 = \eta_2 \xi_2$, $\alpha_4 = \eta_2(1 - \xi_2)$, with the Jacobian $\eta_1 \eta_2$

$$\begin{aligned}
& F_{\Gamma}(s, t; a_1, a_2, a_3, a_4, d) \\
= & i\pi^{d/2} \frac{\Gamma(a + \epsilon - 2)\Gamma(2 - \epsilon - a_1 - a_2)\Gamma(2 - \epsilon - a_3 - a_4)}{\Gamma(4 - 2\epsilon - a) \prod \Gamma(a_l)} \\
\times & \int_0^1 \int_0^1 \frac{\xi_1^{a_1-1} (1 - \xi_1)^{a_2-1} \xi_2^{a_3-1} (1 - \xi_2)^{a_4-1}}{[-s\xi_1\xi_2 - t(1 - \xi_1)(1 - \xi_2) - i0]^{a+\epsilon-2}} d\xi_1 d\xi_2
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Apply the basic formula to separate

$-s\xi_1\xi_2$ and $-t(1 - \xi_1)(1 - \xi_2)$ in the denominator

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& \times \int_0^1 \int_0^1 \frac{\xi_1^{a_1-1} (1 - \xi_1)^{a_2-1} \xi_2^{a_3-1} (1 - \xi_2)^{a_4-1}}{[-s\xi_1\xi_2 - t(1 - \xi_1)(1 - \xi_2) - i0]^{a+\epsilon-2}} d\xi_1 d\xi_2
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Change the order of integration over z and ξ -parameters,
evaluate parametric integrals in terms of gamma functions

$$\begin{aligned}
F_{\Gamma}(s, t; a_1, a_2, a_3, a_4, d) &= \frac{i\pi^{d/2}}{\Gamma(4 - 2\epsilon - a) \prod \Gamma(a_l) (-s)^{a+\epsilon-2}} \\
&\times \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathbf{d}z \left(\frac{t}{s}\right)^z \Gamma(a + \epsilon - 2 + z) \Gamma(a_2 + z) \Gamma(a_4 + z) \Gamma(-z) \\
&\times \Gamma(2 - a_1 - a_2 - a_4 - \epsilon - z) \Gamma(2 - a_2 - a_3 - a_4 - \epsilon - z)
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$$\begin{aligned}
F_{\Gamma}(s, t; 1, 1, 1, 1, d) &= \frac{i\pi^{d/2}}{\Gamma(-2\epsilon) (-s)^{2+\epsilon}} \\
&\times \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathbf{d}z \left(\frac{t}{s}\right)^z \Gamma(2 + \epsilon + z) \Gamma(1+z)^2 \Gamma(-1 - \epsilon - z)^2 \Gamma(-z)
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\end{aligned}$$

Take minus residue at $z = -1 - \epsilon$ and shift the contour

Example 3.

$$f(\epsilon) = \frac{1}{2\pi i} \int_{C_1} \int_{C_2} \Gamma(\epsilon + z_1 + z_2) \Gamma(-z_1) \Gamma(-z_2) x^{z_1} y^{z_2} \mathbf{d}z_1 \mathbf{d}z_2$$

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$$\begin{aligned} \Gamma(\epsilon + z_1 + z_2) \Gamma(-z_2) &\rightarrow \Gamma(\epsilon + z_1), \\ \Gamma(\epsilon + z_1) \Gamma(-z_1) &\rightarrow \Gamma(\epsilon) \end{aligned}$$

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Shift contours and take residues

$$\begin{aligned}
f(\epsilon) &= \frac{1}{2\pi i} \int_{C_1} \int_{C'_2} \Gamma(\epsilon + z_1 + z_2) \Gamma(-z_1) \Gamma(-z_2) x^{z_1} y^{z_2} \mathbf{d}z_1 \mathbf{d}z_2 \\
&+ \frac{1}{2\pi i} \int_{C_1} \Gamma(\epsilon + z_1) \Gamma(-z_1) x^{z_1} y^{\epsilon - z_1} \mathbf{d}z_1 ,
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&\frac{1}{2\pi i} \int_{C_1} \Gamma(\epsilon + z_1) \Gamma(-z_1) x^{z_1} y^{\epsilon - z_1} \mathbf{d}z_1 \\
&= \frac{1}{2\pi i} \int_{C'_1} \Gamma(\epsilon + z_1) \Gamma(-z_1) x^{z_1} y^{\epsilon - z_1} \mathbf{d}z_1 + \Gamma(\epsilon) x^{-\epsilon} y^{2\epsilon}
\end{aligned}$$

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Two strategies: Strategy A and Strategy B

● Strategy A

[V.A. Smirnov'99]

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The general rule: $\Gamma(a+z)\Gamma(b-z)$, where a and b depend on the rest of the variables, generates a pole of the type $\Gamma(a+b)$

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The general rule: $\Gamma(a+z)\Gamma(b-z)$, where a and b depend on the rest of the variables, generates a pole of the type $\Gamma(a+b)$

Identifying **key** gamma functions (responsible for the generation of poles in ϵ).

Let $\Gamma(A_i)$ with $A_i = a_i + b_i\epsilon + \sum_j c_{ij}z_j$
be one of the key gamma functions. Consider ϵ real.

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changing rules for the contours)

$$\mathbf{Re} A_i > 0 \rightarrow -1 < \mathbf{Re} A_i < 0$$
$$\Gamma(A_i) \rightarrow \Gamma^{(1)}(A_i)$$

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Changing more:

$$-n < \operatorname{Re} A_i < -n + 1 \text{ for } n = 2, 3, \dots$$
$$\Gamma(A_i) \rightarrow \Gamma^{(n)}(A_i)$$

Let $\Gamma(A_i)$ with $A_i = a_i + b_i\epsilon + \sum_j c_{ij}z_j$
be one of the key gamma functions. Consider ϵ real.

‘Changing the nature’ of these key gamma functions (i.e. changing rules for the contours)

$$\text{Re} A_i > 0 \rightarrow -1 < \text{Re} A_i < 0$$
$$\Gamma(A_i) \rightarrow \Gamma^{(1)}(A_i)$$

Changing more:

$$-n < \text{Re} A_i < -n + 1 \text{ for } n = 2, 3, \dots$$
$$\Gamma(A_i) \rightarrow \Gamma^{(n)}(A_i)$$

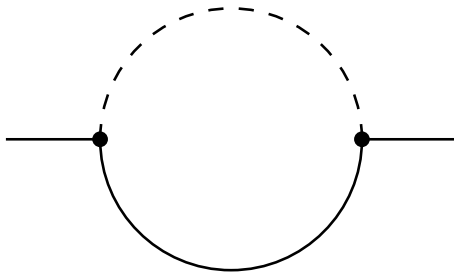
Taking residues and shifting contours.

For each resulting residue, which involves one integration less, apply a similar procedure, etc.

● Strategy B

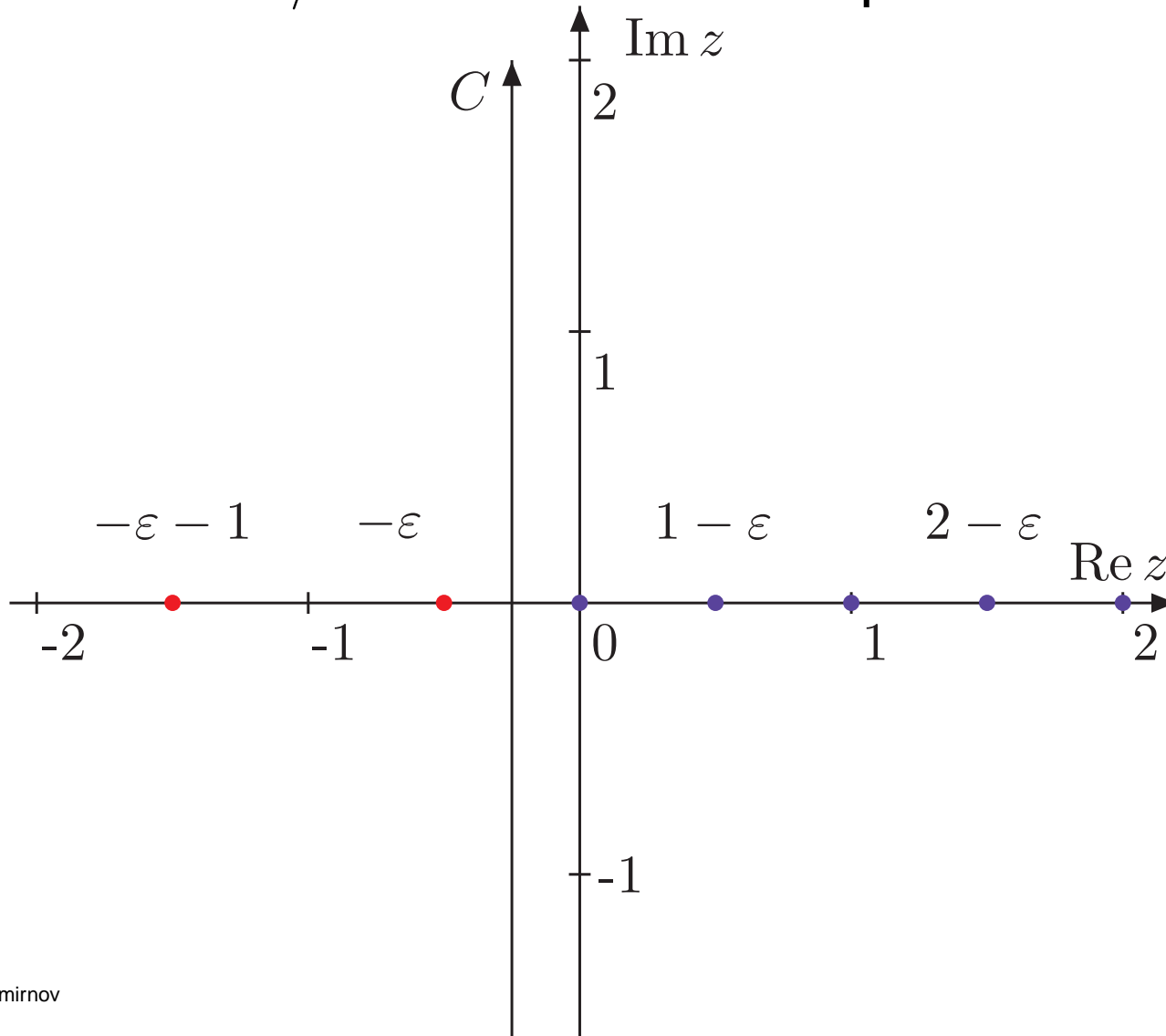
[J.B. Tausk'99, Anastasiou'05, Czakon'05].

Example 1 (again)

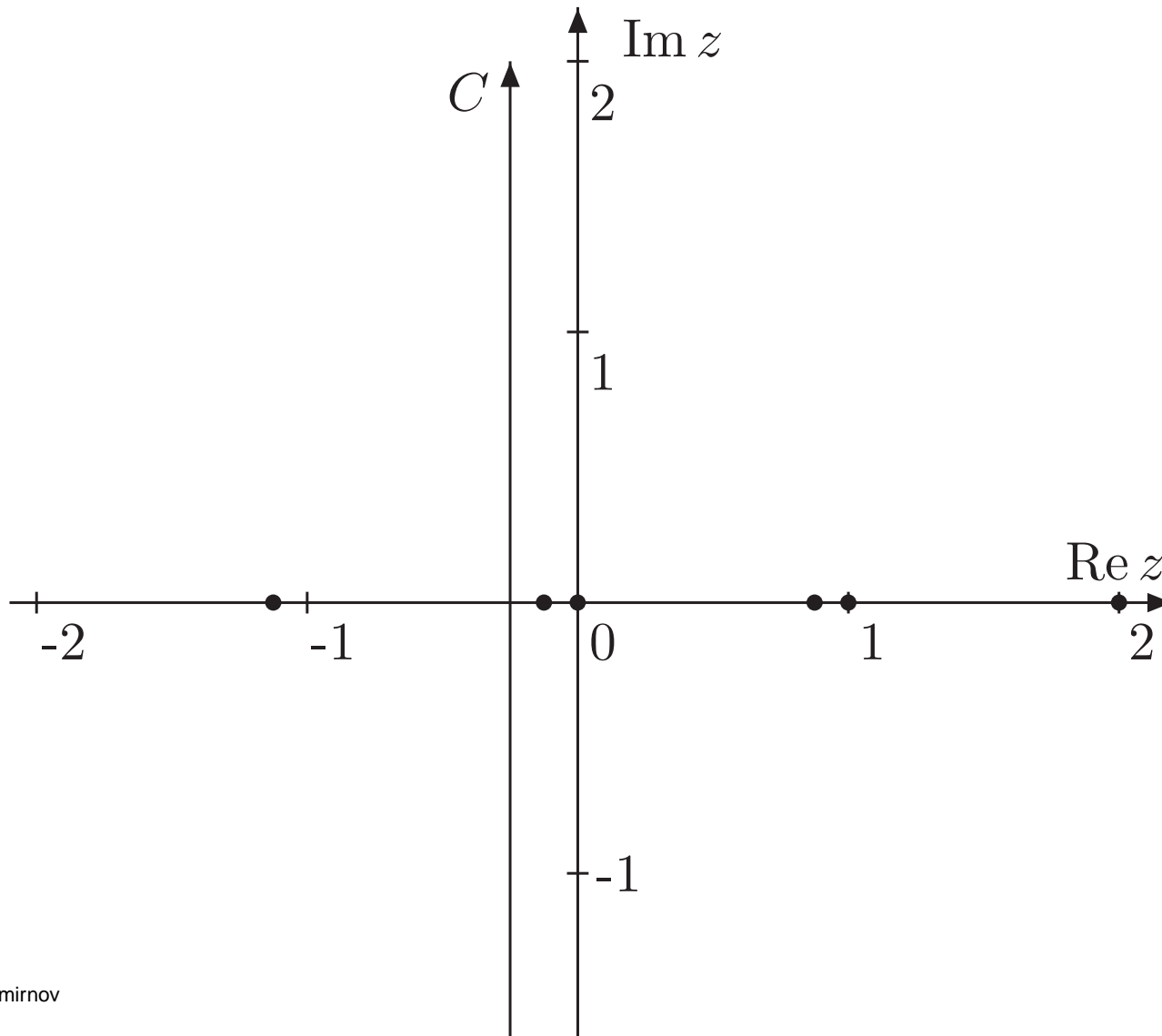


$$F_{\Gamma}(q^2, m^2; 1, 1, d) = \frac{i\pi^{d/2}\Gamma(1 - \epsilon)}{(-q^2)^{\epsilon}} \\ \times \frac{1}{2\pi i} \int_C \mathbf{d}z \left(\frac{m^2}{-q^2} \right)^z \frac{\Gamma(\epsilon + z)\Gamma(-z)\Gamma(1 - \epsilon - z)}{\Gamma(2 - 2\epsilon - z)}$$

Take ϵ real. Choose ϵ and a straight contour, e.g. $\epsilon = 1/2$, $\operatorname{Re} z = -1/4$. The contour is kept fixed. Tend ϵ to zero.



Whenever a pole of some gamma function is crossed add a residue and tend ϵ to zero further



● Strategy B

[J.B. Tausk'99, Anastasiou'05, Czakon'05].

- Strategy B

[J.B. Tausk'99, Anastasiou'05, Czakon'05].

Choose a domain of ϵ and $\text{Re}z_i, \dots, \text{Re}w_i$ in such a way that *all* the integrations over the MB variables can be performed over straight lines parallel to imaginary axis.

● Strategy B

[J.B. Tausk'99, Anastasiou'05, Czakon'05].

Choose a domain of ϵ and $\text{Re}z_i, \dots, \text{Re}w_i$ in such a way that *all* the integrations over the MB variables can be performed over straight lines parallel to imaginary axis.

Let $\epsilon \rightarrow 0$. Whenever a pole of some gamma function is crossed, take into account the corresponding residue.

● Strategy B

[J.B. Tausk'99, Anastasiou'05, Czakon'05].

Choose a domain of ϵ and $\text{Re}z_i, \dots, \text{Re}w_i$ in such a way that *all* the integrations over the MB variables can be performed over straight lines parallel to imaginary axis.

Let $\epsilon \rightarrow 0$. Whenever a pole of some gamma function is crossed, take into account the corresponding residue.

For every resulting residue, which involves one integration less, apply a similar procedure, etc.

Two algorithmic descriptions

[C. Anastasiou'05, M. Czakon'05]

Two algorithmic descriptions

[C. Anastasiou'05, M. Czakon'05]

The Czakon's version `MB.m` implemented in `Mathematica` is public.

To be put on <http://projects.hepforge.org/mbtools/>

(as well as other codes related to MB)

In[2]:= << MB/MB.m

MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08

(* The integrand of the MB integral for the one-loop propagator diagram with m1=m and m2=0 *)

In[3]:= MB[a1_, a2_] := (-1)^(a1 + a2) / QQ^(a1 + a2 + ep - 2) / Gamma[a1] / Gamma[a2] Gamma[2 - ep - a2] Gamma[a1 + a2 + ep - 2 + z] Gamma[2 - ep - a1 - z] Gamma[-z] / Gamma[4 - 2 ep - a1 - a2 - z] mm^z / QQ^z;

(* Notation: mm=m^2, QQ=-q^2;
I Pi^(d/2) is pulled out *)

(* The diagram with a1=1 and a2=1 *)

In[4]:= P1 = MB[1, 1]

Out[4]=
$$\frac{mm^z QQ^{-ep-z} \Gamma[1-ep] \Gamma[1-ep-z] \Gamma[-z] \Gamma[ep+z]}{\Gamma[2-2ep-z]}$$

In[5]:= P1Rules = MBOptimizedRules[P1, ep → 0, {}, {ep}]

MBrules::norules : no rules could be found to regulate this integral

Out[5]=
$$\left\{ \left\{ ep \rightarrow \frac{3}{4} \right\}, \left\{ z \rightarrow -\frac{1}{2} \right\} \right\}$$

In[6]:= P1cont = MBcontinue[P1, ep → 0, P1Rules]

Level 1

Taking +residue in z = -ep

Level 2

Integral {1}

2 integral(s) found

Out[6]=
$$\left\{ \left\{ MBint \left[\frac{mm^{-ep} \Gamma[1-ep] \Gamma[ep]}{\Gamma[2-ep]}, \{ep \rightarrow 0\}, \{\}\right] \right\}, \right. \\ \left. MBint \left[\frac{mm^z QQ^{-ep-z} \Gamma[1-ep] \Gamma[1-ep-z] \Gamma[-z] \Gamma[ep+z]}{\Gamma[2-2ep-z]}, \{ep \rightarrow 0\}, \left\{ z \rightarrow -\frac{1}{2} \right\} \right] \right\}$$

In[7]:= P1select = MBpreselect[P1cont, {ep, 0, 0}]

Out[7]=
$$\left\{ MBint \left[\frac{mm^{-ep} \Gamma[1-ep] \Gamma[ep]}{\Gamma[2-ep]}, \{ep \rightarrow 0\}, \{\}\right] \right. \\ \left. MBint \left[\frac{mm^z QQ^{-ep-z} \Gamma[1-ep] \Gamma[1-ep-z] \Gamma[-z] \Gamma[ep+z]}{\Gamma[2-2ep-z]}, \{ep \rightarrow 0\}, \left\{ z \rightarrow -\frac{1}{2} \right\} \right] \right\}$$

In[8]:= P1exp = MBexpand[P1select, E^(EulerGamma ep), {ep, 0, 0}]

Out[8]=
$$\left\{ MBint \left[1 + \frac{1}{ep} - \text{Log}[mm], \{ep \rightarrow 0\}, \{\}\right] \right. \\ \left. MBint \left[\frac{mm^z QQ^{-z} \Gamma[1-z] \Gamma[-z] \Gamma[z]}{\Gamma[2-z]}, \{ep \rightarrow 0\}, \left\{ z \rightarrow -\frac{1}{2} \right\} \right] \right\}$$

In[9]:= **Plexp[[1]]**

Out[9]= $\text{MBint}\left[1 + \frac{1}{\text{ep}} - \text{Log}[\text{mm}], \{\{\text{ep} \rightarrow 0\}, \{\}\}\right]$

In[10]:= **Plexp[[2]]**

Out[10]= $\text{MBint}\left[\frac{\text{mm}^z \text{QQ}^{-z} \text{Gamma}[1-z] \text{Gamma}[-z] \text{Gamma}[z]}{\text{Gamma}[2-z]}, \{\{\text{ep} \rightarrow 0\}, \{z \rightarrow -\frac{1}{2}\}\}\right]$

In[11]:= **int1 = Plexp[[2]] [[1]] //.**

{Gamma[2-z] -> (1-z) Gamma[1-z], Gamma[1-z] -> -z Gamma[-z], Gamma[z] -> Gamma[1+z]/z}

Out[11]= $\frac{\text{mm}^z \text{QQ}^{-z} \text{Gamma}[-z] \text{Gamma}[1+z]}{(1-z) z}$

(* The MB integral can be evaluated by closing the integration contour to the right in the complex z-plane. *)

(* First two residues *)

In[12]:= **res12 = -Residue[int1, {z, 0}] - Residue[int1, {z, 1}]**

Out[12]= $1 + \text{Log}[\text{mm}] - \text{Log}[\text{QQ}] - \frac{\text{mm} - \text{mm} \text{Log}[\text{mm}] + \text{mm} \text{Log}[\text{QQ}]}{\text{QQ}}$

(* Now we take residues at z=2,3,... *)

In[13]:= **int1 /. {Gamma[-z] Gamma[1+z] -> -pi Csc[pi z]}**

Out[13]= $-\frac{\text{mm}^z \pi \text{QQ}^{-z} \text{Csc}[\pi z]}{(1-z) z}$

(* Now we take (minus) the residue at z=n, To do this we shift the variable and then take a residue at z=0 *)

In[14]:= **% /. z -> z + n**

Out[14]= $-\frac{\text{mm}^{n+z} \pi \text{QQ}^{-n-z} \text{Csc}[\pi (n+z)]}{(1-n-z) (n+z)}$

In[15]:= **% /. {Csc[pi (n+z)] -> (-1)^n Csc[pi z]}**

Out[15]= $-\frac{(-1)^n \text{mm}^{n+z} \pi \text{QQ}^{-n-z} \text{Csc}[\pi z]}{(1-n-z) (n+z)}$

In[16]:= **-Residue[%, {z, 0}]**

Out[16]= $-\frac{(-1)^n \text{mm}^n \text{QQ}^{-n}}{(-1+n) n}$

(* Sum up contributions of the residues at z=2,3,... *)

In[17]:= **Sum[%, {n, 2, Infinity}]**

Out[17]= $-\frac{-\text{mm} + \text{mm} \text{Log}\left[1 + \frac{\text{mm}}{\text{QQ}}\right] + \text{QQ} \text{Log}\left[1 + \frac{\text{mm}}{\text{QQ}}\right]}{\text{QQ}}$

(* We add the above contributions from the residues at z=0 and z=1 as well as Blexp[[1]] *)

In[18]:= **Simplify**[% + res12 + P1exp[[1]][[1]]]

$$\text{Out[18]} = \frac{1}{\text{ep QQ}} \left(\text{QQ} + 2 \text{ep QQ} + \text{ep mm} \text{Log}[\text{mm}] - \text{ep mm} \text{Log}[\text{QQ}] - \text{ep QQ} \text{Log}[\text{QQ}] - \text{ep} (\text{mm} + \text{QQ}) \text{Log} \left[\frac{\text{mm} + \text{QQ}}{\text{QQ}} \right] \right)$$

In[19]:= % /. **Log** $\left[\frac{\text{mm} + \text{QQ}}{\text{QQ}} \right] \rightarrow \text{Log}[\text{mm} + \text{QQ}] - \text{Log}[\text{QQ}]$

$$\text{Out[19]} = \frac{1}{\text{ep QQ}} \left(\text{QQ} + 2 \text{ep QQ} + \text{ep mm} \text{Log}[\text{mm}] - \text{ep mm} \text{Log}[\text{QQ}] - \text{ep QQ} \text{Log}[\text{QQ}] - \text{ep} (\text{mm} + \text{QQ}) (-\text{Log}[\text{QQ}] + \text{Log}[\text{mm} + \text{QQ}]) \right)$$

In[20]:= **res** = % /. **Log** $[\text{mm} + \text{QQ}] \rightarrow \text{Log}[(\text{mm} + \text{QQ}) / \text{mm}] + \text{Log}[\text{mm}]$

$$\text{Out[20]} = \frac{1}{\text{ep QQ}} \left(\text{QQ} + 2 \text{ep QQ} + \text{ep mm} \text{Log}[\text{mm}] - \text{ep mm} \text{Log}[\text{QQ}] - \text{ep QQ} \text{Log}[\text{QQ}] - \text{ep} (\text{mm} + \text{QQ}) \left(\text{Log}[\text{mm}] - \text{Log}[\text{QQ}] + \text{Log} \left[\frac{\text{mm} + \text{QQ}}{\text{mm}} \right] \right) \right)$$

(* result *)

In[21]:= **resSimpl** = $2 + \frac{1}{\text{ep}} - \text{Log}[\text{mm}] - \left(1 + \frac{\text{mm}}{\text{QQ}} \right) \text{Log} \left[1 + \frac{\text{QQ}}{\text{mm}} \right];$

(* check *)

In[22]:= **Simplify**[res - resSimpl]

Out[22]= 0

In[2]:= << MB/MB.m

MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08

(* The integrand of the MB integral for the one-loop massless box diagram with $p1^2=p2^2=p3^2=p4^2=0$ *)

In[3]:= Box1[a1_, a2_, a3_, a4_] :=
(S^{2-a1-a2-a3-a4-ep-z} T^z Gamma[a1+a2+a3+a4-2+ep+z] Gamma[a2+z] Gamma[a4+z]
Gamma[2-a1-a2-a4-ep-z] Gamma[2-a2-a3-a4-ep-z] Gamma[-z]) /
(Gamma[a1] Gamma[a2] Gamma[a3] Gamma[a4] Gamma[4-a1-a2-a3-a4-2ep]);

(* Notation:
s=(p1+p2)^2=-S, t=(p1+p2)^2=-T;
I Pi^(d/2) is pulled out, as always *)

(* The box with the powers of the propagators equal to one *)

In[4]:= Box1[1, 1, 1, 1]

Out[4]=
$$\frac{S^{-2-ep-z} T^z \Gamma[-1-ep-z]^2 \Gamma[-z] \Gamma[1+z]^2 \Gamma[2+ep+z]}{\Gamma[-2ep]}$$

In[5]:= B1 = % /. {S -> 1, T -> x}

Out[5]=
$$\frac{x^z \Gamma[-1-ep-z]^2 \Gamma[-z] \Gamma[1+z]^2 \Gamma[2+ep+z]}{\Gamma[-2ep]}$$

In[6]:= B1Rules = MBOptimizedRules[B1, ep -> 0, {}, {ep}]

MBRules::norules : no rules could be found to regulate this integral

Out[6]=
$$\left\{ \left\{ ep \rightarrow -1 \right\}, \left\{ z \rightarrow -\frac{1}{2} \right\} \right\}$$

In[7]:= B1cont = MBcontinue[B1, ep -> 0, B1Rules]

Level 1

Taking -residue in z = -1 - ep

Level 2

Integral {1}

2 integral(s) found

Out[7]=
$$\left\{ \left\{ MBint \left[-\frac{EulerGamma x^{-1-ep} \Gamma[-ep]^2 \Gamma[1+ep]}{\Gamma[-2ep]} - \frac{x^{-1-ep} \Gamma[-ep]^2 \Gamma[1+ep] \text{Log}[x]}{\Gamma[-2ep]} - \frac{2 x^{-1-ep} \Gamma[-ep]^2 \Gamma[1+ep] \text{PolyGamma}[0, -ep]}{\Gamma[-2ep]} + \frac{x^{-1-ep} \Gamma[-ep]^2 \Gamma[1+ep] \text{PolyGamma}[0, 1+ep]}{\Gamma[-2ep]}, \{ \{ ep \rightarrow 0 \}, \{ \} \} \right\} \right\},$$

$$MBint \left[\frac{x^z \Gamma[-1-ep-z]^2 \Gamma[-z] \Gamma[1+z]^2 \Gamma[2+ep+z]}{\Gamma[-2ep]}, \left\{ \{ ep \rightarrow 0 \}, \left\{ z \rightarrow -\frac{1}{2} \right\} \right\} \right]$$

In[8]:= **Blselect = MBpreselect[Blcont, {ep, 0, 0}]**

$$\text{Out[8]} = \left\{ \text{MBint} \left[- \frac{\text{EulerGamma } x^{-1-\text{ep}} \text{Gamma}[-\text{ep}]^2 \text{Gamma}[1+\text{ep}]}{\text{Gamma}[-2\text{ep}]} - \frac{x^{-1-\text{ep}} \text{Gamma}[-\text{ep}]^2 \text{Gamma}[1+\text{ep}] \text{Log}[x]}{\text{Gamma}[-2\text{ep}]} - \frac{2 x^{-1-\text{ep}} \text{Gamma}[-\text{ep}]^2 \text{Gamma}[1+\text{ep}] \text{PolyGamma}[0, -\text{ep}]}{\text{Gamma}[-2\text{ep}]} + \frac{x^{-1-\text{ep}} \text{Gamma}[-\text{ep}]^2 \text{Gamma}[1+\text{ep}] \text{PolyGamma}[0, 1+\text{ep}]}{\text{Gamma}[-2\text{ep}]} \right], \{\{\text{ep} \rightarrow 0\}, \{\}\} \right\}$$

In[9]:= **Blselect = MBpreselect[Blcont, {ep, 0, 1}]**

$$\text{Out[9]} = \left\{ \text{MBint} \left[- \frac{\text{EulerGamma } x^{-1-\text{ep}} \text{Gamma}[-\text{ep}]^2 \text{Gamma}[1+\text{ep}]}{\text{Gamma}[-2\text{ep}]} - \frac{x^{-1-\text{ep}} \text{Gamma}[-\text{ep}]^2 \text{Gamma}[1+\text{ep}] \text{Log}[x]}{\text{Gamma}[-2\text{ep}]} - \frac{2 x^{-1-\text{ep}} \text{Gamma}[-\text{ep}]^2 \text{Gamma}[1+\text{ep}] \text{PolyGamma}[0, -\text{ep}]}{\text{Gamma}[-2\text{ep}]} + \frac{x^{-1-\text{ep}} \text{Gamma}[-\text{ep}]^2 \text{Gamma}[1+\text{ep}] \text{PolyGamma}[0, 1+\text{ep}]}{\text{Gamma}[-2\text{ep}]} \right], \{\{\text{ep} \rightarrow 0\}, \{\}\} \right\}, \\ \text{MBint} \left[\frac{x^z \text{Gamma}[-1-\text{ep}-z]^2 \text{Gamma}[-z] \text{Gamma}[1+z]^2 \text{Gamma}[2+\text{ep}+z]}{\text{Gamma}[-2\text{ep}]}, \{\{\text{ep} \rightarrow 0\}, \{z \rightarrow -\frac{1}{2}\}\} \right]$$

In[10]:= **Blexp = MBexpand[Blselect, E^(EulerGamma ep), {ep, 0, 1}]**

$$\text{Out[10]} = \left\{ \text{MBint} \left[\frac{4}{\text{ep}^2 x} - \frac{4 \pi^2}{3 x} - \frac{2 \text{Log}[x]}{\text{ep} x} + \frac{7 \text{ep} \pi^2 \text{Log}[x]}{6 x} + \frac{\text{ep} \text{Log}[x]^3}{3 x} + \frac{17 \text{ep} \text{PolyGamma}[2, 1]}{3 x} \right], \{\{\text{ep} \rightarrow 0\}, \{\}\} \right\}, \\ \text{MBint} \left[-2 \text{ep} x^2 \text{Gamma}[-1-z]^2 \text{Gamma}[-z] \text{Gamma}[1+z]^2 \text{Gamma}[2+z], \{\{\text{ep} \rightarrow 0\}, \{z \rightarrow -\frac{1}{2}\}\} \right]$$

In[11]:= **res1 = Blexp[[1]][[1]]**

$$\text{Out[11]} = \frac{4}{\text{ep}^2 x} - \frac{4 \pi^2}{3 x} - \frac{2 \text{Log}[x]}{\text{ep} x} + \frac{7 \text{ep} \pi^2 \text{Log}[x]}{6 x} + \frac{\text{ep} \text{Log}[x]^3}{3 x} + \frac{17 \text{ep} \text{PolyGamma}[2, 1]}{3 x}$$

In[12]:= **Blexp[[2]]**

$$\text{Out[12]} = \text{MBint} \left[-2 \text{ep} x^2 \text{Gamma}[-1-z]^2 \text{Gamma}[-z] \text{Gamma}[1+z]^2 \text{Gamma}[2+z], \{\{\text{ep} \rightarrow 0\}, \{z \rightarrow -\frac{1}{2}\}\} \right]$$

In[13]:= **int1 = Blexp[[2]][[1]]**

$$\text{Out[13]} = -2 \text{ep} x^2 \text{Gamma}[-1-z]^2 \text{Gamma}[-z] \text{Gamma}[1+z]^2 \text{Gamma}[2+z]$$

In[14]:= **Simplify[% //. {Gamma[-1-z] -> Gamma[-z] / (-1-z), Gamma[2+z] -> Gamma[1+z] (1+z)}]**

$$\text{Out[14]} = - \frac{2 \text{ep} x^2 \text{Gamma}[-z]^3 \text{Gamma}[1+z]^3}{1+z}$$

In[15]:= **% /. {Gamma[-z]^3 Gamma[1+z]^3 -> -pi^3 Csc[pi z]^3}**

$$\text{Out[15]} = \frac{2 \text{ep} \pi^3 x^2 \text{Csc}[\pi z]^3}{1+z}$$

(* Now we take residues at z=0,1,2,... *)

In[16]:= **% /. z -> z+n**

$$\text{Out[16]} = \frac{2 \text{ep} \pi^3 x^{n+z} \text{Csc}[\pi (n+z)]^3}{1+n+z}$$

In[17]:= % /. {Csc[π (n + z)] \rightarrow (-1) ^ n Csc[π z]}

$$\text{Out[17]} = \frac{2 (-1)^{3n} \text{ep } \pi^3 x^{n+z} \text{Csc}[\pi z]^3}{1 + n + z}$$

In[18]:= % /. {(-1) ^ {3n} \rightarrow (-1) ^ n}

$$\text{Out[18]} = \frac{2 (-1)^n \text{ep } \pi^3 x^{n+z} \text{Csc}[\pi z]^3}{1 + n + z}$$

In[19]:= -Residue[%, {z, 0}]

$$\text{Out[19]} = -\frac{1}{(1+n)^3} (-1)^n \text{ep } x^n (2 + \pi^2 + 2n\pi^2 + n^2\pi^2 - 2\text{Log}[x] - 2n\text{Log}[x] + \text{Log}[x]^2 + 2n\text{Log}[x]^2 + n^2\text{Log}[x]^2)$$

In[20]:= Apart[% /. n \rightarrow n - 1, n]

$$\text{Out[20]} = \frac{2 (-1)^n \text{ep } x^{-1+n}}{n^3} - \frac{2 (-1)^n \text{ep } x^{-1+n} \text{Log}[x]}{n^2} + \frac{(-1)^n \text{ep } x^{-1+n} (\pi^2 + \text{Log}[x]^2)}{n}$$

In[21]:= res2 = Sum[%, {n, 1, Infinity}]

$$\text{Out[21]} = \frac{1}{x} (-\text{ep } \pi^2 \text{Log}[1+x] - \text{ep } \text{Log}[x]^2 \text{Log}[1+x] - 2\text{ep } \text{Log}[x] \text{PolyLog}[2, -x] + 2\text{ep } \text{PolyLog}[3, -x])$$

(* Numerical check *)

In[22]:= % / ep /. {x \rightarrow 0.76, ep \rightarrow 0.3}

$$\text{Out[22]} = -9.70976$$

In[23]:= NIntegrate[int1 / ep /. {x \rightarrow 0.76, ep \rightarrow 0.3, z \rightarrow -0.5 + I * y1}, {y1, -Infinity, Infinity}] / 2 / Pi

$$\text{Out[23]} = -9.70976 + 2.67167 \times 10^{-13} i$$

In[24]:= res1 + res2

$$\text{Out[24]} = \frac{4}{\text{ep}^2 x} - \frac{4\pi^2}{3x} - \frac{2\text{Log}[x]}{\text{ep } x} + \frac{7\text{ep } \pi^2 \text{Log}[x]}{6x} + \frac{\text{ep } \text{Log}[x]^3}{3x} + \frac{17\text{ep } \text{PolyGamma}[2, 1]}{3x} + \frac{1}{x} (-\text{ep } \pi^2 \text{Log}[1+x] - \text{ep } \text{Log}[x]^2 \text{Log}[1+x] - 2\text{ep } \text{Log}[x] \text{PolyLog}[2, -x] + 2\text{ep } \text{PolyLog}[3, -x])$$

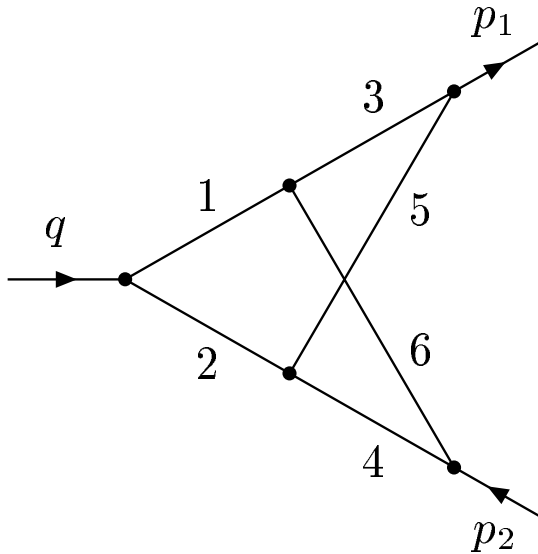
(* This is our result (up to I Pi^(d/2)) *)

In[25]:= (% /. x \rightarrow T / S) S^{-2-ep}

$$\text{Out[25]} = S^{-2-\text{ep}} \left(\frac{4S}{\text{ep}^2 T} - \frac{4\pi^2 S}{3T} - \frac{2S \text{Log}\left[\frac{T}{S}\right]}{\text{ep } T} + \frac{7\text{ep } \pi^2 S \text{Log}\left[\frac{T}{S}\right]}{6T} + \frac{\text{ep } S \text{Log}\left[\frac{T}{S}\right]^3}{3T} + \frac{17\text{ep } S \text{PolyGamma}[2, 1]}{3T} + \frac{1}{T} S \left(-\text{ep } \pi^2 \text{Log}\left[1 + \frac{T}{S}\right] - \text{ep } \text{Log}\left[\frac{T}{S}\right]^2 \text{Log}\left[1 + \frac{T}{S}\right] - 2\text{ep } \text{Log}\left[\frac{T}{S}\right] \text{PolyLog}\left[2, -\frac{T}{S}\right] + 2\text{ep } \text{PolyLog}\left[3, -\frac{T}{S}\right] \right) \right)$$

Non-planar two-loop massless vertex diagram with

$$p_1^2 = p_2^2 = 0, \quad Q^2 = -(p_1 - p_2)^2 = 2p_1 \cdot p_2$$



$$F_{\Gamma}(Q^2; a_1, \dots, a_6, d) = \int \int \frac{d^d k d^d l}{1 \left[(k+l)^2 - 2p_1 \cdot (k+l) \right]^{a_1}}$$

$$\times \frac{1}{\left[(k+l)^2 - 2p_2 \cdot (k+l) \right]^{a_2} (k^2 - 2p_1 \cdot k)^{a_3} (l^2 - 2p_2 \cdot l)^{a_4} (k^2)^{a_5} (l^2)^{a_6}}$$

$$\frac{1}{(k^2 - 2p_1 \cdot k)^{a_3} (k^2)^{a_5}} = \frac{(-1)^{a_3+a_5} \Gamma(a_3 + a_5)}{\Gamma(a_3) \Gamma(a_5)} \times \int_0^1 \frac{d\xi_1 \xi_1^{a_3-1} (1 - \xi_1)^{a_5-1}}{[-(k - \xi_1 p_1)^2 - i0]^{a_3+a_5}}$$

and, similarly, for the second pair, with the replacements

$$\xi_1 \rightarrow \xi_2, p_1 \rightarrow p_2, k \rightarrow l, a_3 \rightarrow a_4, a_5 \rightarrow a_6$$

Change the integration variable $l \rightarrow r = k + l$ and integrate over k by means of our massless one-loop formula

$$\int \frac{dk}{[-(k - \xi_1 p_1)^2]^{a_3+a_5} [-(r - \xi_2 p_2 - k)^2]^{a_4+a_6}}$$

$$= i\pi^{d/2} \frac{\Gamma(a_3 + a_5, a_4 + a_6)}{[-(r - \xi_1 p_1 - \xi_2 p_2)^2]^{a_3+a_4+a_5+a_6+\epsilon-2}}$$

Apply Feynman parametric formula to the propagators 1 and 2 and the propagator arising from the previous integration, with a resulting integral over r evaluated in terms of gamma functions:

$$\int \frac{d^d r}{[-(r^2 - Q^2 A(\xi_1, \xi_2, \xi_3, \xi_4))]^{a+\epsilon-2}}$$

$$= i\pi^{d/2} \frac{\Gamma(a + 2\epsilon - 4)}{\Gamma(a + \epsilon - 2)} \frac{1}{(Q^2)^{a+2\epsilon-4} A(\xi_1, \xi_2, \xi_3, \xi_4)^{a+2\epsilon-4}}$$

where $a = a_1 + \dots + a_6$ and

$$A(\xi_1, \xi_2, \xi_3, \xi_4) = \xi_3 \xi_4 + (1 - \xi_3 - \xi_4)[\xi_2 \xi_3 (1 - \xi_1) + \xi_1 \xi_4 (1 - \xi_2)]$$

Gonsalves'83:

$$F_{\Gamma}(Q^2; a_1, \dots, a_6, d) = \frac{(-1)^a \left(i\pi^{d/2}\right)^2 \Gamma(2 - \epsilon - a_{35}) \Gamma(2 - \epsilon - a_{46})}{(Q^2)^{a+2\epsilon-4} \prod \Gamma(a_l) \Gamma(4 - 2\epsilon - a_{3456})}$$

$$\times \Gamma(a + 2\epsilon - 4) \int_0^1 d\xi_1 \dots \int_0^1 d\xi_4 \xi_1^{a_3-1} (1 - \xi_1)^{a_5-1} \xi_2^{a_4-1} (1 - \xi_2)^{a_6-1}$$

$$\times \xi_3^{a_1-1} \xi_4^{a_2-1} (1 - \xi_3 - \xi_4)_+^{a_{3456} + \epsilon - 3} A(\xi_1, \xi_2, \xi_3, \xi_4)^{4-2\epsilon-a}$$

$$\begin{aligned}
& \frac{\Gamma(a + 2\epsilon - 4)}{[\eta\xi(1 - \xi) + (1 - \eta)(\xi\xi_2(1 - \xi_1) + (1 - \xi)\xi_1(1 - \xi_2))]^{a+2\epsilon-4}} \\
&= \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathbf{d}z_1 \frac{\Gamma(-z_1) \eta^{z_1} \xi^{z_1} (1 - \xi)^{z_1}}{(1 - \eta)^{a+2\epsilon-4+z_1}} \\
&\quad \times \frac{\Gamma(a + 2\epsilon - 4 + z_1)}{[\xi\xi_2(1 - \xi_1) + (1 - \xi)\xi_1(1 - \xi_2)]^{a+2\epsilon-4+z_1}}
\end{aligned}$$

The last line \rightarrow

$$\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \mathbf{d}z_2 \frac{\Gamma(a + 2\epsilon - 4 + z_1 + z_2) \Gamma(-z_2) \xi^{z_2} \xi_2^{z_2} (1 - \xi_1)^{z_2}}{(1 - \xi)^{a+2\epsilon-4+z_1+z_2} \xi_1^{a+2\epsilon-4+z_1+z_2} (1 - \xi_2)^{a+2\epsilon-4+z_1+z_2}}$$

$$\begin{aligned}
F_{\Gamma}(Q^2; a_1, \dots, a_6, d) &= \frac{(-1)^a \left(i\pi^{d/2}\right)^2 \Gamma(2 - \epsilon - a_{35})}{(Q^2)^{a+2\epsilon-4} \Gamma(6 - 3\epsilon - a) \prod \Gamma(a_l)} \\
&\times \frac{\Gamma(2 - \epsilon - a_{46})}{\Gamma(4 - 2\epsilon - a_{3456})} \frac{1}{(2\pi i)^2} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} dz_1 dz_2 \Gamma(a + 2\epsilon - 4 + z_1 + z_2) \\
&\quad \times \Gamma(-z_1) \Gamma(-z_2) \Gamma(a_4 + z_2) \Gamma(a_5 + z_2) \Gamma(a_1 + z_1 + z_2) \\
&\quad \times \frac{\Gamma(2 - \epsilon - a_{12} - z_1) \Gamma(4 - 2\epsilon + a_2 - a - z_2)}{\Gamma(4 - 2\epsilon - a_{1235} - z_1) \Gamma(4 - 2\epsilon - a_{1246} - z_1)} \\
&\quad \times \Gamma(4 - 2\epsilon + a_3 - a - z_1 - z_2) \Gamma(4 - 2\epsilon + a_6 - a - z_1 - z_2) ,
\end{aligned}$$

where $a_{3456} = a_3 + a_4 + a_5 + a_6$, etc.

The first Barnes lemma

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda_1 + z) \Gamma(\lambda_2 + z) \Gamma(\lambda_3 - z) \Gamma(\lambda_4 - z) \\ &= \frac{\Gamma(\lambda_1 + \lambda_3) \Gamma(\lambda_1 + \lambda_4) \Gamma(\lambda_2 + \lambda_3) \Gamma(\lambda_2 + \lambda_4)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)} \end{aligned}$$

Multiple corollaries, e.g.,

$$\begin{aligned} & \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \Gamma(\lambda_1 + z) \Gamma^*(\lambda_2 + z) \Gamma(-\lambda_2 - z) \Gamma(\lambda_3 - z) \\ &= \Gamma(\lambda_1 - \lambda_2) \Gamma(\lambda_2 + \lambda_3) [\psi(\lambda_1 - \lambda_2) - \psi(\lambda_1 + \lambda_3)] \end{aligned}$$

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \frac{dz}{z} \Gamma(\lambda_1 + z) \Gamma(\lambda_2 + z) \Gamma(\lambda_3 - z) \Gamma(\lambda_4 - z) \\
& \quad = \frac{\Gamma(2 - \lambda_1 - \lambda_3) \Gamma(1 - \lambda_2 - \lambda_3) \Gamma(\lambda_1 + \lambda_3 - 1) \Gamma(\lambda_2 + \lambda_3)}{\Gamma(1 - \lambda_1) \Gamma(1 - \lambda_2)} \\
& \quad \times [\Gamma(1 - \lambda_1) \Gamma(1 - \lambda_2) - \Gamma(2 - \lambda_1 - \lambda_2 - \lambda_3) \Gamma(\lambda_3)]
\end{aligned}$$

The second Barnes lemma

$$\begin{aligned}
& \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} dz \frac{\Gamma(\lambda_1 + z) \Gamma(\lambda_2 + z) \Gamma(\lambda_3 + z) \Gamma(\lambda_4 - z) \Gamma(\lambda_5 - z)}{\Gamma(\lambda_6 + z)} \\
& \quad = \frac{\Gamma(\lambda_1 + \lambda_4) \Gamma(\lambda_2 + \lambda_4) \Gamma(\lambda_3 + \lambda_4) \Gamma(\lambda_1 + \lambda_5)}{\Gamma(\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5) \Gamma(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_5)} \\
& \quad \times \frac{\Gamma(\lambda_2 + \lambda_5) \Gamma(\lambda_3 + \lambda_5)}{\Gamma(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)}, \quad \lambda_6 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5
\end{aligned}$$

```

In[2]:= << MB/MB.m
MB 1.1
by Michal Czakon
more info in hep-ph/0511200
last modified 06 Mar 08

(* 2fold MB representation for the non-planar vertex massless diagram.
The factor  $QQ^{4-a_1-a_2-a_3-a_4-a_5-a_6-2\epsilon}$  is omitted.
 $QQ=-(p_1-p_2)^2$ .
The factor  $(i\pi^{d/2})^2$  is also omitted as usually. *)

In[3]:= NPMB[a1_, a2_, a3_, a4_, a5_, a6_] := ((-1)^(a1 + a2 + a3 + a4 + a5 + a6) /
(Gamma[a1] Gamma[a2] Gamma[a3] Gamma[a4] Gamma[a5] Gamma[a6])
Gamma[2 - epsilon - a3 - a5] Gamma[2 - epsilon - a4 - a6] / Gamma[4 - 2 epsilon - a3 - a4 - a5 - a6] /
Gamma[6 - 3 epsilon - a1 - a2 - a3 - a4 - a5 - a6]
Gamma[a1 + a2 + a3 + a4 + a5 + a6 + 2 epsilon - 4 + z1 + z2] Gamma[-z1] Gamma[-z2]
Gamma[2 - epsilon - a1 - a2 - z1] Gamma[a4 + z2] Gamma[a1 + z1 + z2]
Gamma[4 - 2 epsilon - a1 - a3 - a4 - a5 - a6 - z2] Gamma[4 - 2 epsilon - a1 - a2 - a4 - a5 - a6 - z1 - z2]
Gamma[a5 + z2] Gamma[4 - 2 epsilon - a1 - a2 - a3 - a4 - a5 - z1 - z2] /
Gamma[4 - 2 epsilon - a1 - a2 - a4 - a6 - z1] / Gamma[4 - 2 epsilon - a1 - a2 - a3 - a5 - z1]);

(* The diagram with all powers of the propagators equal
to one. We shall evaluate it in expansion in epsilon up to epsilon^0. *)

In[4]:= V2 = NPMB[1, 1, 1, 1, 1, 1]

Out[4]:= (Gamma[-epsilon]^2 Gamma[-epsilon - z1] Gamma[-z1] Gamma[-1 - 2 epsilon - z2] Gamma[-1 - 2 epsilon - z1 - z2]^2
Gamma[-z2] Gamma[1 + z2]^2 Gamma[1 + z1 + z2] Gamma[2 + 2 epsilon + z1 + z2]) /
(Gamma[-3 epsilon] Gamma[-2 epsilon] Gamma[-2 epsilon - z1]^2)

In[5]:= V2rules = MBOptimizedRules[V2, epsilon -> 0, {}, {epsilon}]

Out[5]:= {{epsilon -> -5/8}, {z1 -> -1/4, z2 -> -1/4}}

In[6]:= V2cont = MBcontinue[V2, epsilon -> 0, V2rules];
Level 1
Taking -residue in z2 = -1 - 2 epsilon
Taking -residue in z2 = -1 - 2 epsilon - z1
Level 2
Integral {1}
Taking +residue in z1 = 2 epsilon
Integral {2}
Level 3
Integral {1, 1}
4 integral(s) found

In[7]:= V2select = MBpreselect[MBmerge[V2cont], {epsilon, 0, 0}]

```

In[8]:= **V2exp = Simplify[MBexpand[V2select, Exp[2 ep EulerGamma], {ep, 0, 0}]]**

$$\text{Out[8]= } \left\{ \text{MBint} \left[\frac{1}{\text{ep}^4} - \frac{\pi^2}{2 \text{ep}^2} - \frac{41 \pi^4}{40} + \frac{55 \text{PolyGamma}[2, 1]}{3 \text{ep}}, \{ \{\text{ep} \rightarrow 0\}, \{\} \} \right], \right. \\ \text{MBint} \left[\frac{1}{4 \text{ep}^2} \text{Gamma}[-z1]^2 \text{Gamma}[z1] \text{Gamma}[1+z1] \left(12 + 12 \text{ep EulerGamma} + 6 \text{ep}^2 \text{EulerGamma}^2 - \right. \right. \\ \left. \left. 7 \text{ep}^2 \pi^2 + 6 \text{ep}^2 \text{PolyGamma}[0, -z1]^2 + 12 \text{ep}^2 \text{PolyGamma}[0, z1]^2 - 12 \text{ep}^2 \text{PolyGamma}[0, 1+z1]^2 - \right. \right. \\ \left. \left. 24 \text{ep PolyGamma}[0, z1] (1 + \text{ep EulerGamma} + \text{ep PolyGamma}[0, 1+z1]) + 12 \text{ep} \right. \right. \\ \left. \left. \text{PolyGamma}[0, -z1] (3 + 3 \text{ep EulerGamma} - 2 \text{ep PolyGamma}[0, z1] + 4 \text{ep PolyGamma}[0, 1+z1]) - \right. \right. \\ \left. \left. 66 \text{ep}^2 \text{PolyGamma}[1, -z1] + 12 \text{ep}^2 \text{PolyGamma}[1, z1] - 12 \text{ep}^2 \text{PolyGamma}[1, 1+z1] \right), \right. \\ \left. \{ \{\text{ep} \rightarrow 0\}, \left\{ z1 \rightarrow -\frac{1}{4} \right\} \} \right], \text{MBint} \left[6 \text{Gamma}[-1-z2] \text{Gamma}[-1-z1-z2]^2 \text{Gamma}[-z2] \right. \\ \left. \text{Gamma}[1+z2]^2 \text{Gamma}[1+z1+z2] \text{Gamma}[2+z1+z2], \{ \{\text{ep} \rightarrow 0\}, \left\{ z1 \rightarrow -\frac{1}{4}, z2 \rightarrow -\frac{1}{4} \right\} \} \} \right] \left. \right\}$$

In[9]:= **Length[V2exp]**

Out[9]= 3

In[10]:= **res1 = V2exp[[1]][[1]]**

$$\text{Out[10]= } \frac{1}{\text{ep}^4} - \frac{\pi^2}{2 \text{ep}^2} - \frac{41 \pi^4}{40} + \frac{55 \text{PolyGamma}[2, 1]}{3 \text{ep}}$$

In[11]:= **V2exp[[3]]**

$$\text{Out[11]= } \text{MBint} \left[6 \text{Gamma}[-1-z2] \text{Gamma}[-1-z1-z2]^2 \text{Gamma}[-z2] \text{Gamma}[1+z2]^2 \right. \\ \left. \text{Gamma}[1+z1+z2] \text{Gamma}[2+z1+z2], \{ \{\text{ep} \rightarrow 0\}, \left\{ z1 \rightarrow -\frac{1}{4}, z2 \rightarrow -\frac{1}{4} \right\} \} \right]$$

In[12]:= **Barnes1[V2exp[[3]], z1]**

$$\text{Out[12]= } \text{MBint} \left[\pi^2 \text{Gamma}[-1-z2] \text{Gamma}[-z2] \text{Gamma}[1+z2]^2, \{ \{\text{ep} \rightarrow 0\}, \left\{ z2 \rightarrow -\frac{1}{4} \right\} \} \right]$$

In[13]:= **res2 = Barnes1[%, z2][[1]]**

$$\text{Out[13]= } -\frac{\pi^4}{6}$$

In[14]:= **V2exp[[2]]**

$$\text{Out[14]= } \text{MBint} \left[\frac{1}{4 \text{ep}^2} \right. \\ \left. \text{Gamma}[-z1]^2 \text{Gamma}[z1] \text{Gamma}[1+z1] \left(12 + 12 \text{ep EulerGamma} + 6 \text{ep}^2 \text{EulerGamma}^2 - 7 \text{ep}^2 \pi^2 + \right. \right. \\ \left. \left. 6 \text{ep}^2 \text{PolyGamma}[0, -z1]^2 + 12 \text{ep}^2 \text{PolyGamma}[0, z1]^2 - 12 \text{ep}^2 \text{PolyGamma}[0, 1+z1]^2 - \right. \right. \\ \left. \left. 24 \text{ep PolyGamma}[0, z1] (1 + \text{ep EulerGamma} + \text{ep PolyGamma}[0, 1+z1]) + 12 \text{ep} \right. \right. \\ \left. \left. \text{PolyGamma}[0, -z1] (3 + 3 \text{ep EulerGamma} - 2 \text{ep PolyGamma}[0, z1] + 4 \text{ep PolyGamma}[0, 1+z1]) - \right. \right. \\ \left. \left. 66 \text{ep}^2 \text{PolyGamma}[1, -z1] + 12 \text{ep}^2 \text{PolyGamma}[1, z1] - \right. \right. \\ \left. \left. 12 \text{ep}^2 \text{PolyGamma}[1, 1+z1] \right), \{ \{\text{ep} \rightarrow 0\}, \left\{ z1 \rightarrow -\frac{1}{4} \right\} \} \right]$$

In[15]:= **CoeffEps[X_, n_] := (X /. X[[1]] → Simplify[Coefficient[X[[1]], ep, n]);**

In[16]:= **CoeffEps[V2exp[[2]], -2]**

$$\text{Out[16]= } \text{MBint} \left[3 \text{Gamma}[-z1]^2 \text{Gamma}[z1] \text{Gamma}[1+z1], \{ \{\text{ep} \rightarrow 0\}, \left\{ z1 \rightarrow -\frac{1}{4} \right\} \} \right]$$

In[17]:= **res32 = Barnes1[CoeffEps[V2exp[[2]], -2], z1][[1]]**

$$\text{Out[17]} = -\frac{\pi^2}{2}$$

In[18]:= **CoeffEps[V2exp[[2]], -1]**

$$\text{Out[18]} = \text{MBint}\left[3 \Gamma[-z1]^2 \Gamma[z1] \Gamma[1+z1] \left(\text{EulerGamma} + 3 \text{PolyGamma}[0, -z1] - 2 \text{PolyGamma}[0, z1]\right), \left\{\{ep \rightarrow 0\}, \left\{z1 \rightarrow -\frac{1}{4}\right\}\right\}\right]$$

In[19]:= **res31 = 9 Zeta[3];**

In[20]:= **res31 // N**

Out[20]= 10.8185

In[21]:= **NIntegrate[CoeffEps[V2exp[[2]], -1][[1]] / (2 Pi) /. {z1 -> -1/4 + I * y1}, {y1, -Infinity, Infinity}]**

Out[21]= 10.8185 + 2.13163 × 10⁻¹⁴ i

In[22]:= **CoeffEps[V2exp[[2]], 0]**

$$\text{Out[22]} = \text{MBint}\left[\frac{1}{4} \Gamma[-z1]^2 \Gamma[z1] \Gamma[1+z1] \left(6 \text{EulerGamma}^2 - 7 \pi^2 + 6 \text{PolyGamma}[0, -z1]^2 + 12 \text{PolyGamma}[0, z1]^2 - 12 \text{PolyGamma}[0, 1+z1]^2 - 24 \text{PolyGamma}[0, z1] (\text{EulerGamma} + \text{PolyGamma}[0, 1+z1]) + 12 \text{PolyGamma}[0, -z1] (3 \text{EulerGamma} - 2 \text{PolyGamma}[0, z1] + 4 \text{PolyGamma}[0, 1+z1]) - 66 \text{PolyGamma}[1, -z1] + 12 \text{PolyGamma}[1, z1] - 12 \text{PolyGamma}[1, 1+z1]\right), \left\{\{ep \rightarrow 0\}, \left\{z1 \rightarrow -\frac{1}{4}\right\}\right\}\right]$$

In[23]:= **res30 = 7 π⁴ / 10;**

In[24]:= **res30 // N**

Out[24]= 68.1864

In[25]:= **NIntegrate[CoeffEps[V2exp[[2]], 0][[1]] / (2 Pi) /. {z1 -> -1/4 + I * y1}, {y1, -Infinity, Infinity}]**

Out[25]= 68.1864 + 0. i

(* result *)

In[26]:= **FullSimplify[res1 + res2 + res32 / ep^2 + res31 / ep + res30]**

$$\text{Out[26]} = \frac{1}{ep^4} - \frac{\pi^2}{ep^2} - \frac{59 \pi^4}{120} - \frac{83 \text{Zeta}[3]}{3 ep}$$

```
In[3]:= SetDirectory["c:/diskE/job2008/Zurich"];
```

```
In[4]:= << MB/MB.m
```

MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08

(* Example 4a *)

```
In[5]:= F = Gamma[3/2 + ep + z] Gamma[-1 - 2 ep - z]
Gamma[4 ep + z] Gamma[-z] Gamma[1/2 - ep - z] / Gamma[1 - 2 ep - z]
```

```
Out[5]= 
$$\frac{1}{\Gamma[1 - 2 ep - z]}$$

```

```
Gamma[-1 - 2 ep - z] Gamma[1/2 - ep - z] Gamma[-z] Gamma[3/2 + ep + z] Gamma[4 ep + z]
```

(* Strategy #2 *)

```
In[6]:= Frules = MBOptimizedRules[F, ep -> 0, {}, {ep}]
```

MBrules::norules : no rules could be found to regulate this integral

```
Out[6]= {}
```

(* Strategy #1 *)

(* The two residues *)

```
In[7]:= -Residue[F, {z, -1 - 2 ep}] + Residue[F, {z, -4 ep}]
```

```
Out[7]= 
$$\frac{\Gamma\left[\frac{1}{2} - ep\right] \Gamma\left[\frac{3}{2} + ep\right] \Gamma[-1 + 2 ep] \Gamma[1 + 2 ep] + \Gamma\left[\frac{3}{2} - 3 ep\right] \Gamma[4 ep] \Gamma[-1 + 2 ep] \Gamma\left[\frac{1}{2} + 3 ep\right]}{\Gamma[1 + 2 ep]}$$

```

(* plus an integral with the first poles of Gamma[4 ep+z] and Gamma[-1-2 ep-z] of the opposite nature *)

(* Strategy #2: introduce an auxiliary analytic regularization *)

```
In[8]:= F = Gamma[3/2 + ep + z] Gamma[-1 - 2 ep - z + y]
Gamma[4 ep + z] Gamma[-z] Gamma[1/2 - ep - z] / Gamma[1 - 2 ep - z]
```

```
Out[8]= 
$$\frac{1}{\Gamma[1 - 2 ep - z]}$$

```

```
Gamma[1/2 - ep - z] Gamma[-1 - 2 ep + y - z] Gamma[-z] Gamma[3/2 + ep + z] Gamma[4 ep + z]
```

```
In[9]:= Steplrules = MBOptimizedRules[F, y -> 0, {}, {ep, y}]
```

MBrules::norules : no rules could be found to regulate this integral

```
Out[9]= {{ep -> 1/2, y -> 7/4}, {z -> -5/4}}
```

```
In[10]:= con1 = MBcontinue[F, y -> 0, Steplrules]
```

Level 1

Taking -residue in $z = -1 - 2\epsilon p + y$

Level 2

Integral {1}

2 integral(s) found

$$\text{Out[10]= } \left\{ \left\{ \text{MBint} \left[\frac{\text{Gamma} \left[\frac{3}{2} + \epsilon p - y \right] \text{Gamma} [1 + 2 \epsilon p - y] \text{Gamma} \left[\frac{1}{2} - \epsilon p + y \right] \text{Gamma} [-1 + 2 \epsilon p + y]}{\text{Gamma} [2 - y]} \right. \right. \right. \\ \left. \left. \left. \left\{ \left\{ \epsilon p \rightarrow \frac{1}{2}, y \rightarrow 0 \right\}, \{ \} \right\} \right\}, \text{MBint} \left[\frac{1}{\text{Gamma} [1 - 2 \epsilon p - z]} \right. \right. \\ \left. \left. \text{Gamma} \left[\frac{1}{2} - \epsilon p - z \right] \text{Gamma} [-1 - 2 \epsilon p + y - z] \text{Gamma} [-z] \text{Gamma} \left[\frac{3}{2} + \epsilon p + z \right] \text{Gamma} [4 \epsilon p + z], \right. \right. \\ \left. \left. \left. \left\{ \left\{ \epsilon p \rightarrow \frac{1}{2}, y \rightarrow 0 \right\}, \left\{ z \rightarrow -\frac{5}{4} \right\} \right\} \right\} \right\} \right\}$$

In[11]= **exp1 = MBexpand[con1, 1, {y, 0, 0}]**

$$\text{Out[11]= } \left\{ \text{MBint} \left[\text{Gamma} \left[\frac{1}{2} - \epsilon p \right] \text{Gamma} \left[\frac{3}{2} + \epsilon p \right] \text{Gamma} [-1 + 2 \epsilon p] \text{Gamma} [1 + 2 \epsilon p], \left\{ \left\{ \epsilon p \rightarrow \frac{1}{2}, y \rightarrow 0 \right\}, \{ \} \right\} \right], \right. \\ \left. \text{MBint} \left[\frac{1}{\text{Gamma} [1 - 2 \epsilon p - z]} \text{Gamma} [-1 - 2 \epsilon p - z] \text{Gamma} \left[\frac{1}{2} - \epsilon p - z \right] \right. \right. \\ \left. \left. \text{Gamma} [-z] \text{Gamma} \left[\frac{3}{2} + \epsilon p + z \right] \text{Gamma} [4 \epsilon p + z], \left\{ \left\{ \epsilon p \rightarrow \frac{1}{2}, y \rightarrow 0 \right\}, \left\{ z \rightarrow -\frac{5}{4} \right\} \right\} \right\} \right\}$$

In[12]= **con2 = Table[MBcontinue[exp1[[i, 1]], \epsilon p \to 0, exp1[[i, 2]]], {i, Length[exp1]}**

Level 1

1 integral(s) found

Level 1

Taking +residue in $z = -1 - 2 \text{ ep}$

Taking +residue in $z = -4 \text{ ep}$

Taking +residue in $z = -1 - 4 \text{ ep}$

Level 2

Integral {1}

Integral {2}

Integral {3}

4 integral(s) found

Out[12]= $\left\{ \left\{ \text{MBint} \left[\text{Gamma} \left[\frac{1}{2} - \text{ep} \right] \text{Gamma} \left[\frac{3}{2} + \text{ep} \right] \text{Gamma} [-1 + 2 \text{ ep}] \text{Gamma} [1 + 2 \text{ ep}], \{ \{ \text{ep} \rightarrow 0, \text{y} \rightarrow 0 \}, \{ \} \} \right] \right\}, \right.$
 $\left\{ \left\{ \text{MBint} \left[-\text{Gamma} \left[\frac{1}{2} - \text{ep} \right] \text{Gamma} \left[\frac{3}{2} + \text{ep} \right] \text{Gamma} [-1 + 2 \text{ ep}] \text{Gamma} [1 + 2 \text{ ep}], \{ \{ \text{ep} \rightarrow 0, \text{y} \rightarrow 0 \}, \{ \} \} \right] \right\}, \right.$
 $\left\{ \text{MBint} \left[\frac{\text{Gamma} \left[\frac{3}{2} - 3 \text{ ep} \right] \text{Gamma} [4 \text{ ep}] \text{Gamma} [-1 + 2 \text{ ep}] \text{Gamma} \left[\frac{1}{2} + 3 \text{ ep} \right]}{\text{Gamma} [1 + 2 \text{ ep}]}, \{ \{ \text{ep} \rightarrow 0, \text{y} \rightarrow 0 \}, \{ \} \} \right] \right\},$
 $\left\{ \text{MBint} \left[-\frac{\text{Gamma} \left[\frac{1}{2} - 3 \text{ ep} \right] \text{Gamma} [2 \text{ ep}] \text{Gamma} \left[\frac{3}{2} + 3 \text{ ep} \right] \text{Gamma} [1 + 4 \text{ ep}]}{\text{Gamma} [2 + 2 \text{ ep}]}, \{ \{ \text{ep} \rightarrow 0, \text{y} \rightarrow 0 \}, \{ \} \} \right] \right\},$
 $\text{MBint} \left[\frac{1}{\text{Gamma} [1 - 2 \text{ ep} - z]} \text{Gamma} [-1 - 2 \text{ ep} - z] \text{Gamma} \left[\frac{1}{2} - \text{ep} - z \right] \text{Gamma} [-z] \text{Gamma} \left[\frac{3}{2} + \text{ep} + z \right] \text{Gamma} [4 \text{ ep} + z], \{ \{ \text{ep} \rightarrow 0, \text{y} \rightarrow 0 \}, \{ z \rightarrow -\frac{5}{4} \} \} \right] \right\}$

In[13]= **MBmerge [%]**

Out[13]= $\left\{ \text{MBint} \left[\frac{\text{Gamma} \left[\frac{3}{2} - 3 \text{ ep} \right] \text{Gamma} [4 \text{ ep}] \text{Gamma} [-1 + 2 \text{ ep}] \text{Gamma} \left[\frac{1}{2} + 3 \text{ ep} \right]}{\text{Gamma} [1 + 2 \text{ ep}]} - \right.$
 $\frac{\text{Gamma} \left[\frac{1}{2} - 3 \text{ ep} \right] \text{Gamma} [2 \text{ ep}] \text{Gamma} \left[\frac{3}{2} + 3 \text{ ep} \right] \text{Gamma} [1 + 4 \text{ ep}]}{\text{Gamma} [2 + 2 \text{ ep}]}, \{ \{ \text{ep} \rightarrow 0, \text{y} \rightarrow 0 \}, \{ \} \} \right],$
 $\text{MBint} \left[\frac{1}{\text{Gamma} [1 - 2 \text{ ep} - z]} \text{Gamma} [-1 - 2 \text{ ep} - z] \text{Gamma} \left[\frac{1}{2} - \text{ep} - z \right] \text{Gamma} [-z] \text{Gamma} \left[\frac{3}{2} + \text{ep} + z \right] \text{Gamma} [4 \text{ ep} + z], \{ \{ \text{ep} \rightarrow 0, \text{y} \rightarrow 0 \}, \{ z \rightarrow -\frac{5}{4} \} \} \right] \right\}$

In[14]= **exp2 = MBexpand [%, Exp[2 ep EulerGamma], {ep, 0, 0}]**

In[15]= **MBmerge [%]**

$$\text{Out[15]= } \left\{ \text{MBint} \left[-\frac{1}{96 \text{ep}^2} \pi \left(6 - 6 \text{ep} \left(-6 + 2 \text{EulerGamma} - 3 \text{PolyGamma} \left[0, \frac{1}{2} \right] + 3 \text{PolyGamma} \left[0, \frac{3}{2} \right] \right) \right) + \right. \right. \\ \left. \left. \text{ep}^2 \left(12 \text{EulerGamma}^2 + 35 \pi^2 - 36 \text{EulerGamma} \left(2 + \text{PolyGamma} \left[0, \frac{1}{2} \right] - \text{PolyGamma} \left[0, \frac{3}{2} \right] \right) \right) + \right. \right. \\ \left. \left. 3 \left(-44 + 9 \text{PolyGamma} \left[0, \frac{1}{2} \right]^2 + 12 \text{PolyGamma} \left[0, \frac{3}{2} \right] + 9 \text{PolyGamma} \left[0, \frac{3}{2} \right]^2 - \right. \right. \right. \\ \left. \left. \left. 6 \text{PolyGamma} \left[0, \frac{1}{2} \right] \left(2 + 3 \text{PolyGamma} \left[0, \frac{3}{2} \right] \right) \right) \right) \right], \{ \{ \text{ep} \rightarrow 0, \text{y} \rightarrow 0 \}, \{ \} \} \right\}, \\ \text{MBint} \left[\frac{\text{Gamma}[-1-z] \text{Gamma}[\frac{1}{2}-z] \text{Gamma}[-z] \text{Gamma}[z] \text{Gamma}[\frac{3}{2}+z]}{\text{Gamma}[1-z]}, \right. \\ \left. \{ \{ \text{ep} \rightarrow 0, \text{y} \rightarrow 0 \}, \right. \\ \left. \left. \{ z \rightarrow -\frac{5}{4} \} \} \right\} \right]$$

(***** **Example 4b** *****)

In[16]= **F = Gamma[-1/2 + ep + z] Gamma[1 + ep + z] Gamma[3/2 - ep - z] Gamma[-z]**

$$\text{Out[16]= } \text{Gamma} \left[\frac{3}{2} - \text{ep} - z \right] \text{Gamma}[-z] \text{Gamma} \left[-\frac{1}{2} + \text{ep} + z \right] \text{Gamma}[1 + \text{ep} + z]$$

(***** **Strategy #1:**
there are no poles. Expand the integrand in epsilon. However,
the contour cannot be a straight line. *****)

(***** **Strategy #2** *****)

In[17]= **F = Gamma[-1/2 + ep + z] Gamma[1 + ep + z] Gamma[3/2 - ep - z] Gamma[-z]**

$$\text{Out[17]= } \text{Gamma} \left[\frac{3}{2} - \text{ep} - z \right] \text{Gamma}[-z] \text{Gamma} \left[-\frac{1}{2} + \text{ep} + z \right] \text{Gamma}[1 + \text{ep} + z]$$

In[18]= **Frules = MBOptimizedRules[F, ep → 0, {}, {ep}]**

MBResidues::contour : contour starts and/or ends on a pole of Gamma[1 + ep + z]

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MBResidues::contour : contour starts and/or ends on a pole of Gamma[1 + ep + z]

General::stop : Further output of MBResidues::contour will be suppressed during this calculation. >>

Out[18]= \$Aborted

(***The integral of Gamma[a+s] Gamma[b+s] Gamma[c-s] Gamma[dd-s]***)

In[19]= **Mel140[a_, b_, c_, d_] := Gamma[a + c] Gamma[a + d] Gamma[b + c] Gamma[b + d] / Gamma[a + b + c + d];**

In[20]= **Mel140[-1/2 + ep, 1 + ep, 3/2 - ep, 0]**

$$\text{Out[20]= } \frac{3 \sqrt{\pi} \text{Gamma} \left[-\frac{1}{2} + \text{ep} \right] \text{Gamma}[1 + \text{ep}]}{4 \text{Gamma}[2 + \text{ep}]}$$

- Strategy **A** in a slightly modified form

[A.V. Smirnov & V.A. Smirnov'08]

- Strategy A in a slightly modified form

[A.V. Smirnov & V.A. Smirnov'08]

Strategy B: straight contours in the beginning

Strategy A: straight contours in the end

- Strategy A in a slightly modified form

[A.V. Smirnov & V.A. Smirnov'08]

Strategy B: straight contours in the beginning

Strategy A: straight contours in the end

Set $\epsilon = 0$

- Strategy A in a slightly modified form

[A.V. Smirnov & V.A. Smirnov'08]

Strategy B: straight contours in the beginning

Strategy A: straight contours in the end

Set $\epsilon = 0$

Look for straight contours (i.e. $\text{Re} z_i$) for which gamma functions are changed in a minimal way

- Strategy A in a slightly modified form

[A.V. Smirnov & V.A. Smirnov'08]

Strategy B: straight contours in the beginning

Strategy A: straight contours in the end

Set $\epsilon = 0$

Look for straight contours (i.e. $\text{Re} z_i$) for which gamma functions are changed in a minimal way

Let $\prod \Gamma(A_i)$ with $A_i = a_i + b_i \epsilon + \sum_j c_{ij} z_j$

be the numerator of a multiple MB integral

Let $\sigma(x) = [(1 - x)_+]$ where $[. . .]$ is the integer part of a number and $x_+ = x$ for $x > 0$ and 0 otherwise.

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In other words, if $-n < x < -n + 1$ then $\sigma(x) = n$ for $n > 0$ and $\sigma(x) = 0$ for $n \leq 0$.

Let $\sigma(x) = [(1 - x)_+]$ where $[...]$ is the integer part of a number and $x_+ = x$ for $x > 0$ and 0 otherwise.

In other words, if $-n < x < -n + 1$ then $\sigma(x) = n$ for $n > 0$ and $\sigma(x) = 0$ for $n \leq 0$.

Choose contours, i.e. $\text{Re}z_i$, for which

$$\sum_i \sigma(\text{Re}A_i|_{\epsilon=0}) \equiv \sum_i \sigma\left(a_i + \sum_j c_{ij} \text{Re}z_j\right)$$

is minimal

After such a choice is done identify gamma functions which should be changed

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The second step in Strategy A is the same as in the old version:

take care of the distinguished gamma functions, i.e. take a residue and replace Γ by $\Gamma^{(1)}(A_i)$ (and, possibly, $\Gamma^{(1)}(A_i)$ by $\Gamma^{(2)}(A_i)$ etc.)

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Proceed iteratively: every residue is considered from the stretch, i.e. treated in the same way as the initial MB integral

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Proceed iteratively: every residue is considered from the stretch, i.e. treated in the same way as the initial MB integral

[MBresolve.m](#)

<http://science.sander.su>

```

ln[1]:= SetDirectory["c:/diskE/job2008/Zurich"];

(* http://www-ttp.particle.uni-karlsruhe.de/~asmirnov *)

ln[2]:= << MB/MB.m;
<< MB/MBresolve.m

MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08

MBresolve 1.0

by Alexander Smirnov

last modified 22 Oct 08

(* The integrand of the MB integral for the one-
loop propagator diagram with m1=m and m2=0 *)

ln[4]:= MB[a1_, a2_] := (-1)^(a1 + a2) / QQ^(a1 + a2 + ep - 2) / Gamma[a1] / Gamma[a2]
Gamma[2 - ep - a2] Gamma[a1 + a2 + ep - 2 + z] Gamma[2 - ep - a1 - z]
Gamma[-z] / Gamma[4 - 2 ep - a1 - a2 - z] mm^z / QQ^z;

(* Notation: mm=m^2, QQ=-q^2;
I Pi^(d/2) is pulled out *)

(* The diagram with a1=1 and a2=1 *)

ln[5]:= P1 = MB[1, 1]

Out[5]= 
$$\frac{mm^z QQ^{-ep-z} \Gamma[1-ep] \Gamma[1-ep-z] \Gamma[-z] \Gamma[ep+z]}{\Gamma[2-2ep-z]}$$


ln[6]:= MBresolve[P1, ep]

CREATING RESIDUES LIST.....0.2812 seconds
EVALUATING RESIDUES.....0.1875 seconds

Out[6]= 
$$\left\{ \text{MBint} \left[ \frac{QQ^{-ep} \Gamma[1-ep]^2 \Gamma[ep]}{\Gamma[2-2ep]}, \{ \{ep \rightarrow 0\}, \{\} \} \right], \text{MBint} \left[ \frac{mm^z QQ^{-ep-z} \Gamma[1-ep] \Gamma[1-ep-z] \Gamma[-z] \Gamma[ep+z]}{\Gamma[2-2ep-z]}, \{ \{ep \rightarrow 0\}, \{z \rightarrow 0.511912\} \} \right] \right\}$$


ln[7]:= Box1[a1_, a2_, a3_, a4_] :=
(S^{2-a1-a2-a3-a4-ep-z} T^z Gamma[a1 + a2 + a3 + a4 - 2 + ep + z] Gamma[a2 + z] Gamma[a4 + z]
Gamma[2 - a1 - a2 - a4 - ep - z] Gamma[2 - a2 - a3 - a4 - ep - z] Gamma[-z]) /
(Gamma[a1] Gamma[a2] Gamma[a3] Gamma[a4] Gamma[4 - a1 - a2 - a3 - a4 - 2 ep]);

ln[8]:= Box1[1, 1, 1, 1]

Out[8]= 
$$\frac{S^{-2-ep-z} T^z \Gamma[-1-ep-z]^2 \Gamma[-z] \Gamma[1+z]^2 \Gamma[2+ep+z]}{\Gamma[-2ep]}$$


ln[9]:= B1 = % /. {S -> 1, T -> x}

Out[9]= 
$$\frac{x^z \Gamma[-1-ep-z]^2 \Gamma[-z] \Gamma[1+z]^2 \Gamma[2+ep+z]}{\Gamma[-2ep]}$$


```

In[10]:= **MBresolve**[B1, ep]

CREATING RESIDUES LIST.....0.4688 seconds
EVALUATING RESIDUES.....0.3125 seconds

Out[10]=
$$\left\{ \text{MBint} \left[-\frac{\text{EulerGamma} \Gamma[-\text{ep}]^2 \Gamma[1+\text{ep}]}{x \Gamma[-2\text{ep}]} + \frac{\Gamma[-\text{ep}]^2 \Gamma[1+\text{ep}] \text{Log}[x]}{x \Gamma[-2\text{ep}]} - \frac{2 \Gamma[-\text{ep}]^2 \Gamma[1+\text{ep}] \text{PolyGamma}[0, -\text{ep}]}{x \Gamma[-2\text{ep}]} + \frac{\Gamma[-\text{ep}]^2 \Gamma[1+\text{ep}] \text{PolyGamma}[0, 1+\text{ep}]}{x \Gamma[-2\text{ep}]} \right], \{\{\text{ep} \rightarrow 0\}, \{\}\} \right\},$$

$$\left\{ \text{MBint} \left[\frac{x^2 \Gamma[-1-\text{ep}-z]^2 \Gamma[-z] \Gamma[1+z]^2 \Gamma[2+\text{ep}+z]}{\Gamma[-2\text{ep}]} \right], \{\{\text{ep} \rightarrow 0\}, \{z \rightarrow -1.81305\}\} \right\} \right\}$$

In[11]:= **NPMB**[a1_, a2_, a3_, a4_, a5_, a6_] :=
$$\frac{(-1)^{(a1+a2+a3+a4+a5+a6)} \Gamma[a1] \Gamma[a2] \Gamma[a3] \Gamma[a4] \Gamma[a5] \Gamma[a6]}{\Gamma[2-\text{ep}-a3-a5] \Gamma[2-\text{ep}-a4-a6] \Gamma[4-2\text{ep}-a3-a4-a5-a6] \Gamma[6-3\text{ep}-a1-a2-a3-a4-a5-a6]} \frac{\Gamma[a1+a2+a3+a4+a5+a6+2\text{ep}-4+z1+z2] \Gamma[-z1] \Gamma[-z2]}{\Gamma[2-\text{ep}-a1-a2-z1] \Gamma[a4+z2] \Gamma[a1+z1+z2]} \frac{\Gamma[4-2\text{ep}-a1-a3-a4-a5-a6-z2] \Gamma[4-2\text{ep}-a1-a2-a4-a5-a6-z1-z2] \Gamma[a5+z2] \Gamma[4-2\text{ep}-a1-a2-a3-a4-a5-z1-z2]}{\Gamma[4-2\text{ep}-a1-a2-a4-a6-z1] \Gamma[4-2\text{ep}-a1-a2-a3-a5-z1]} ;$$

In[12]:= **V2** = **NPMB**[1, 1, 1, 1, 1, 1]

Out[12]=
$$\frac{\left(\Gamma[-\text{ep}]^2 \Gamma[-\text{ep}-z1] \Gamma[-z1] \Gamma[-1-2\text{ep}-z2] \Gamma[-1-2\text{ep}-z1-z2]^2 \Gamma[-z2] \Gamma[1+z2]^2 \Gamma[1+z1+z2] \Gamma[2+2\text{ep}+z1+z2] \right)}{\left(\Gamma[-3\text{ep}] \Gamma[-2\text{ep}] \Gamma[-2\text{ep}-z1]^2 \right)}$$

In[13]:= **MBresolve**[V2, ep]

CREATING RESIDUES LIST.....0.625 seconds

EVALUATING RESIDUES.....0.25 seconds

Out[13]= $\left\{ \text{MBint} \left[\frac{\Gamma[-2 \text{ep}]^4 \Gamma[-\text{ep}]^2 \Gamma[1+2 \text{ep}]^2}{\Gamma[-4 \text{ep}]^2}, \{\{\text{ep} \rightarrow 0\}, \{\}\}\right], \right.$
 $\text{MBint} \left[\left(\frac{\Gamma[-2 \text{ep}] \Gamma[-\text{ep}]^2 \Gamma[1+2 \text{ep}] \Gamma[-\text{ep}-z1] \Gamma[-z1]^3 \Gamma[1+z1] \Gamma[-2 \text{ep}+z1]}{\Gamma[-3 \text{ep}] \Gamma[-2 \text{ep}-z1]^2} \right), \{\{\text{ep} \rightarrow 0\}, \{z1 \rightarrow -0.859981\}\} \right],$
 $\text{MBint} \left[-\frac{1}{\Gamma[-3 \text{ep}]} \text{EulerGamma} \Gamma[-\text{ep}]^2 \Gamma[-1-2 \text{ep}-z2] \Gamma[-z2] \right.$
 $\Gamma[1+\text{ep}+z2] \Gamma[1+2 \text{ep}+z2] - \frac{1}{\Gamma[-3 \text{ep}]} \Gamma[-\text{ep}]^2 \Gamma[-1-2 \text{ep}-z2]$
 $\Gamma[-z2] \Gamma[1+\text{ep}+z2] \Gamma[1+2 \text{ep}+z2] \text{PolyGamma}[0, -2 \text{ep}] - \frac{1}{\Gamma[-3 \text{ep}]}$
 $2 \Gamma[-\text{ep}]^2 \Gamma[-1-2 \text{ep}-z2] \Gamma[-z2] \Gamma[1+\text{ep}+z2] \Gamma[1+2 \text{ep}+z2]$
 $\text{PolyGamma}[0, 1+z2] + \frac{1}{\Gamma[-3 \text{ep}]} \Gamma[-\text{ep}]^2 \Gamma[-1-2 \text{ep}-z2]$
 $\Gamma[-z2] \Gamma[1+\text{ep}+z2] \Gamma[1+2 \text{ep}+z2] \text{PolyGamma}[0, 1+\text{ep}+z2] +$
 $\frac{1}{\Gamma[-3 \text{ep}]} \Gamma[-\text{ep}]^2 \Gamma[-1-2 \text{ep}-z2] \Gamma[-z2] \Gamma[1+\text{ep}+z2]$
 $\left. \Gamma[1+2 \text{ep}+z2] \text{PolyGamma}[0, 1+2 \text{ep}+z2], \{\{\text{ep} \rightarrow 0\}, \{z2 \rightarrow -0.859981\}\} \right],$
 $\text{MBint} \left[\left(\frac{\Gamma[-\text{ep}]^2 \Gamma[-\text{ep}-z1] \Gamma[-z1] \Gamma[-1-2 \text{ep}-z2] \Gamma[-1-2 \text{ep}-z1-z2]^2}{\Gamma[-z2] \Gamma[1+z2]^2 \Gamma[1+z1+z2] \Gamma[2+2 \text{ep}+z1+z2]} \right) / \right.$
 $\left. \left(\Gamma[-3 \text{ep}] \Gamma[-2 \text{ep}] \Gamma[-2 \text{ep}-z1]^2 \right), \{\{\text{ep} \rightarrow 0\}, \{z1 \rightarrow -0.72274, z2 \rightarrow -0.274294\}\} \right\}$

In[14]:= **Simplify**[%]

Out[14]= $\left\{ \text{MBint} \left[\frac{\Gamma[-2 \text{ep}]^4 \Gamma[-\text{ep}]^2 \Gamma[1+2 \text{ep}]^2}{\Gamma[-4 \text{ep}]^2}, \{\{\text{ep} \rightarrow 0\}, \{\}\}\right], \right.$
 $\text{MBint} \left[\left(\frac{\Gamma[-2 \text{ep}] \Gamma[-\text{ep}]^2 \Gamma[1+2 \text{ep}] \Gamma[-\text{ep}-z1] \Gamma[-z1]^3 \Gamma[1+z1] \Gamma[-2 \text{ep}+z1]}{\Gamma[-3 \text{ep}] \Gamma[-2 \text{ep}-z1]^2} \right), \{\{\text{ep} \rightarrow 0\}, \{z1 \rightarrow -0.859981\}\} \right],$
 $\text{MBint} \left[\frac{1}{\Gamma[-3 \text{ep}]} \Gamma[-\text{ep}]^2 \Gamma[-1-2 \text{ep}-z2] \Gamma[-z2] \Gamma[1+\text{ep}+z2] \right.$
 $\Gamma[1+2 \text{ep}+z2] (-\text{EulerGamma} - \text{PolyGamma}[0, -2 \text{ep}] - 2 \text{PolyGamma}[0, 1+z2] +$
 $\text{PolyGamma}[0, 1+\text{ep}+z2] + \text{PolyGamma}[0, 1+2 \text{ep}+z2]), \{\{\text{ep} \rightarrow 0\}, \{z2 \rightarrow -0.859981\}\} \right],$
 $\text{MBint} \left[\left(\frac{\Gamma[-\text{ep}]^2 \Gamma[-\text{ep}-z1] \Gamma[-z1] \Gamma[-1-2 \text{ep}-z2] \Gamma[-1-2 \text{ep}-z1-z2]^2}{\Gamma[-z2] \Gamma[1+z2]^2 \Gamma[1+z1+z2] \Gamma[2+2 \text{ep}+z1+z2]} \right) / \right.$
 $\left. \left(\Gamma[-3 \text{ep}] \Gamma[-2 \text{ep}] \Gamma[-2 \text{ep}-z1]^2 \right), \{\{\text{ep} \rightarrow 0\}, \{z1 \rightarrow -0.72274, z2 \rightarrow -0.274294\}\} \right\}$

(* **Example 4a** *)

In[15]:= $F = \text{Gamma}\left[\frac{3}{2} + \text{ep} + z\right] \text{Gamma}\left[-1 - 2 \text{ep} - z\right]$
 $\text{Gamma}\left[4 \text{ep} + z\right] \text{Gamma}\left[-z\right] \text{Gamma}\left[\frac{1}{2} - \text{ep} - z\right] / \text{Gamma}\left[1 - 2 \text{ep} - z\right]$

Out[15]= $\frac{1}{\text{Gamma}\left[1 - 2 \text{ep} - z\right]}$
 $\text{Gamma}\left[-1 - 2 \text{ep} - z\right] \text{Gamma}\left[\frac{1}{2} - \text{ep} - z\right] \text{Gamma}\left[-z\right] \text{Gamma}\left[\frac{3}{2} + \text{ep} + z\right] \text{Gamma}\left[4 \text{ep} + z\right]$

In[16]:= **MBresolve[F, ep]**

CREATING RESIDUES LIST.....0.3281 seconds
 EVALUATING RESIDUES.....0.3125 seconds

Out[16]= $\left\{ \text{MBint}\left[\frac{\text{Gamma}\left[\frac{3}{2} - 3 \text{ep}\right] \text{Gamma}\left[4 \text{ep}\right] \text{Gamma}\left[-1 + 2 \text{ep}\right] \text{Gamma}\left[\frac{1}{2} + 3 \text{ep}\right]}{\text{Gamma}\left[1 + 2 \text{ep}\right]}, \{\{\text{ep} \rightarrow 0\}, \{\}\}\right], \right.$
 $\text{MBint}\left[-\frac{\text{Gamma}\left[\frac{1}{2} - 3 \text{ep}\right] \text{Gamma}\left[2 \text{ep}\right] \text{Gamma}\left[\frac{3}{2} + 3 \text{ep}\right] \text{Gamma}\left[1 + 4 \text{ep}\right]}{\text{Gamma}\left[2 + 2 \text{ep}\right]}, \{\{\text{ep} \rightarrow 0\}, \{\}\}\right],$
 $\left. \text{MBint}\left[\left(\text{Gamma}\left[-1 - 2 \text{ep} - z\right] \text{Gamma}\left[\frac{1}{2} - \text{ep} - z\right] \text{Gamma}\left[-z\right] \text{Gamma}\left[\frac{3}{2} + \text{ep} + z\right] \text{Gamma}\left[4 \text{ep} + z\right]\right) / \right.$
 $\left. \text{Gamma}\left[1 - 2 \text{ep} - z\right], \{\{\text{ep} \rightarrow 0\}, \{z \rightarrow -1.48302\}\}\right] \right\}$

In[17]:= **MBmerge[%]**

Out[17]= $\left\{ \text{MBint}\left[\frac{\text{Gamma}\left[\frac{3}{2} - 3 \text{ep}\right] \text{Gamma}\left[4 \text{ep}\right] \text{Gamma}\left[-1 + 2 \text{ep}\right] \text{Gamma}\left[\frac{1}{2} + 3 \text{ep}\right]}{\text{Gamma}\left[1 + 2 \text{ep}\right]} - \right.$
 $\left. \frac{\text{Gamma}\left[\frac{1}{2} - 3 \text{ep}\right] \text{Gamma}\left[2 \text{ep}\right] \text{Gamma}\left[\frac{3}{2} + 3 \text{ep}\right] \text{Gamma}\left[1 + 4 \text{ep}\right]}{\text{Gamma}\left[2 + 2 \text{ep}\right]}, \{\{\text{ep} \rightarrow 0\}, \{\}\}\right],$
 $\left. \text{MBint}\left[\left(\text{Gamma}\left[-1 - 2 \text{ep} - z\right] \text{Gamma}\left[\frac{1}{2} - \text{ep} - z\right] \text{Gamma}\left[-z\right] \text{Gamma}\left[\frac{3}{2} + \text{ep} + z\right] \text{Gamma}\left[4 \text{ep} + z\right]\right) / \right.$
 $\left. \text{Gamma}\left[1 - 2 \text{ep} - z\right], \{\{\text{ep} \rightarrow 0\}, \{z \rightarrow -1.48302\}\}\right] \right\}$

(* **Example 4b** *)

In[18]:= $F = \text{Gamma}\left[-1/2 + \text{ep} + z\right] \text{Gamma}\left[1 + \text{ep} + z\right] \text{Gamma}\left[3/2 - \text{ep} - z\right] \text{Gamma}\left[-z\right]$

Out[18]= $\text{Gamma}\left[\frac{3}{2} - \text{ep} - z\right] \text{Gamma}\left[-z\right] \text{Gamma}\left[-\frac{1}{2} + \text{ep} + z\right] \text{Gamma}\left[1 + \text{ep} + z\right]$

In[19]:= **MBresolve[F, ep]**

CREATING RESIDUES LIST.....0.2188 seconds
 EVALUATING RESIDUES.....0.1562 seconds

Out[19]= $\left\{ \text{MBint}\left[\frac{1}{2} \sqrt{\pi} \text{Gamma}\left[-\frac{1}{2} + \text{ep}\right], \{\{\text{ep} \rightarrow 0\}, \{\}\}\right], \text{MBint}\left[\right.$
 $\left. \text{Gamma}\left[\frac{3}{2} - \text{ep} - z\right] \text{Gamma}\left[-z\right] \text{Gamma}\left[-\frac{1}{2} + \text{ep} + z\right] \text{Gamma}\left[1 + \text{ep} + z\right], \{\{\text{ep} \rightarrow 0\}, \{z \rightarrow -0.294497\}\}\right] \right\}$

Part 2.

Evaluating by Mellin-Barnes representation:
advanced examples and various additional tools

The massless box diagram with two legs on shell,
 $p_3^2 = p_4^2 = 0$, and two legs off shell, $p_1^2, p_2^2 \neq 0$

$$\begin{aligned}
 B_{1100} &= i\pi^{d/2} \frac{\Gamma(a + \epsilon - 2)}{\prod \Gamma(a_l)} \\
 &\times \int_0^\infty \cdots \int_0^\infty \left(\prod_{l=1}^4 \alpha_l^{a_l - 1} d\alpha_l \right) \delta \left(\sum_{l=1}^4 \alpha_l - 1 \right) \\
 &\times (-s\alpha_1\alpha_3 - t\alpha_2\alpha_4 - p_1^2\alpha_1\alpha_2 - p_2^2\alpha_2\alpha_3 - i0)^{2-a-\epsilon}
 \end{aligned}$$

Apply

$$\frac{1}{(X_1 + \dots + X_n)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{(2\pi i)^{n-1}} \int_{-i\infty}^{+i\infty} \dots \int_{-i\infty}^{+i\infty} dz_2 \dots dz_n \prod_{i=2}^n X_i^{z_i} \\ \times X_1^{-\lambda - z_2 - \dots - z_n} \Gamma(\lambda + z_2 + \dots + z_n) \prod_{i=2}^n \Gamma(-z_i)$$

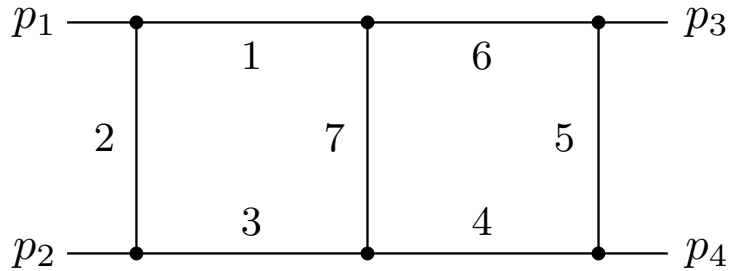
Separate terms with p_1^2 and p_2^2 , turn to new variables by

$$\alpha_1 = \eta_1 \xi_1, \quad \alpha_2 = \eta_1 (1 - \xi_1), \quad \alpha_3 = \eta_2 \xi_2, \quad \alpha_4 = \eta_2 (1 - \xi_2)$$

and evaluate integrals over parameters to obtain a three fold MB representation

$$\begin{aligned}
B_{1100} &= \frac{i\pi^{d/2}}{\Gamma(4 - 2\epsilon - a) \prod \Gamma(a_l) (-s)^{a+\epsilon-2}} \\
&\times \frac{1}{(2\pi i)^3} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} \int_{-i\infty}^{+i\infty} dz_2 dz_3 dz_4 \frac{(-p_1^2)^{z_2} (-p_2^2)^{z_3} (-t)^{z_4}}{(-s)^{z_2+z_3+z_4}} \\
&\times \Gamma(a + \epsilon - 2 + z_2 + z_3 + z_4) \Gamma(a_2 + z_2 + z_3 + z_4) \Gamma(a_4 + z_4) \\
&\times \Gamma(2 - \epsilon - a_{234} - z_3 - z_4) \Gamma(2 - \epsilon - a_{124} - z_2 - z_4) \\
&\times \Gamma(-z_2) \Gamma(-z_3) \Gamma(-z_4) .
\end{aligned}$$

Double box with irreducible numerator $(k + p_1 + p_2 + p_4)^2$



$$\begin{aligned}
 B_2(s, t; a_1, \dots, a_8, \epsilon) &= \int \int \frac{\mathbf{d}^d k \mathbf{d}^d l}{(k^2)^{a_1} [(k + p_1)^2]^{a_2} [(k + p_1 + p_2)^2]^{a_3} \\
 &\quad \times \frac{[(k + p_1 + p_2 + p_4)^2]^{-a_8}}{[(l + p_1 + p_2)^2]^{a_4} [(l + p_1 + p_2 + p_4)^2]^{a_5} (l^2)^{a_6} [(k - l)^2]^{a_7}}
 \end{aligned}$$

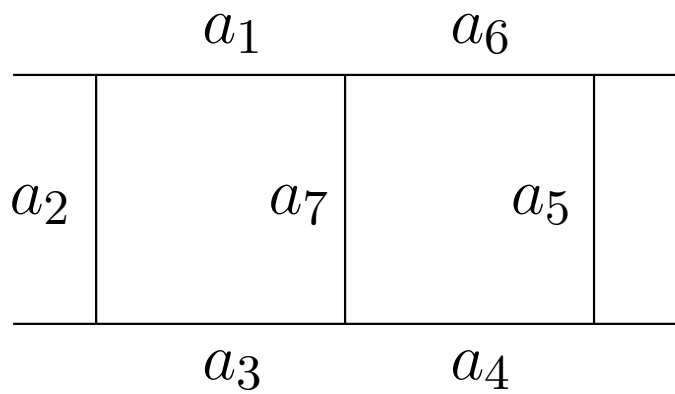
$$B_2(s, t; a_1, \dots, a_8, \epsilon) = \int \frac{\mathbf{d}^d k [(k + p_1 + p_2 + p_4)^2]^{-a_8}}{(k^2)^{a_1} [(k + p_1)^2]^{a_2} [(k + p_1 + p_2)^2]^{a_3}} \\ \times B_{1100}(s, (k + p_1 + p_2 + p_4)^2, k^2, (k + p_1 + p_2)^2; a_6, a_7, a_4, a_5, d)$$

After using the threefold MB representation for B_{1100} and changing the order of integration we obtain an on-shell box integral with indices shifted by z -variables. Apply then the onefold representation for the this box.

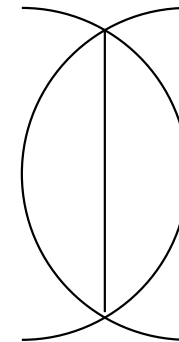
Derivation of MB representation loop-by-loop.

AMBRE

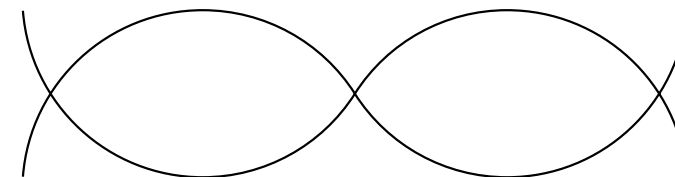
[J. Gluza, K. Kajda & T. Riemann'07]



$$a_1, a_3, a_4, a_6 \rightarrow 0$$



$$a_2, a_5, a_7 \rightarrow 0$$



```

In[1]:= SetDirectory["c:/diskE/job2008/Zurich"];

In[2]:= << MB/MB.m;
        << MB/MBresolve.m

MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08

MBresolve 1.0

by Alexander Smirnov

last modified 22 Oct 08

(* The end of derivation of MB
representation for the planar massless double box diagram at
p1^2=p2^2=p3^2=p4^2=0.

Notation: S=-s=-(p1+p2)^2, T=-t=-(p1+p3)^2 *)

(* This is MB representation for the box *)

In[4]:= Box1[a1_, a2_, a3_, a4_] :=
  (S^{2-a1-a2-a3-a4-ep-z1} T^{z1} Gamma[a1 + a2 + a3 + a4 - 2 + ep + z1] Gamma[a2 + z1] Gamma[a4 + z1]
  Gamma[2 - a1 - a2 - a4 - ep - z1] Gamma[2 - a2 - a3 - a4 - ep - z1] Gamma[-z1]) /
  (Gamma[a1] Gamma[a2] Gamma[a3] Gamma[a4] Gamma[4 - a1 - a2 - a3 - a4 - 2 ep]);

In[5]:= (1 / S^{a4 + a5 + a6 + a7 + ep - 2 + z2 + z3 + z4} /
  (Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7] Gamma[4 - a4 - a5 - a6 - a7 - 2 ep]
  Gamma[a4 + a5 + a6 + a7 - 2 + ep + z2 + z3 + z4] Gamma[a7 + z2 + z3 + z4] Gamma[a5 + z4]
  Gamma[2 - a4 - a5 - a7 - ep - z3 - z4] Gamma[2 - a5 - a6 - a7 - ep - z2 - z4]
  Gamma[-z2] Gamma[-z3] Gamma[-z4]) Box1[a1 - z2, a2, a3 - z3, a8 - z4];

In[6]:= Simplify[%]

Out[6]:= (S^{4-a1-a2-a3-a4-a5-a6-a7-a8-2 ep-z1} T^{z1} Gamma[-z1] Gamma[a2 + z1] Gamma[-z2] Gamma[-z3]
  Gamma[a8 + z1 - z4] Gamma[2 - a5 - a6 - a7 - ep - z2 - z4] Gamma[2 - a4 - a5 - a7 - ep - z3 - z4]
  Gamma[-2 + a1 + a2 + a3 + a8 + ep + z1 - z2 - z3 - z4] Gamma[-z4] Gamma[a5 + z4]
  Gamma[2 - a1 - a2 - a8 - ep - z1 + z2 + z4] Gamma[2 - a2 - a3 - a8 - ep - z1 + z3 + z4]
  Gamma[a7 + z2 + z3 + z4] Gamma[-2 + a4 + a5 + a6 + a7 + ep + z2 + z3 + z4]) /
  (Gamma[a2] Gamma[a4] Gamma[a5] Gamma[a6] Gamma[a7] Gamma[4 - a4 - a5 - a6 - a7 - 2 ep]
  Gamma[a1 - z2] Gamma[a3 - z3] Gamma[a8 - z4] Gamma[4 - a1 - a2 - a3 - a8 - 2 ep + z2 + z3 + z4])

(* Changing variables *)

In[7]:= % /. {z2 -> z2 - z4, z3 -> z3 - z4};

In[8]:= % /. {z3 -> z3 + z1};

In[9]:= % /. z4 -> z4 + z1;

In[10]:= % /. z2 -> z2 + z1;

```

In[11]:= **Simplify**[% /. z4 → z4 + z2 + z3]

Out[11]=
$$\left(S^{4-a_1-a_2-a_3-a_4-a_5-a_6-a_7-a_8-2\text{ep}-z_1} T^{z_1} \Gamma[-z_1] \Gamma[a_2+z_1] \Gamma[2-a_5-a_6-a_7-\text{ep}-z_1-z_2] \right. \\ \Gamma[2-a_1-a_2-a_8-\text{ep}+z_2] \Gamma[2-a_4-a_5-a_7-\text{ep}-z_1-z_3] \\ \Gamma[2-a_2-a_3-a_8-\text{ep}+z_3] \Gamma[a_7+z_1-z_4] \Gamma[-2+a_4+a_5+a_6+a_7+\text{ep}+z_1-z_4] \\ \Gamma[a_8-z_2-z_3-z_4] \Gamma[-z_1-z_2-z_3-z_4] \Gamma[-2+a_1+a_2+a_3+a_8+\text{ep}+z_4] \\ \left. \Gamma[z_2+z_4] \Gamma[z_3+z_4] \Gamma[a_5+z_1+z_2+z_3+z_4] \right) / \\ (\Gamma[a_2] \Gamma[a_4] \Gamma[a_5] \Gamma[a_6] \Gamma[a_7] \\ \Gamma[4-a_4-a_5-a_6-a_7-2\text{ep}] \Gamma[4-a_1-a_2-a_3-a_8-2\text{ep}+z_1-z_4] \\ \Gamma[a_8-z_1-z_2-z_3-z_4] \Gamma[a_3+z_2+z_4] \Gamma[a_1+z_3+z_4])$$

In[12]:= **B2**[a1_, a2_, a3_, a4_, a5_, a6_, a7_, a8_] :=

$$\left(S^{4-a_1-a_2-a_3-a_4-a_5-a_6-a_7-a_8-2\text{ep}-z_1} T^{z_1} \Gamma[-z_1] \Gamma[a_2+z_1] \Gamma[2-a_5-a_6-a_7-\text{ep}-z_1-z_2] \right. \\ \Gamma[2-a_1-a_2-a_8-\text{ep}+z_2] \Gamma[2-a_4-a_5-a_7-\text{ep}-z_1-z_3] \\ \Gamma[2-a_2-a_3-a_8-\text{ep}+z_3] \Gamma[a_7+z_1-z_4] \Gamma[-2+a_4+a_5+a_6+a_7+\text{ep}+z_1-z_4] \\ \Gamma[a_8-z_2-z_3-z_4] \Gamma[-z_1-z_2-z_3-z_4] \Gamma[-2+a_1+a_2+a_3+a_8+\text{ep}+z_4] \\ \left. \Gamma[z_2+z_4] \Gamma[z_3+z_4] \Gamma[a_5+z_1+z_2+z_3+z_4] \right) / \\ (\Gamma[a_2] \Gamma[a_4] \Gamma[a_5] \Gamma[a_6] \Gamma[a_7] \Gamma[4-a_4-a_5-a_6-a_7-2\text{ep}] \\ \Gamma[4-a_1-a_2-a_3-a_8-2\text{ep}+z_1-z_4] \\ \Gamma[a_8-z_1-z_2-z_3-z_4] \Gamma[a_3+z_2+z_4] \Gamma[a_1+z_3+z_4])$$

(* the function in the one-loop integration formula *)

In[13]:= **G**[a1_, a2_] := $\Gamma[a_1+a_2+\text{ep}-2] \Gamma[2-\text{ep}-a_1]$

$\Gamma[2-\text{ep}-a_2] / \Gamma[a_1] / \Gamma[a_2] / \Gamma[4-2\text{ep}-a_1-a_2];$

(* a vertical check: shrink vertical lines,
a2,a5,a7→0

*)

In[14]:= **B2**[a1, a2, a3, a4, a5, a6, a7, 0]

Out[14]=
$$\left(S^{4-a_1-a_2-a_3-a_4-a_5-a_6-a_7-2\text{ep}-z_1} T^{z_1} \Gamma[-z_1] \Gamma[a_2+z_1] \Gamma[2-a_5-a_6-a_7-\text{ep}-z_1-z_2] \right. \\ \Gamma[2-a_1-a_2-\text{ep}+z_2] \Gamma[2-a_4-a_5-a_7-\text{ep}-z_1-z_3] \Gamma[2-a_2-a_3-\text{ep}+z_3] \\ \Gamma[a_7+z_1-z_4] \Gamma[-2+a_4+a_5+a_6+a_7+\text{ep}+z_1-z_4] \Gamma[-z_2-z_3-z_4] \\ \Gamma[-2+a_1+a_2+a_3+\text{ep}+z_4] \Gamma[z_2+z_4] \Gamma[z_3+z_4] \Gamma[a_5+z_1+z_2+z_3+z_4] \left. \right) / \\ (\Gamma[a_2] \Gamma[a_4] \Gamma[a_5] \Gamma[a_6] \Gamma[a_7] \Gamma[4-a_4-a_5-a_6-a_7-2\text{ep}] \\ \Gamma[4-a_1-a_2-a_3-2\text{ep}+z_1-z_4] \Gamma[a_3+z_2+z_4] \Gamma[a_1+z_3+z_4])$$

(* a2→0; $\Gamma[-z_1] \Gamma[a_2+z_1]$ *)

In[15]:= **-Residue**[B2[a1, a2, a3, a4, a5, a6, a7, 0], {z1, 0}]

Out[15]=
$$\left(S^{4-a_1-a_2-a_3-a_4-a_5-a_6-a_7-2\text{ep}} \Gamma[2-a_5-a_6-a_7-\text{ep}-z_2] \right. \\ \Gamma[2-a_1-a_2-\text{ep}+z_2] \Gamma[2-a_4-a_5-a_7-\text{ep}-z_3] \Gamma[2-a_2-a_3-\text{ep}+z_3] \\ \Gamma[a_7-z_4] \Gamma[-2+a_4+a_5+a_6+a_7+\text{ep}-z_4] \Gamma[-z_2-z_3-z_4] \\ \Gamma[-2+a_1+a_2+a_3+\text{ep}+z_4] \Gamma[z_2+z_4] \Gamma[z_3+z_4] \Gamma[a_5+z_2+z_3+z_4] \left. \right) / \\ (\Gamma[a_4] \Gamma[a_5] \Gamma[a_6] \Gamma[a_7] \Gamma[4-a_4-a_5-a_6-a_7-2\text{ep}] \\ \Gamma[4-a_1-a_2-a_3-2\text{ep}-z_4] \Gamma[a_3+z_2+z_4] \Gamma[a_1+z_3+z_4])$$

In[16]:= % /. a2 → 0

Out[16]=
$$\left(S^{4-a_1-a_3-a_4-a_5-a_6-a_7-2\text{ep}} \Gamma[2-a_5-a_6-a_7-\text{ep}-z_2] \right. \\ \Gamma[2-a_1-\text{ep}+z_2] \Gamma[2-a_4-a_5-a_7-\text{ep}-z_3] \Gamma[2-a_3-\text{ep}+z_3] \\ \Gamma[a_7-z_4] \Gamma[-2+a_4+a_5+a_6+a_7+\text{ep}-z_4] \Gamma[-z_2-z_3-z_4] \\ \Gamma[-2+a_1+a_3+\text{ep}+z_4] \Gamma[z_2+z_4] \Gamma[z_3+z_4] \Gamma[a_5+z_2+z_3+z_4] \left. \right) / \\ (\Gamma[a_4] \Gamma[a_5] \Gamma[a_6] \Gamma[a_7] \Gamma[4-a_4-a_5-a_6-a_7-2\text{ep}] \\ \Gamma[4-a_1-a_3-2\text{ep}-z_4] \Gamma[a_3+z_2+z_4] \Gamma[a_1+z_3+z_4])$$

(* a5→0; $\Gamma[a_5+z_2+z_3+z_4] \Gamma[-z_2-z_3-z_4]$ *)

In[17]:= **-Residue**[% , {z4, -z2 - z3}]

Out[17]=
$$\left(S^{4-a_1-a_3-a_4-a_5-a_6-a_7-2ep} \Gamma[2-a_5-a_6-a_7-ep-z_2] \Gamma[-z_2] \Gamma[2-a_1-ep+z_2] \right. \\ \Gamma[2-a_4-a_5-a_7-ep-z_3] \Gamma[-2+a_1+a_3+ep-z_2-z_3] \Gamma[-z_3] \\ \left. \Gamma[2-a_3-ep+z_3] \Gamma[a_7+z_2+z_3] \Gamma[-2+a_4+a_5+a_6+a_7+ep+z_2+z_3] \right) / \\ (\Gamma[a_4] \Gamma[a_6] \Gamma[a_7] \Gamma[4-a_4-a_5-a_6-a_7-2ep] \\ \Gamma[a_1-z_2] \Gamma[a_3-z_3] \Gamma[4-a_1-a_3-2ep+z_2+z_3])$$

In[18]:= % /. a5 → 0

Out[18]=
$$\left(S^{4-a_1-a_3-a_4-a_6-a_7-2ep} \Gamma[2-a_6-a_7-ep-z_2] \Gamma[-z_2] \Gamma[2-a_1-ep+z_2] \right. \\ \Gamma[2-a_4-a_7-ep-z_3] \Gamma[-2+a_1+a_3+ep-z_2-z_3] \Gamma[-z_3] \\ \left. \Gamma[2-a_3-ep+z_3] \Gamma[a_7+z_2+z_3] \Gamma[-2+a_4+a_6+a_7+ep+z_2+z_3] \right) / \\ (\Gamma[a_4] \Gamma[a_6] \Gamma[a_7] \Gamma[4-a_4-a_6-a_7-2ep] \\ \Gamma[a_1-z_2] \Gamma[a_3-z_3] \Gamma[4-a_1-a_3-2ep+z_2+z_3])$$

In[19]:= **-Residue**[% , {z3, 0}]

Out[19]=
$$\left(S^{4-a_1-a_3-a_4-a_6-a_7-2ep} \Gamma[2-a_3-ep] \Gamma[2-a_4-a_7-ep] \right. \\ \Gamma[2-a_6-a_7-ep-z_2] \Gamma[-2+a_1+a_3+ep-z_2] \Gamma[-z_2] \\ \left. \Gamma[a_7+z_2] \Gamma[2-a_1-ep+z_2] \Gamma[-2+a_4+a_6+a_7+ep+z_2] \right) / \\ (\Gamma[a_3] \Gamma[a_4] \Gamma[a_6] \Gamma[a_7] \Gamma[4-a_4-a_6-a_7-2ep] \\ \Gamma[a_1-z_2] \Gamma[4-a_1-a_3-2ep+z_2])$$

In[20]:= **-Residue**[% , {z2, 0}]

Out[20]=
$$\left(S^{4-a_1-a_3-a_4-a_6-a_7-2ep} \Gamma[2-a_1-ep] \Gamma[2-a_3-ep] \Gamma[2-a_4-a_7-ep] \right. \\ \Gamma[2-a_6-a_7-ep] \Gamma[-2+a_1+a_3+ep] \Gamma[-2+a_4+a_6+a_7+ep] \left. \right) / \\ (\Gamma[a_1] \Gamma[a_3] \Gamma[a_4] \Gamma[a_6] \Gamma[4-a_1-a_3-2ep] \Gamma[4-a_4-a_6-a_7-2ep])$$

In[21]:= % /. a7 → 0

Out[21]=
$$\left(S^{4-a_1-a_3-a_4-a_6-2ep} \Gamma[2-a_1-ep] \Gamma[2-a_3-ep] \Gamma[2-a_4-ep] \right. \\ \Gamma[2-a_6-ep] \Gamma[-2+a_1+a_3+ep] \Gamma[-2+a_4+a_6+ep] \left. \right) / \\ (\Gamma[a_1] \Gamma[a_3] \Gamma[a_4] \Gamma[a_6] \Gamma[4-a_1-a_3-2ep] \Gamma[4-a_4-a_6-2ep])$$

In[22]:= **G**[a1, a3] **G**[a4, a6] / %

Out[22]= $S^{-4+a_1+a_3+a_4+a_6+2ep}$

(* a7→0; **Gamma**[a7+z2+z3] **Gamma**[-z2]**Gamma**[-z3] *****)

(* a horizontal check: shrink horizontal lines,
a1,a3,a4,a6→0

*****)

In[23]:= **B2**[a1, a2, a3, a4, a5, a6, a7, 0]

Out[23]=
$$\left(S^{4-a_1-a_2-a_3-a_4-a_5-a_6-a_7-2ep-z_1} T^{z_1} \Gamma[-z_1] \Gamma[a_2+z_1] \Gamma[2-a_5-a_6-a_7-ep-z_1-z_2] \right. \\ \Gamma[2-a_1-a_2-ep+z_2] \Gamma[2-a_4-a_5-a_7-ep-z_1-z_3] \Gamma[2-a_2-a_3-ep+z_3] \\ \Gamma[a_7+z_1-z_4] \Gamma[-2+a_4+a_5+a_6+a_7+ep+z_1-z_4] \Gamma[-z_2-z_3-z_4] \\ \Gamma[-2+a_1+a_2+a_3+ep+z_4] \Gamma[z_2+z_4] \Gamma[z_3+z_4] \Gamma[a_5+z_1+z_2+z_3+z_4] \left. \right) / \\ (\Gamma[a_2] \Gamma[a_4] \Gamma[a_5] \Gamma[a_6] \Gamma[a_7] \Gamma[4-a_4-a_5-a_6-a_7-2ep] \\ \Gamma[4-a_1-a_2-a_3-2ep+z_1-z_4] \Gamma[a_3+z_2+z_4] \Gamma[a_1+z_3+z_4])$$

Gamma[-2+a4+a5+a6+a7+ep+z1-z4] **Gamma**[2-a4-a5-a7-ep-z1-z3] **Gamma**[z3+z4]

-2+a4+a5+a6+a7+ep+z1-z4+2-a4-a5-a7-ep-z1-z3+z3+z4

a6

In[24]:= **Residue[B2[a1, a2, a3, a4, a5, a6, a7, 0], {z3, -z4}]**

Out[24]=
$$\left(S^{4-a_1-a_2-a_3-a_4-a_5-a_6-a_7-2\text{ep}-z_1} T^{z_1} \Gamma[-z_1] \Gamma[a_2+z_1] \right. \\ \Gamma[2-a_5-a_6-a_7-\text{ep}-z_1-z_2] \Gamma[-z_2] \Gamma[2-a_1-a_2-\text{ep}+z_2] \Gamma[a_5+z_1+z_2] \\ \Gamma[2-a_2-a_3-\text{ep}-z_4] \Gamma[a_7+z_1-z_4] \Gamma[-2+a_4+a_5+a_6+a_7+\text{ep}+z_1-z_4] \\ \left. \Gamma[-2+a_1+a_2+a_3+\text{ep}+z_4] \Gamma[2-a_4-a_5-a_7-\text{ep}-z_1+z_4] \Gamma[z_2+z_4] \right) / \\ (\Gamma[a_1] \Gamma[a_2] \Gamma[a_4] \Gamma[a_5] \Gamma[a_6] \Gamma[a_7] \\ \Gamma[4-a_4-a_5-a_6-a_7-2\text{ep}] \Gamma[4-a_1-a_2-a_3-2\text{ep}+z_1-z_4] \Gamma[a_3+z_2+z_4])$$

In[25]:= **-Residue[%, {z4, -2+a4+a5+a6+a7+ep+z1}]**

Out[25]=
$$\left(S^{4-a_1-a_2-a_3-a_4-a_5-a_6-a_7-2\text{ep}-z_1} T^{z_1} \Gamma[2-a_4-a_5-a_6-\text{ep}] \right. \\ \Gamma[4-a_2-a_3-a_4-a_5-a_6-a_7-2\text{ep}-z_1] \Gamma[-z_1] \\ \Gamma[a_2+z_1] \Gamma[-4+a_1+a_2+a_3+a_4+a_5+a_6+a_7+2\text{ep}+z_1] \\ \Gamma[2-a_5-a_6-a_7-\text{ep}-z_1-z_2] \Gamma[-z_2] \Gamma[2-a_1-a_2-\text{ep}+z_2] \\ \left. \Gamma[a_5+z_1+z_2] \Gamma[-2+a_4+a_5+a_6+a_7+\text{ep}+z_1+z_2] \right) / \\ (\Gamma[a_1] \Gamma[a_2] \Gamma[a_4] \Gamma[a_5] \Gamma[a_7] \\ \Gamma[6-a_1-a_2-a_3-a_4-a_5-a_6-a_7-3\text{ep}] \Gamma[4-a_4-a_5-a_6-a_7-2\text{ep}] \\ \Gamma[-2+a_3+a_4+a_5+a_6+a_7+\text{ep}+z_1+z_2])$$

In[26]:= **% /. a6 -> 0**

Out[26]=
$$\left(S^{4-a_1-a_2-a_3-a_4-a_5-a_7-2\text{ep}-z_1} T^{z_1} \Gamma[2-a_4-a_5-\text{ep}] \right. \\ \Gamma[4-a_2-a_3-a_4-a_5-a_7-2\text{ep}-z_1] \Gamma[-z_1] \Gamma[a_2+z_1] \\ \Gamma[-4+a_1+a_2+a_3+a_4+a_5+a_7+2\text{ep}+z_1] \Gamma[2-a_5-a_7-\text{ep}-z_1-z_2] \Gamma[-z_2] \\ \left. \Gamma[2-a_1-a_2-\text{ep}+z_2] \Gamma[a_5+z_1+z_2] \Gamma[-2+a_4+a_5+a_7+\text{ep}+z_1+z_2] \right) / \\ (\Gamma[a_1] \Gamma[a_2] \Gamma[a_4] \Gamma[a_5] \Gamma[a_7] \Gamma[6-a_1-a_2-a_3-a_4-a_5-a_7-3\text{ep}] \\ \Gamma[4-a_4-a_5-a_7-2\text{ep}] \Gamma[-2+a_3+a_4+a_5+a_7+\text{ep}+z_1+z_2])$$

(* a1,a3,a4->0 *)

(* let a4->0 *)

Gamma[-2+a4+a5+a7+ep+z1+z2] Gamma[2-a5-a7-ep-z1-z2]

-2+a4+a5+a7+ep+z1+z2+2-a5-a7-ep-z1-z2

a4

In[27]:= **-Residue[%, {z2, 2-a5-a7-ep-z1}]**

Out[27]=
$$\left(S^{4-a_1-a_2-a_3-a_4-a_5-a_7-2\text{ep}-z_1} T^{z_1} \Gamma[2-a_4-a_5-\text{ep}] \Gamma[2-a_7-\text{ep}] \right. \\ \Gamma[4-a_1-a_2-a_5-a_7-2\text{ep}-z_1] \Gamma[4-a_2-a_3-a_4-a_5-a_7-2\text{ep}-z_1] \Gamma[-z_1] \\ \Gamma[a_2+z_1] \Gamma[-2+a_5+a_7+\text{ep}+z_1] \Gamma[-4+a_1+a_2+a_3+a_4+a_5+a_7+2\text{ep}+z_1] \left. \right) / \\ (\Gamma[a_1] \Gamma[a_2] \Gamma[a_3+a_4] \Gamma[a_5] \Gamma[a_7] \\ \Gamma[6-a_1-a_2-a_3-a_4-a_5-a_7-3\text{ep}] \Gamma[4-a_4-a_5-a_7-2\text{ep}])$$

In[28]:= **% /. a4 -> 0**

Out[28]=
$$\left(S^{4-a_1-a_2-a_3-a_5-a_7-2\text{ep}-z_1} T^{z_1} \Gamma[2-a_5-\text{ep}] \Gamma[2-a_7-\text{ep}] \right. \\ \Gamma[4-a_1-a_2-a_5-a_7-2\text{ep}-z_1] \Gamma[4-a_2-a_3-a_5-a_7-2\text{ep}-z_1] \Gamma[-z_1] \\ \Gamma[a_2+z_1] \Gamma[-2+a_5+a_7+\text{ep}+z_1] \Gamma[-4+a_1+a_2+a_3+a_5+a_7+2\text{ep}+z_1] \left. \right) / \\ (\Gamma[a_1] \Gamma[a_2] \Gamma[a_3] \Gamma[a_5] \Gamma[a_7] \\ \Gamma[6-a_1-a_2-a_3-a_5-a_7-3\text{ep}] \Gamma[4-a_5-a_7-2\text{ep}])$$

(* a1,a3->0 *)

(* let a1->0 *)

Gamma[-4+a1+a2+a3+a5+a7+2ep+z1] Gamma[4-a2-a3-a5-a7-2ep-z1]

$-4 + a1 + a2 + a3 + a5 + a7 + 2 \text{ ep} + z1 + 4 - a2 - a3 - a5 - a7 - 2 \text{ ep} - z1$

a1

In[29]:= **-Residue** [%, {z1, 4 - a2 - a3 - a5 - a7 - 2 ep}]

Out[29]= $\left(S^{-a1} T^{4-a2-a3-a5-a7-2 \text{ ep}} \text{Gamma}[-a1 + a3] \text{Gamma}[4 - a3 - a5 - a7 - 2 \text{ ep}] \text{Gamma}[2 - a2 - a3 - \text{ep}] \right. \\ \left. \text{Gamma}[2 - a5 - \text{ep}] \text{Gamma}[2 - a7 - \text{ep}] \text{Gamma}[-4 + a2 + a3 + a5 + a7 + 2 \text{ ep}] \right) / \left(\text{Gamma}[a2] \right. \\ \left. \text{Gamma}[a3] \text{Gamma}[a5] \text{Gamma}[a7] \text{Gamma}[6 - a1 - a2 - a3 - a5 - a7 - 3 \text{ ep}] \text{Gamma}[4 - a5 - a7 - 2 \text{ ep}] \right)$

In[30]:= % /. a1 → 0

Out[30]= $\left(T^{4-a2-a3-a5-a7-2 \text{ ep}} \text{Gamma}[4 - a3 - a5 - a7 - 2 \text{ ep}] \text{Gamma}[2 - a2 - a3 - \text{ep}] \right. \\ \left. \text{Gamma}[2 - a5 - \text{ep}] \text{Gamma}[2 - a7 - \text{ep}] \text{Gamma}[-4 + a2 + a3 + a5 + a7 + 2 \text{ ep}] \right) / \\ \left(\text{Gamma}[a2] \text{Gamma}[a5] \text{Gamma}[a7] \text{Gamma}[6 - a2 - a3 - a5 - a7 - 3 \text{ ep}] \text{Gamma}[4 - a5 - a7 - 2 \text{ ep}] \right)$

In[31]:= % /. a3 → 0

Out[31]= $\left(T^{4-a2-a5-a7-2 \text{ ep}} \text{Gamma}[2 - a2 - \text{ep}] \right. \\ \left. \text{Gamma}[2 - a5 - \text{ep}] \text{Gamma}[2 - a7 - \text{ep}] \text{Gamma}[-4 + a2 + a5 + a7 + 2 \text{ ep}] \right) / \\ \left(\text{Gamma}[a2] \text{Gamma}[a5] \text{Gamma}[a7] \text{Gamma}[6 - a2 - a5 - a7 - 3 \text{ ep}] \right)$

In[32]:= **G**[a2, a7] **G**[a2 + a7 + ep - 2, a5] / %

Out[32]= $T^{-4+a2+a5+a7+2 \text{ ep}}$

(* In addition to the usual factor (I Pi^(d/2))^2,
let us pull out the factor

$s^{4-a1-a2-a3-a4-a5-a6-a7-a8-2 \text{ ep}}$.

Let us turn to the variable $x=T/S = t/s$ *)

In[33]:= **K2**[a1_, a2_, a3_, a4_, a5_, a6_, a7_, a8_] :=

$\left(x^{z1} \text{Gamma}[-z1] \text{Gamma}[a2 + z1] \text{Gamma}[2 - a5 - a6 - a7 - \text{ep} - z1 - z2] \right. \\ \left. \text{Gamma}[2 - a1 - a2 - a8 - \text{ep} + z2] \text{Gamma}[2 - a4 - a5 - a7 - \text{ep} - z1 - z3] \right. \\ \left. \text{Gamma}[2 - a2 - a3 - a8 - \text{ep} + z3] \text{Gamma}[a7 + z1 - z4] \text{Gamma}[-2 + a4 + a5 + a6 + a7 + \text{ep} + z1 - z4] \right. \\ \left. \text{Gamma}[a8 - z2 - z3 - z4] \text{Gamma}[-z1 - z2 - z3 - z4] \text{Gamma}[-2 + a1 + a2 + a3 + a8 + \text{ep} + z4] \right. \\ \left. \text{Gamma}[z2 + z4] \text{Gamma}[z3 + z4] \text{Gamma}[a5 + z1 + z2 + z3 + z4] \right) / \\ \left(\text{Gamma}[a2] \text{Gamma}[a4] \text{Gamma}[a5] \text{Gamma}[a6] \text{Gamma}[a7] \text{Gamma}[4 - a4 - a5 - a6 - a7 - 2 \text{ ep}] \right. \\ \left. \text{Gamma}[4 - a1 - a2 - a3 - a8 - 2 \text{ ep} + z1 - z4] \right. \\ \left. \text{Gamma}[a8 - z1 - z2 - z3 - z4] \text{Gamma}[a3 + z2 + z4] \text{Gamma}[a1 + z3 + z4] \right)$

(* The 2-box with the powers of the propagators equal to one *)

In[34]:= **B2** = x **K2**[1, 1, 1, 1, 1, 1, 1, 0]

Out[34]= $\left(x^{1+z1} \text{Gamma}[-z1] \text{Gamma}[1 + z1] \text{Gamma}[-1 - \text{ep} - z1 - z2] \text{Gamma}[-\text{ep} + z2] \text{Gamma}[-1 - \text{ep} - z1 - z3] \right. \\ \left. \text{Gamma}[-\text{ep} + z3] \text{Gamma}[1 + z1 - z4] \text{Gamma}[2 + \text{ep} + z1 - z4] \text{Gamma}[-z2 - z3 - z4] \right. \\ \left. \text{Gamma}[1 + \text{ep} + z4] \text{Gamma}[z2 + z4] \text{Gamma}[z3 + z4] \text{Gamma}[1 + z1 + z2 + z3 + z4] \right) / \\ \left(\text{Gamma}[-2 \text{ ep}] \text{Gamma}[1 - 2 \text{ ep} + z1 - z4] \text{Gamma}[1 + z2 + z4] \text{Gamma}[1 + z3 + z4] \right)$

(* auxiliary functions *)

In[35]:= **SortByDimension**[l_List] := **Sort**[l, **Length**[#1[[2, 2]]] > **Length**[#2[[2, 2]]] &];
CoeffEps[X_, n_] := (X /. X[[1]] → **Simplify**[**Coefficient**[X[[1]], ep, n]]);
MBDimension[int_MBint] := **Length**[int[[2, 2]]];

```
In[38]:= B2rules = MBOptimizedRules[B2, ep → 0, {}, {ep}]
```

```
Out[38]= {{ep → - $\frac{9}{16}$ }, {z1 → - $\frac{1}{2}$ , z2 → - $\frac{5}{16}$ , z3 → - $\frac{3}{8}$ , z4 →  $\frac{7}{16}$ }}
```

```
In[39]:= B2cont = MBcontinue[B2, ep → 0, B2rules];
```

```
Level 1
```

```
Taking -residue in z2 = -1 - ep - z1
```

```
Taking +residue in z2 = ep
```

```
Taking -residue in z3 = -1 - ep - z1
```

```
Taking +residue in z3 = ep
```

```
Level 2
```

```
Integral {1}
```

```
Taking -residue in z3 = -1 - ep - z1
```

```
Taking +residue in z4 = 1 + ep + z1
```

```
Integral {2}
```

```
Taking -residue in z1 = -1 - 2 ep
```

```
Taking -residue in z3 = -1 - ep - z1
```

```
Taking -residue in z4 = -ep - z3
```

```
Integral {3}
```

```
Taking +residue in z4 = 1 + ep + z1
```

```
Integral {4}
```

```
Taking -residue in z1 = -1 - 2 ep
```

```
Taking -residue in z2 = -1 - ep - z1
```

```
Taking +residue in z2 = ep
```

```
Taking -residue in z4 = -ep - z2
```

```
Level 3
```

```
Integral {1, 1}
```

```
Taking +residue in z4 = 1 + 2 ep + z1
```

```
Taking +residue in z4 = 1 + ep + z1
```

```
Integral {1, 2}
```

```
Integral {2, 1}
```

```
Taking -residue in z4 = -2 ep
```

```
Taking -residue in z4 = -ep - z3
```

```
Integral {2, 2}
```

```
Taking +residue in z4 = 1 + ep + z1
```

```
Integral {2, 3}
```

```
Integral {3, 1}
```

```
Integral {4, 1}
```

```
Taking -residue in z4 = -2 ep
```

```

Taking -residue in z4 = -ep - z2
Integral {4, 2}
Taking +residue in z4 = 1 + ep + z1
Integral {4, 3}
Taking -residue in z1 = -1 - 2 ep
Taking -residue in z4 = -2 ep
Integral {4, 4}
Level 4
Integral {1, 1, 1}
Integral {1, 1, 2}
Integral {2, 1, 1}
Taking +residue in z3 = 3 ep
Taking +residue in z3 = 2 ep
Integral {2, 1, 2}
Integral {2, 2, 1}
Integral {4, 1, 1}
Taking +residue in z2 = 3 ep
Taking +residue in z2 = 2 ep
Integral {4, 1, 2}
Integral {4, 2, 1}
Integral {4, 3, 1}
Taking -residue in z4 = -4 ep
...no contribution
Taking -residue in z4 = -2 ep
Integral {4, 3, 2}
Level 5
Integral {2, 1, 1, 1}
Integral {2, 1, 1, 2}
Integral {4, 1, 1, 1}
Integral {4, 1, 1, 2}
Integral {4, 3, 1, 1}
30 integral(s) found

In[40]:= B2select = MBpreselect[B2cont, {ep, 0, 0}]
In[41]:= B2exp = Simplify[MBexpand[B2select, E^(2 EulerGamma ep), {ep, 0, 0}]]

```

In[42]:= **B2expS = SortByDimension[MBmerge[B2exp]]**

Out[42]=
$$\left\{ \text{MBint} \left[\frac{\Gamma[-z3] \Gamma[z3] \Gamma[1-z4] \Gamma[-z3-z4] \Gamma[z4] \Gamma[z3+z4]^2}{\Gamma[1+z3+z4]}, \left\{ \{ep \rightarrow 0\}, \left\{ z3 \rightarrow -\frac{3}{8}, z4 \rightarrow \frac{7}{16} \right\} \right\} \right], \right.$$

$$\text{MBint} \left[\frac{\Gamma[-z2] \Gamma[z2] \Gamma[1-z4] \Gamma[-z2-z4] \Gamma[z4] \Gamma[z2+z4]^2}{\Gamma[1+z2+z4]}, \left\{ \{ep \rightarrow 0\}, \left\{ z2 \rightarrow -\frac{5}{16}, z4 \rightarrow \frac{7}{16} \right\} \right\} \right],$$

$$\text{MBint} \left[2 x^{1+z1} \Gamma[-1-z1] \Gamma[-z1] \Gamma[1+z1]^2 \Gamma[-1-z1-z3] \Gamma[-z3] \Gamma[z3] \Gamma[2+z1+z3], \left\{ \{ep \rightarrow 0\}, \left\{ z1 \rightarrow -\frac{1}{2}, z3 \rightarrow -\frac{3}{8} \right\} \right\} \right],$$

$$\text{MBint} \left[2 x^{1+z1} \Gamma[-1-z1] \Gamma[-z1] \Gamma[1+z1]^2 \Gamma[-1-z1-z2] \Gamma[-z2] \Gamma[z2] \Gamma[2+z1+z2], \left\{ \{ep \rightarrow 0\}, \left\{ z1 \rightarrow -\frac{1}{2}, z2 \rightarrow -\frac{5}{16} \right\} \right\} \right],$$

$$\text{MBint} \left[-\frac{1}{2 ep \Gamma[1+z4]} \Gamma[1-z4] \Gamma[-z4] \Gamma[z4]^3 (1 + 4 ep \text{EulerGamma} + ep \text{PolyGamma}[0, 1-z4] + 4 ep \text{PolyGamma}[0, z4] - ep \text{PolyGamma}[0, 1+z4]), \left\{ \{ep \rightarrow 0\}, \left\{ z4 \rightarrow \frac{7}{16} \right\} \right\} \right],$$

$$\text{MBint} \left[\frac{1}{12 \Gamma[1+z3]} \Gamma[-z3]^2 \Gamma[z3] \left(24 \Gamma[z3]^2 + \frac{1}{ep^2} \Gamma[1+z3]^2 (6 + 12 ep \text{EulerGamma} + 12 ep^2 \text{EulerGamma}^2 + 4 ep^2 \pi^2 - 12 ep \text{Log}[x] - 24 ep^2 \text{EulerGamma} \text{Log}[x] + 12 ep^2 \text{Log}[x]^2 + 3 ep^2 \text{PolyGamma}[0, -z3]^2 + 6 ep (1 + 2 ep \text{EulerGamma} - 2 ep \text{Log}[x]) \text{PolyGamma}[0, z3] + 3 ep^2 \text{PolyGamma}[0, z3]^2 + 6 ep \text{PolyGamma}[0, -z3] (1 + 2 ep \text{EulerGamma} - 2 ep \text{Log}[x] + ep \text{PolyGamma}[0, z3]) + 3 ep^2 \text{PolyGamma}[1, -z3] - 21 ep^2 \text{PolyGamma}[1, z3]) \right), \left\{ \{ep \rightarrow 0\}, \left\{ z3 \rightarrow -\frac{3}{8} \right\} \right\} \right],$$

$$\text{MBint} \left[\frac{1}{12 \Gamma[1+z2]} \Gamma[-z2]^2 \Gamma[z2] \left(24 \Gamma[z2]^2 + \frac{1}{ep^2} \Gamma[1+z2]^2 (6 + 12 ep \text{EulerGamma} + 12 ep^2 \text{EulerGamma}^2 + 4 ep^2 \pi^2 - 12 ep \text{Log}[x] - 24 ep^2 \text{EulerGamma} \text{Log}[x] + 12 ep^2 \text{Log}[x]^2 + 3 ep^2 \text{PolyGamma}[0, -z2]^2 + 6 ep (1 + 2 ep \text{EulerGamma} - 2 ep \text{Log}[x]) \text{PolyGamma}[0, z2] + 3 ep^2 \text{PolyGamma}[0, z2]^2 + 6 ep \text{PolyGamma}[0, -z2] (1 + 2 ep \text{EulerGamma} - 2 ep \text{Log}[x] + ep \text{PolyGamma}[0, z2]) + 3 ep^2 \text{PolyGamma}[1, -z2] - 21 ep^2 \text{PolyGamma}[1, z2]) \right), \left\{ \{ep \rightarrow 0\}, \left\{ z2 \rightarrow -\frac{5}{16} \right\} \right\} \right],$$

$$\text{MBint} \left[\frac{1}{ep} 4 x^{1+z1} \Gamma[-1-z1]^2 \Gamma[-z1] \Gamma[1+z1]^2 (3 ep \Gamma[1+z1] + \Gamma[2+z1] (-1 + 4 ep \text{PolyGamma}[0, -1-z1] - 2 ep \text{PolyGamma}[0, 1+z1] - 2 ep \text{PolyGamma}[0, 2+z1])), \left\{ \{ep \rightarrow 0\}, \left\{ z1 \rightarrow -\frac{1}{2} \right\} \right\} \right],$$

$$\text{MBint} \left[-\frac{1}{18 ep^4} (-72 + 42 ep^2 \pi^2 + 20 ep^4 \pi^4 + 6 ep^2 (-6 + 17 ep^2 \pi^2) \text{Log}[x]^2 - 12 ep^3 \text{Log}[x]^3 + 24 ep^4 \text{Log}[x]^4 - 213 ep^3 \text{PolyGamma}[2, 1] + 3 ep \text{Log}[x] (30 - 31 ep^2 \pi^2 + 94 ep^3 \text{PolyGamma}[2, 1])), \{\{ep \rightarrow 0\}, \{\}\} \right]$$

In[43]:= **Length[B2expS]**

Out[43]= 9

In[44]:= **MBDimension** /@ **B2expS**

Out[44]= {2, 2, 2, 2, 1, 1, 1, 1, 0}

In[45]:= **B2expS**[[1]]

Out[45]=
$$\text{MBint} \left[\frac{\text{Gamma}[-z3] \text{Gamma}[z3] \text{Gamma}[1-z4] \text{Gamma}[-z3-z4] \text{Gamma}[z4] \text{Gamma}[z3+z4]^2}{\text{Gamma}[1+z3+z4]}, \right.$$

$$\left. \left\{ \{ep \rightarrow 0\}, \left\{ z3 \rightarrow -\frac{3}{8}, z4 \rightarrow \frac{7}{16} \right\} \right\} \right]$$

In[46]:= **Barnes2**[**B2expS**[[1]], **z4**]

Out[46]=
$$\text{MBint} \left[\frac{1}{6} \pi^2 \text{Gamma}[-z3]^2 \text{Gamma}[z3] \text{Gamma}[1+z3] - \right.$$

$$\left. \text{Gamma}[-z3]^2 \text{Gamma}[z3] \text{Gamma}[1+z3] \text{PolyGamma}[1, 1+z3], \left\{ \{ep \rightarrow 0\}, \left\{ z3 \rightarrow -\frac{3}{8} \right\} \right\} \right]$$

In[47]:= **res01** = $\frac{17}{4}$ **Zeta**[4]

Out[47]= $\frac{17 \pi^4}{360}$

In[48]:= % // **N**

Out[48]= 4.59987

In[49]:= **NIntegrate**[**Barnes2**[**B2expS**[[1]], **z4**][[1]] / (2 Pi) /. {**z3** $\rightarrow -\frac{3}{8} + I * y1$ },
{y1, -Infinity, Infinity}]

Out[49]= 4.59987 + 0. i

In[50]:= **B2expS**[[2]]

Out[50]=
$$\text{MBint} \left[\frac{\text{Gamma}[-z2] \text{Gamma}[z2] \text{Gamma}[1-z4] \text{Gamma}[-z2-z4] \text{Gamma}[z4] \text{Gamma}[z2+z4]^2}{\text{Gamma}[1+z2+z4]}, \right.$$

$$\left. \left\{ \{ep \rightarrow 0\}, \left\{ z2 \rightarrow -\frac{5}{16}, z4 \rightarrow \frac{7}{16} \right\} \right\} \right]$$

In[51]:= **Barnes2**[**B2expS**[[2]], **z4**]

Out[51]=
$$\text{MBint} \left[\frac{1}{6} \pi^2 \text{Gamma}[-z2]^2 \text{Gamma}[z2] \text{Gamma}[1+z2] - \right.$$

$$\left. \text{Gamma}[-z2]^2 \text{Gamma}[z2] \text{Gamma}[1+z2] \text{PolyGamma}[1, 1+z2], \left\{ \{ep \rightarrow 0\}, \left\{ z2 \rightarrow -\frac{5}{16} \right\} \right\} \right]$$

In[52]:= **res02** = $\frac{17}{4}$ **Zeta**[4]

Out[52]= $\frac{17 \pi^4}{360}$

In[53]:= % // **N**

Out[53]= 4.59987

In[54]:= **NIntegrate**[**Barnes2**[**B2expS**[[2]], z4] [[1]] / (2 Pi) /. {z2 → - $\frac{5}{16}$ + I * y1},
{y1, -Infinity, Infinity}]

Out[54]= 4.59987 - 2.88547 × 10⁻¹³ i

In[55]:= **B2expS**[[3]]

Out[55]= **MBint**[2 x^{1+z1} Gamma[-1 - z1] Gamma[-z1] Gamma[1 + z1]² Gamma[-1 - z1 - z3]
Gamma[-z3] Gamma[z3] Gamma[2 + z1 + z3], {{ep → 0}, {z1 → - $\frac{1}{2}$, z3 → - $\frac{3}{8}$ }}]

In[56]:= **Barnes1**[**B2expS**[[3]], z3]

Out[56]= **MBint**[-2 x^{1+z1} Gamma[-1 - z1]² Gamma[-z1] Gamma[1 + z1]³ +
2 EulerGamma x^{1+z1} Gamma[-1 - z1]² Gamma[-z1] Gamma[1 + z1]² Gamma[2 + z1] +
2 x^{1+z1} Gamma[-1 - z1]² Gamma[-z1] Gamma[1 + z1]² Gamma[2 + z1] PolyGamma[0, 2 + z1],
{{ep → 0}, {z1 → - $\frac{1}{2}$ }}]

In[57]:= **B2expS**[[4]]

Out[57]= **MBint**[2 x^{1+z1} Gamma[-1 - z1] Gamma[-z1] Gamma[1 + z1]² Gamma[-1 - z1 - z2]
Gamma[-z2] Gamma[z2] Gamma[2 + z1 + z2], {{ep → 0}, {z1 → - $\frac{1}{2}$, z2 → - $\frac{5}{16}$ }}]

In[58]:= **Barnes1**[**B2expS**[[4]], z2]

Out[58]= **MBint**[-2 x^{1+z1} Gamma[-1 - z1]² Gamma[-z1] Gamma[1 + z1]³ +
2 EulerGamma x^{1+z1} Gamma[-1 - z1]² Gamma[-z1] Gamma[1 + z1]² Gamma[2 + z1] +
2 x^{1+z1} Gamma[-1 - z1]² Gamma[-z1] Gamma[1 + z1]² Gamma[2 + z1] PolyGamma[0, 2 + z1],
{{ep → 0}, {z1 → - $\frac{1}{2}$ }}]

In[59]:= **B2expS**[[5]]

Out[59]= **MBint**[- $\frac{1}{2 \text{ ep Gamma}[1 + z4]}$
Gamma[1 - z4] Gamma[-z4] Gamma[z4]³ (1 + 4 ep EulerGamma + ep PolyGamma[0, 1 - z4] +
4 ep PolyGamma[0, z4] - ep PolyGamma[0, 1 + z4]), {{ep → 0}, {z4 → $\frac{7}{16}$ }}]

In[60]:= **CoeffEps**[**B2expS**[[5]], -1]

Out[60]= **MBint**[- $\frac{\text{Gamma}[1 - z4] \text{Gamma}[-z4] \text{Gamma}[z4]^3}{2 \text{Gamma}[1 + z4]}$, {{ep → 0}, {z4 → $\frac{7}{16}$ }}]

In[61]:= **CoeffEps**[**B2expS**[[5]], -1] /. {Gamma[1 - z4] → -z4 Gamma[-z4], Gamma[1 + z4] → z4 Gamma[z4]}

Out[61]= **MBint**[$\frac{1}{2}$ Gamma[-z4]² Gamma[z4]², {{ep → 0}, {z4 → $\frac{7}{16}$ }}]

In[62]:= **res11 = Barnes1**[% , z4] [[1]]

Out[62]= - $\frac{1}{2}$ PolyGamma[2, 1]

In[63]:= **CoeffEps**[**B2expS**[[5]], 0]

Out[63]=
$$\text{MBint} \left[-\frac{1}{2 \text{Gamma}[1+z4]} \text{Gamma}[1-z4] \text{Gamma}[-z4] \text{Gamma}[z4]^3 \right. \\ \left. (4 \text{EulerGamma} + \text{PolyGamma}[0, 1-z4] + 4 \text{PolyGamma}[0, z4] - \text{PolyGamma}[0, 1+z4]), \right. \\ \left. \left\{ \{ep \rightarrow 0\}, \left\{ z4 \rightarrow \frac{7}{16} \right\} \right\} \right]$$

In[64]:= **res03** = $-\frac{11}{2} \text{Zeta}[4]$

Out[64]= $-\frac{11 \pi^4}{180}$

In[65]:= **% // N**

Out[65]= -5.95278

In[66]:= **NIntegrate**[**CoeffEps**[**B2expS**[[5]], 0][[1]] / (2 Pi) /. {z4 \rightarrow $\frac{7}{16} + I * y1$ },
{y1, -Infinity, Infinity}]

Out[66]= -5.95278 + 0. i

In[67]:= **B2expS**[[6]]

Out[67]=
$$\text{MBint} \left[\frac{1}{12 \text{Gamma}[1+z3]} \text{Gamma}[-z3]^2 \text{Gamma}[z3] \right. \\ \left(24 \text{Gamma}[z3]^2 + \frac{1}{ep^2} \text{Gamma}[1+z3]^2 (6 + 12 ep \text{EulerGamma} + 12 ep^2 \text{EulerGamma}^2 + 4 ep^2 \pi^2 - \right. \\ \left. 12 ep \text{Log}[x] - 24 ep^2 \text{EulerGamma} \text{Log}[x] + 12 ep^2 \text{Log}[x]^2 + 3 ep^2 \text{PolyGamma}[0, -z3]^2 + \right. \\ \left. 6 ep (1 + 2 ep \text{EulerGamma} - 2 ep \text{Log}[x]) \text{PolyGamma}[0, z3] + 3 ep^2 \text{PolyGamma}[0, z3]^2 + \right. \\ \left. 6 ep \text{PolyGamma}[0, -z3] (1 + 2 ep \text{EulerGamma} - 2 ep \text{Log}[x] + ep \text{PolyGamma}[0, z3]) + \right. \\ \left. \left. 3 ep^2 \text{PolyGamma}[1, -z3] - 21 ep^2 \text{PolyGamma}[1, z3] \right) \right], \left\{ \{ep \rightarrow 0\}, \left\{ z3 \rightarrow -\frac{3}{8} \right\} \right\} \right]$$

In[68]:= **B2expS**[[7]]

Out[68]=
$$\text{MBint} \left[\frac{1}{12 \text{Gamma}[1+z2]} \text{Gamma}[-z2]^2 \text{Gamma}[z2] \right. \\ \left(24 \text{Gamma}[z2]^2 + \frac{1}{ep^2} \text{Gamma}[1+z2]^2 (6 + 12 ep \text{EulerGamma} + 12 ep^2 \text{EulerGamma}^2 + 4 ep^2 \pi^2 - \right. \\ \left. 12 ep \text{Log}[x] - 24 ep^2 \text{EulerGamma} \text{Log}[x] + 12 ep^2 \text{Log}[x]^2 + 3 ep^2 \text{PolyGamma}[0, -z2]^2 + \right. \\ \left. 6 ep (1 + 2 ep \text{EulerGamma} - 2 ep \text{Log}[x]) \text{PolyGamma}[0, z2] + 3 ep^2 \text{PolyGamma}[0, z2]^2 + \right. \\ \left. 6 ep \text{PolyGamma}[0, -z2] (1 + 2 ep \text{EulerGamma} - 2 ep \text{Log}[x] + ep \text{PolyGamma}[0, z2]) + \right. \\ \left. \left. 3 ep^2 \text{PolyGamma}[1, -z2] - 21 ep^2 \text{PolyGamma}[1, z2] \right) \right], \left\{ \{ep \rightarrow 0\}, \left\{ z2 \rightarrow -\frac{5}{16} \right\} \right\} \right]$$

In[69]:= **F67** =

MBint [**Simplify** [**B2expS** [[6]] [[1]] /. **z3** → **z2**] + **B2expS** [[7]] [[1]]], { **ep** → 0 }, { **z2** → - $\frac{1}{2}$ }]]

Out[69]= **MBint** [$\frac{1}{6 \Gamma[1+z2]} \Gamma[-z2]^2 \Gamma[z2]$
 $\left(24 \Gamma[z2]^2 + \frac{1}{ep^2} \Gamma[1+z2]^2 (6 + 12 ep \text{EulerGamma} + 12 ep^2 \text{EulerGamma}^2 + 4 ep^2 \pi^2 - \right.$
 $12 ep \text{Log}[x] - 24 ep^2 \text{EulerGamma} \text{Log}[x] + 12 ep^2 \text{Log}[x]^2 + 3 ep^2 \text{PolyGamma}[0, -z2]^2 +$
 $6 ep (1 + 2 ep \text{EulerGamma} - 2 ep \text{Log}[x]) \text{PolyGamma}[0, z2] + 3 ep^2 \text{PolyGamma}[0, z2]^2 +$
 $6 ep \text{PolyGamma}[0, -z2] (1 + 2 ep \text{EulerGamma} - 2 ep \text{Log}[x] + ep \text{PolyGamma}[0, z2]) +$
 $\left. 3 ep^2 \text{PolyGamma}[1, -z2] - 21 ep^2 \text{PolyGamma}[1, z2] \right) \right], \{ \{ep \rightarrow 0\}, \{z2 \rightarrow -\frac{1}{2}\} \}$

In[70]:= **res04** = $-\frac{\pi^2}{6 ep^2} + \frac{1}{ep} \left(\frac{\pi^2}{3} \text{Log}[x] + \text{Zeta}[3] \right) + \left(-\frac{\pi^2}{3} \text{Log}[x]^2 - 2 \text{Zeta}[3] \text{Log}[x] + \frac{\pi^4}{9} \right);$

In[71]:= **res04** /. { **ep** → 0.3, **x** → 0.1 }

Out[71]= -40.6045

In[72]:= **NIntegrate** [**F67** [[1]] / (2 Pi) /. { **ep** → 0.3, **x** → 0.1, **z2** → - $\frac{1}{2} + I * y1$ },
{ **y1**, -**Infinity**, **Infinity** }]

Out[72]= -40.6045 + 0. i

In[73]:= **B2expS** [[8]]

Out[73]= **MBint** [$\frac{1}{ep} 4 x^{1+z1} \Gamma[-1-z1]^2 \Gamma[-z1] \Gamma[1+z1]^2$
 $(3 ep \Gamma[1+z1] + \Gamma[2+z1] (-1 + 4 ep \text{PolyGamma}[0, -1-z1] -$
 $2 ep \text{PolyGamma}[0, 1+z1] - 2 ep \text{PolyGamma}[0, 2+z1])) \right], \{ \{ep \rightarrow 0\}, \{z1 \rightarrow -\frac{1}{2}\} \}$

In[74]:= **B2expS** [[9]] [[1]]

Out[74]= $-\frac{1}{18 ep^4} (-72 + 42 ep^2 \pi^2 + 20 ep^4 \pi^4 + 6 ep^2 (-6 + 17 ep^2 \pi^2) \text{Log}[x]^2 - 12 ep^3 \text{Log}[x]^3 + 24 ep^4 \text{Log}[x]^4 -$
 $213 ep^3 \text{PolyGamma}[2, 1] + 3 ep \text{Log}[x] (30 - 31 ep^2 \pi^2 + 94 ep^3 \text{PolyGamma}[2, 1]))$

(* Collecting result for terms with trivial dependence on x *)

In[75]:= **restr** = **Apart** [**FullSimplify** [**res01** + **res02** + **res11** / **ep** + **res03** + **res04** + **B2expS** [[9]] [[1]]], **ep**]

Out[75]= $\frac{4}{ep^4} - \frac{5 \text{Log}[x]}{ep^3} + \frac{-5 \pi^2 + 4 \text{Log}[x]^2}{2 ep^2} + \frac{33 \pi^2 \text{Log}[x] + 4 \text{Log}[x]^3 - 130 \text{Zeta}[3]}{6 ep} +$
 $\frac{1}{30} (-29 \pi^4 - 180 \pi^2 \text{Log}[x]^2 - 40 \text{Log}[x]^4 + 880 \text{Log}[x] \text{Zeta}[3])$

(* Collecting terms with nontrivial dependence on x *)


```
In[76]:= MBmerge[{Barnes1[B2expS[[3]], z3], Barnes1[B2expS[[4]], z2], B2expS[[8]]}][[1]]
```

```
Out[76]= MBint[ $\frac{1}{\text{ep}}$  4 x1+z1 Gamma[-1-z1]2 Gamma[-z1] Gamma[1+z1]2
(2 ep Gamma[1+z1] + Gamma[2+z1] (-1 + ep EulerGamma + 4 ep PolyGamma[0, -1-z1] -
2 ep PolyGamma[0, 1+z1] - ep PolyGamma[0, 2+z1]))], {{ep → 0}, {z1 → - $\frac{1}{2}$ }}]
```

```
In[77]:= NIntegrate[%[[1]] / (2 Pi) /. {ep → 0.3, x → 0.1, z1 → - $\frac{1}{2}$  + I * y1}, {y1, -Infinity, Infinity}]
```

```
Out[77]= -7.68026 + 0. i
```

```
In[78]:= resnontr = - $\frac{2 \pi^2 \text{Log}[1+x]}{\text{ep}}$  +  $\frac{10}{3} \pi^2 \text{Log}[x] \text{Log}[1+x]$  -  $\frac{2 \text{Log}[x]^2 \text{Log}[1+x]}{\text{ep}}$  +  $\frac{8}{3} \text{Log}[x]^3 \text{Log}[1+x]$  -
 $\pi^2 \text{Log}[1+x]^2 - \text{Log}[x]^2 \text{Log}[1+x]^2 - \frac{20}{3} \pi^2 \text{PolyLog}[2, -x] - \frac{4 \text{Log}[x] \text{PolyLog}[2, -x]}{\text{ep}}$  -
 $2 \text{Log}[x]^2 \text{PolyLog}[2, -x] + \frac{4 \text{PolyLog}[3, -x]}{\text{ep}}$  + 24 Log[x] PolyLog[3, -x] -
 $4 \left( -\frac{1}{2} \text{PolyLog}[2, -x]^2 - \text{Log}[1+x] \text{PolyLog}[3, -x] \right) - 44 \text{PolyLog}[4, -x] +$ 
 $4 \text{Log}[x] (-\text{Log}[1+x] \text{PolyLog}[2, -x] - 2 \text{PolyLog}[1, 2, -x]) + 4 \text{Log}[x] \text{PolyLog}[1, 2, -x] -$ 
 $4 \left( \frac{1}{2} \text{PolyLog}[2, -x]^2 - 2 \text{PolyLog}[2, 2, -x] \right) - 4 \text{PolyLog}[2, 2, -x] - 4 \text{Log}[1+x] \text{Zeta}[3];$ 
```

```
In[79]:= % /. {ep → 0.3, x → 0.1}
```

```
Out[79]= -7.68026 - 3.59491 × 10-15 i
```

```
In[80]:= resnontr + restr;
```

Apart[%, ep]

$$\frac{4}{ep^4} - \frac{5 \operatorname{Log}[x]}{ep^3} + \frac{-5 \pi^2 + 4 \operatorname{Log}[x]^2}{2 ep^2} +$$

$$\frac{1}{6 ep} \left(33 \pi^2 \operatorname{Log}[x] + 4 \operatorname{Log}[x]^3 - 12 \pi^2 \operatorname{Log}[1+x] - 12 \operatorname{Log}[x]^2 \operatorname{Log}[1+x] - \right.$$

$$\left. 24 \operatorname{Log}[x] \operatorname{PolyLog}[2, -x] + 24 \operatorname{PolyLog}[3, -x] - 130 \operatorname{Zeta}[3] \right) +$$

$$\frac{1}{30} \left(-29 \pi^4 - 180 \pi^2 \operatorname{Log}[x]^2 - 40 \operatorname{Log}[x]^4 + 100 \pi^2 \operatorname{Log}[x] \operatorname{Log}[1+x] + 80 \operatorname{Log}[x]^3 \operatorname{Log}[1+x] - \right.$$

$$30 \pi^2 \operatorname{Log}[1+x]^2 - 30 \operatorname{Log}[x]^2 \operatorname{Log}[1+x]^2 - 200 \pi^2 \operatorname{PolyLog}[2, -x] - 60 \operatorname{Log}[x]^2 \operatorname{PolyLog}[2, -x] -$$

$$120 \operatorname{Log}[x] \operatorname{Log}[1+x] \operatorname{PolyLog}[2, -x] + 720 \operatorname{Log}[x] \operatorname{PolyLog}[3, -x] +$$

$$120 \operatorname{Log}[1+x] \operatorname{PolyLog}[3, -x] - 1320 \operatorname{PolyLog}[4, -x] - 120 \operatorname{Log}[x] \operatorname{PolyLog}[1, 2, -x] +$$

$$\left. 120 \operatorname{PolyLog}[2, 2, -x] + 880 \operatorname{Log}[x] \operatorname{Zeta}[3] - 120 \operatorname{Log}[1+x] \operatorname{Zeta}[3] \right)$$

(* the result *)

$$1/x \left(\frac{4}{ep^4} - \frac{5 \operatorname{Log}[x]}{ep^3} + \frac{-5 \pi^2 + 4 \operatorname{Log}[x]^2}{2 ep^2} + \right.$$

$$\frac{1}{6 ep} \left(33 \pi^2 \operatorname{Log}[x] + 4 \operatorname{Log}[x]^3 - 12 \pi^2 \operatorname{Log}[1+x] - 12 \operatorname{Log}[x]^2 \operatorname{Log}[1+x] - \right.$$

$$\left. 24 \operatorname{Log}[x] \operatorname{PolyLog}[2, -x] + 24 \operatorname{PolyLog}[3, -x] - 130 \operatorname{Zeta}[3] \right) +$$

$$\frac{1}{30} \left(-29 \pi^4 - 180 \pi^2 \operatorname{Log}[x]^2 - 40 \operatorname{Log}[x]^4 + 100 \pi^2 \operatorname{Log}[x] \operatorname{Log}[1+x] + 80 \operatorname{Log}[x]^3 \operatorname{Log}[1+x] - \right.$$

$$30 \pi^2 \operatorname{Log}[1+x]^2 - 30 \operatorname{Log}[x]^2 \operatorname{Log}[1+x]^2 - 200 \pi^2 \operatorname{PolyLog}[2, -x] - 60 \operatorname{Log}[x]^2$$

$$\operatorname{PolyLog}[2, -x] - 120 \operatorname{Log}[x] \operatorname{Log}[1+x] \operatorname{PolyLog}[2, -x] + 720 \operatorname{Log}[x] \operatorname{PolyLog}[3, -x] +$$

$$120 \operatorname{Log}[1+x] \operatorname{PolyLog}[3, -x] - 1320 \operatorname{PolyLog}[4, -x] - 120 \operatorname{Log}[x] \operatorname{PolyLog}[1, 2, -x] +$$

$$\left. \left. 120 \operatorname{PolyLog}[2, 2, -x] + 880 \operatorname{Log}[x] \operatorname{Zeta}[3] - 120 \operatorname{Log}[1+x] \operatorname{Zeta}[3] \right) \right)$$

Summing up series with nested sums

$$S_i(n) = \sum_{j=1}^n \frac{1}{j^i}, \quad S_{ik}(n) = \sum_{j=1}^n \frac{S_k(j)}{j^i},$$

$$S_{ikl}(n) = \sum_{j=1}^n \frac{S_{kl}(j)}{j^i}, \quad S_{iklm}(n) = \sum_{j=1}^n \frac{S_{klm}(j)}{j^i}$$

E.g., with one index:

$$\psi(n) = S_1(n-1) - \gamma_E,$$

$$\psi^{(k)}(n) = (-1)^k k! (S_{k+1}(n-1) - \zeta(k+1)), \quad k = 1, 2, \dots,$$

SUMMER

XSummer

[J.A.M. Vermaseren'00]

[S. Moch and P. Uwer'00]

Harmonic polylogarithms (HPL)

$H_{a_1, a_2, \dots, a_n}(x) \equiv H(a_1, a_2, \dots, a_n; x)$, with $a_i = 1, 0, -1$

[E. Remiddi & J.A.M. Vermaseren'00]

are generalizations of the usual polylogarithms $\text{Li}_a(z)$ and Nielsen polylogarithms $S_{a,b}(z)$

$$H(a_1, a_2, \dots, a_n; x) = \int_0^x f(a_1; t) H(a_2, \dots, a_n; t) dt,$$

where $f(\pm 1; t) = 1/(1 \mp t)$, $f(0; t) = 1/t$,

$$H(\pm 1; x) = \mp \ln(1 \mp x), \quad H(0; x) = \ln x,$$

with $a_i = 1, 0, -1$.

HPL are implemented in Mathematica

[D. Maitre'06]

$$\begin{aligned}
& \% /. \left\{ \right. \\
& \quad \frac{(-1)^n z^n S3[-1+n]}{n} \rightarrow -\text{Log}[1+z] \text{PolyLog}[3, -z] - 1/2 \text{PolyLog}[2, -z]^2, \\
& \quad \frac{(-1)^n z^n S12[-1+n]}{n} \rightarrow 1/2 \text{Log}[1+z]^2 \text{PolyLog}[2, -z] + 2 \text{Log}[1+z] \text{PolyLog}[1, 2, -z] + \\
& \quad 3 \text{PolyLog}[1, 3, -z] - \text{Log}[1+z] \text{PolyLog}[3, -z] - 1/2 \text{PolyLog}[2, -z]^2, \\
& \quad \frac{(-1)^n z^n S1[-1+n] S2[-1+n]}{n} \rightarrow 1/2 \text{Log}[1+z]^2 \text{PolyLog}[2, -z] - \\
& \quad \text{Log}[1+z] \text{PolyLog}[3, -z] - 1/2 \text{PolyLog}[2, -z]^2 + \text{Log}[1+z] \text{PolyLog}[1, 2, -z], \\
& \quad \frac{(-1)^n z^n S1[-1+n]^3}{n} \rightarrow 1/4 \text{Log}[1+z]^4 - \text{Log}[1+z] \text{PolyLog}[3, -z] - \\
& \quad 1/2 \text{PolyLog}[2, -z]^2 + 3/2 \text{Log}[1+z]^2 \text{PolyLog}[2, -z] + 3 \text{Log}[1+z] \text{PolyLog}[1, 2, -z] \\
& \left. \right\};
\end{aligned}$$

$$\begin{aligned}
& \% /. \left\{ \right. \\
& \quad \frac{(-1)^n z^n S2[-1+n]}{n} \rightarrow -\text{Log}[1+z] \text{PolyLog}[2, -z] - 2 \text{PolyLog}[1, 2, -z], \\
& \quad \frac{(-1)^n z^n S2[-1+n]}{n^2} \rightarrow 1/2 \text{PolyLog}[2, -z]^2 - 2 \text{PolyLog}[2, 2, -z], \\
& \quad \frac{(-1)^n z^n S1[-1+n]^2}{n} \rightarrow \\
& \quad -\text{Log}[1+z] \text{PolyLog}[2, -z] - 1/3 \text{Log}[1+z]^3 - 2 \text{PolyLog}[1, 2, -z], \\
& \quad \frac{(-1)^n z^n S1[-1+n]^2}{n^2} \rightarrow 1/2 \text{PolyLog}[2, -z]^2 + 2 \text{PolyLog}[1, 3, -z] - 2 \text{PolyLog}[2, 2, -z] \\
& \left. \right\};
\end{aligned}$$

$$\begin{aligned}
& \% /. \left\{ \right. \\
& \quad \frac{(-1)^n z^n S1[-1+n]}{n^3} \rightarrow \text{PolyLog}[2, 2, -z], \\
& \quad \frac{(-1)^n z^n S1[-1+n]}{n^2} \rightarrow \text{PolyLog}[1, 2, -z], \\
& \quad \frac{(-1)^n z^n S1[-1+n]}{n} \rightarrow \frac{1}{2} \text{Log}[1+z]^2 \left. \right\};
\end{aligned}$$

$$\begin{aligned}
& \% /. \left\{ \right. \\
& \quad \frac{(-1)^n z^n}{n^4} \rightarrow \text{PolyLog}[4, -z], \quad \frac{(-1)^n z^n}{n^3} \rightarrow \text{PolyLog}[3, -z], \\
& \quad \frac{(-1)^n z^n}{n^2} \rightarrow \text{PolyLog}[2, -z], \quad \frac{(-1)^n z^n}{n} \rightarrow -\text{Log}[1+z] \left. \right\};
\end{aligned}$$

$$\begin{aligned}
& \% /. \left\{ \right. \\
& \quad \frac{1}{n^6} \rightarrow \frac{\pi^6}{945}, \\
& \quad \frac{S1[-1+n]}{n^5} \rightarrow \frac{\pi^6}{1260} - \frac{\text{Zeta}[3]^2}{2}, \\
& \quad \frac{S2[-1+n]}{n^4} \rightarrow -\frac{4 \pi^6}{2835} + \text{Zeta}[3]^2, \\
& \quad \frac{S1[-1+n]^2}{n^4} \rightarrow \frac{37 \pi^6}{22680} - \text{Zeta}[3]^2, \\
& \quad \frac{S3[-1+n]}{n^3} \rightarrow -\frac{\pi^6}{1890} + \frac{\text{Zeta}[3]^2}{2}, \\
& \quad \frac{S4[-1+n]}{n^2} \rightarrow \frac{5 \pi^6}{2268} - \text{Zeta}[3]^2, \\
& \left. \right\}
\end{aligned}$$

$$\frac{S13[-1+n]}{n^2} \rightarrow \frac{61 \pi^6}{45360},$$

$$\frac{S2[-1+n]^2}{n^2} \rightarrow \frac{59 \pi^6}{22680} - \text{Zeta}[3]^2,$$

$$\frac{S1[-1+n]^3}{n^3} \rightarrow -\frac{11 \pi^6}{5040} + 2 \text{Zeta}[3]^2,$$

$$\frac{S1[-1+n] S2[-1+n]}{n^3} \rightarrow -\frac{121 \pi^6}{45360} + 2 \text{Zeta}[3]^2,$$

$$\frac{S12[-1+n]}{n^3} \rightarrow \frac{41 \pi^6}{22680} - \text{Zeta}[3]^2,$$

$$\frac{S1[-1+n] S3[-1+n]}{n^2} \rightarrow \frac{167 \pi^6}{45360} - \frac{3 \text{Zeta}[3]^2}{2},$$

$$\frac{S1[-1+n]^2 S2[-1+n]}{n^2} \rightarrow \frac{23 \pi^6}{3780} - \text{Zeta}[3]^2,$$

$$\frac{S1[-1+n]^4}{n^2} \rightarrow \frac{859 \pi^6}{22680} + 3 \text{Zeta}[3]^2,$$

$$\frac{S112[-1+n]}{n^2} \rightarrow \frac{17 \pi^6}{4536} - \text{Zeta}[3]^2,$$

$$\frac{S1[-1+n] S12[-1+n]}{n^2} \rightarrow \frac{313 \pi^6}{45360} - 2 \text{Zeta}[3]^2 \};$$

% /. { $\frac{1}{n^5} \rightarrow \text{Zeta}[5],$

$$\frac{S1[-1+n]}{n^4} \rightarrow \frac{1}{6} (-\pi^2 \text{Zeta}[3] + 12 \text{Zeta}[5]), \frac{S2[-1+n]}{n^3} \rightarrow \frac{1}{2} (\pi^2 \text{Zeta}[3] - 11 \text{Zeta}[5]),$$

$$\frac{S1[-1+n]^2}{n^3} \rightarrow \frac{1}{6} (\pi^2 \text{Zeta}[3] - 9 \text{Zeta}[5]), \frac{S3[-1+n]}{n^2} \rightarrow \frac{1}{6} (-2 \pi^2 \text{Zeta}[3] + 27 \text{Zeta}[5]),$$

$$\frac{S1[-1+n]^3}{n^2} \rightarrow \frac{1}{6} (\pi^2 \text{Zeta}[3] + 45 \text{Zeta}[5]), \frac{S1[-1+n] S2[-1+n]}{n^2} \rightarrow$$

$$\frac{1}{6} (-\pi^2 \text{Zeta}[3] + 21 \text{Zeta}[5]), \frac{S12[-1+n]}{n^2} \rightarrow \frac{1}{3} (-2 \pi^2 \text{Zeta}[3] + 27 \text{Zeta}[5]) \};$$

% /. { $\frac{S2[-1+n]}{n^2} \rightarrow \frac{\pi^4}{120}, \frac{S1[-1+n]^2}{n^2} \rightarrow \frac{11 \pi^4}{360},$

$$\frac{S1[-1+n]}{n^3} \rightarrow \frac{\pi^4}{360}, \frac{1}{n^4} \rightarrow \frac{\pi^4}{90} \};$$

% /. { $\frac{S1[-1+n]}{n^2} \rightarrow \text{Zeta}[3], \frac{1}{n^3} \rightarrow \text{Zeta}[3] \};$

Expand [% /. $\frac{1}{n^2} \rightarrow \frac{\pi^2}{6}$];

```

In[1]:= SetDirectory["c:/diskE/job2008/Zurich"];

In[2]:= << MB/MB.m

MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08

In[3]:= SortByDimension[l_List] := Sort[l, Length[#1[[2, 2]]] > Length[#2[[2, 2]]] &];
CoeffEps[X_, n_] := (X /. X[[1]] -> Simplify[Coefficient[X[[1]], ep, n]));
MBDimension[int_MBint] := Length[int[[2, 2]]];

(* a 4fold MB representation for the on-shell 2loop
non-planar vertex diagram derived loop by loop;
sg=-1 *)

In[7]:= V2 = (sg^z4 Gamma[-1 - ep - z1 - z2] Gamma[-z2] Gamma[1 + z1 + z2] Gamma[-1 - ep - z1 - z3]
Gamma[-z3] Gamma[1 + z1 + z3] Gamma[-1 - 2 ep - z2 - z4] Gamma[-1 - 2 ep - z3 - z4]
Gamma[-z4] Gamma[2 + 2 ep + z4] Gamma[-z1 + z4] Gamma[2 + ep + z1 + z2 + z3 + z4]) /
(Gamma[-3 ep] Gamma[-2 ep] Gamma[1 - z2] Gamma[1 - z3]);

In[8]:= V2rules = MBOptimizedRules[V2, ep -> 0, {}, {ep}]

MBrules::norules : no rules could be found to regulate this integral
MBrules::norules : no rules could be found to regulate this integral
MBrules::norules : no rules could be found to regulate this integral

General::stop : Further output of MBrules::norules will be suppressed during this calculation. >>

Out[8]= {{ep -> -5/8}, {z1 -> -1/4, z2 -> -1/2, z3 -> -5/16, z4 -> -1/8}}

In[9]:= V2cont = MBcontinue[V2, ep -> 0, V2rules];

```

Level 1

Taking -residue in $z_2 = -1 - \epsilon p - z_1$

Taking -residue in $z_3 = -1 - \epsilon p - z_1$

Taking -residue in $z_4 = -1 - 2 \epsilon p - z_2$

Taking -residue in $z_4 = -1 - 2 \epsilon p - z_3$

Level 2

Integral {1}

Taking -residue in $z_4 = -\epsilon p + z_1$

Integral {2}

Taking -residue in $z_2 = -1 - \epsilon p - z_1$

Taking -residue in $z_4 = -\epsilon p + z_1$

Taking -residue in $z_4 = -1 - 2 \epsilon p - z_2$

Integral {3}

Taking -residue in $z_2 = -1 - 2 \epsilon p - z_1$

Integral {4}

Taking -residue in $z_2 = -1 - \epsilon p - z_1$

Taking -residue in $z_3 = -1 - 2 \epsilon p - z_1$

Level 3

Integral {1, 1}

Integral {2, 1}

Taking -residue in $z_4 = -\epsilon p + z_1$

Integral {2, 2}

Integral {2, 3}

Taking -residue in $z_2 = -1 - 2 \epsilon p - z_1$

Integral {3, 1}

Integral {4, 1}

Taking -residue in $z_3 = -1 - 2 \epsilon p - z_1$

Integral {4, 2}

Taking -residue in $z_2 = -1 - 2 \epsilon p - z_1$

Level 4

Integral {2, 1, 1}

Integral {2, 3, 1}

Integral {4, 1, 1}

Integral {4, 2, 1}

16 integral(s) found

(* no $1/\epsilon^4$ poles??? *)

In[10]:= **V2select4 = MBpreselect [MBmerge [V2cont], {ep, 0, -4}]**

Out[10]= {}

In[11]:= **V2select0 = MBpreselect [MBmerge [V2cont], {ep, 0, 0}]**

In[12]:= **V2select0S = Simplify [SortByDimension [V2select0]]**

Out[12]=
$$\left\{ \text{MBint} \left[\left(\text{sg}^{z_4} \Gamma[-\text{ep}]^2 \Gamma[1 + \text{ep} + z_1]^2 \Gamma[-\text{ep} + z_1 - z_4]^2 \right. \right. \right.$$

$$\left. \left. \Gamma[-z_4] \Gamma[2 + 2 \text{ep} + z_4] \Gamma[-z_1 + z_4] \Gamma[-\text{ep} - z_1 + z_4] \right) / \right.$$

$$\left. \left(\Gamma[-3 \text{ep}] \Gamma[-2 \text{ep}] \Gamma[2 + \text{ep} + z_1]^2 \right), \left\{ \{\text{ep} \rightarrow 0\}, \left\{ z_1 \rightarrow -\frac{1}{4}, z_4 \rightarrow -\frac{1}{8} \right\} \right\} \right],$$

$$\text{MBint} \left[\left(\text{sg}^{-1-2 \text{ep}-z_3} \Gamma[-1 - \text{ep} - z_1 - z_3] \Gamma[-z_3] \Gamma[1 + z_1 + z_3] \right. \right.$$

$$\left(\text{sg}^{1+\text{ep}+z_1+z_3} \Gamma[-\text{ep}]^2 \Gamma[\text{ep} - z_1] \Gamma[1 + \text{ep} + z_1] \right.$$

$$\Gamma[2 + \text{ep} + z_1] \Gamma[-1 - \text{ep} - z_1 - z_3] \Gamma[1 - \text{ep} + z_1 + z_3] +$$

$$\Gamma[-2 \text{ep}] \Gamma[-1 - 2 \text{ep} - z_1 - z_3] \left(\text{sg}^{1+2 \text{ep}+z_1+z_3} \Gamma[\text{ep}] \Gamma[-z_1] \right.$$

$$\Gamma[2 + \text{ep} + z_1] \Gamma[1 + 2 \text{ep} + z_1] \Gamma[1 - \text{ep} + z_1 + z_3] + \Gamma[-\text{ep}]$$

$$\left. \left. \Gamma[1 + \text{ep} + z_1] \Gamma[1 - z_3] \Gamma[1 + 2 \text{ep} + z_3] \Gamma[1 + \text{ep} + z_1 + z_3] \right) \right) /$$

$$\left(\Gamma[-3 \text{ep}] \Gamma[-2 \text{ep}] \Gamma[2 + \text{ep} + z_1] \Gamma[1 - z_3] \right), \left\{ \{\text{ep} \rightarrow 0\}, \right.$$

$$\left. \left\{ z_1 \rightarrow -\frac{1}{4}, z_3 \rightarrow -\frac{5}{16} \right\} \right\},$$

$$\text{MBint} \left[\left(\text{sg}^{-1-2 \text{ep}-z_2} \Gamma[-1 - \text{ep} - z_1 - z_2] \Gamma[-z_2] \Gamma[1 + z_1 + z_2] \right. \right.$$

$$\left(\text{sg}^{1+\text{ep}+z_1+z_2} \Gamma[-\text{ep}]^2 \Gamma[\text{ep} - z_1] \Gamma[1 + \text{ep} + z_1] \right.$$

$$\Gamma[2 + \text{ep} + z_1] \Gamma[-1 - \text{ep} - z_1 - z_2] \Gamma[1 - \text{ep} + z_1 + z_2] +$$

$$\Gamma[-2 \text{ep}] \Gamma[-1 - 2 \text{ep} - z_1 - z_2] \left(\text{sg}^{1+2 \text{ep}+z_1+z_2} \Gamma[\text{ep}] \Gamma[-z_1] \right.$$

$$\Gamma[2 + \text{ep} + z_1] \Gamma[1 + 2 \text{ep} + z_1] \Gamma[1 - \text{ep} + z_1 + z_2] + \Gamma[-\text{ep}]$$

$$\left. \left. \Gamma[1 + \text{ep} + z_1] \Gamma[1 - z_2] \Gamma[1 + 2 \text{ep} + z_2] \Gamma[1 + \text{ep} + z_1 + z_2] \right) \right) /$$

$$\left(\Gamma[-3 \text{ep}] \Gamma[-2 \text{ep}] \Gamma[2 + \text{ep} + z_1] \Gamma[1 - z_2] \right), \left\{ \{\text{ep} \rightarrow 0\}, \right.$$

$$\left. \left\{ z_1 \rightarrow -\frac{1}{4}, z_2 \rightarrow -\frac{1}{2} \right\} \right\},$$

$$\text{MBint} \left[\left(\text{sg}^{-\text{ep}+z_1} \left(\text{sg}^{\text{ep}} \Gamma[-3 \text{ep}] \Gamma[-2 \text{ep}] \Gamma[\text{ep}]^2 \Gamma[-z_1] \Gamma[2 + \text{ep} + z_1] \right. \right. \right.$$

$$\Gamma[1 + 2 \text{ep} + z_1]^2 - \Gamma[-\text{ep}]^2 \Gamma[1 + \text{ep} + z_1] \Gamma[2 + 2 \text{ep} + z_1]$$

$$\left. \left. \left(-2 \text{sg}^{\text{ep}} \Gamma[-2 \text{ep}] \Gamma[\text{ep}] \Gamma[-z_1] \Gamma[1 + 2 \text{ep} + z_1] + \Gamma[-\text{ep}] \right. \right. \right.$$

$$\Gamma[\text{ep} - z_1] \Gamma[1 + \text{ep} + z_1] \left(2 \text{EulerGamma} + \text{Log}[\text{sg}] + \text{PolyGamma}[0, -2 \text{ep}] + \right.$$

$$\left. \left. \text{PolyGamma}[0, -\text{ep}] - \text{PolyGamma}[0, \text{ep} - z_1] + \text{PolyGamma}[0, 2 + \text{ep} + z_1] \right) \right) \right) /$$

$$\left(\Gamma[-3 \text{ep}] \Gamma[2 + \text{ep} + z_1] \Gamma[2 + 2 \text{ep} + z_1] \right), \left\{ \{\text{ep} \rightarrow 0\}, \right.$$

$$\left. \left\{ z_1 \rightarrow -\frac{1}{4} \right\} \right\} \right\}$$

In[13]:= **MBDimension /@ V2select0S**

Out[13]= {2, 2, 2, 1}

(* One-dimensional contribution V2select0S[[4]] *)

In[14]:= **V2select0S[[4]]**

$$\text{Out[14]} = \text{MBInt} \left[\left(\text{sg}^{-\text{ep}+\text{z1}} \left(\text{sg}^{\text{ep}} \Gamma[-3 \text{ ep}] \Gamma[-2 \text{ ep}] \Gamma[\text{ep}]^2 \Gamma[-\text{z1}] \Gamma[2 + \text{ep} + \text{z1}] \right. \right. \right. \\ \left. \left. \left. \Gamma[1 + 2 \text{ ep} + \text{z1}]^2 - \Gamma[-\text{ep}]^2 \Gamma[1 + \text{ep} + \text{z1}] \Gamma[2 + 2 \text{ ep} + \text{z1}] \right. \right. \\ \left. \left. \left. (-2 \text{ sg}^{\text{ep}} \Gamma[-2 \text{ ep}] \Gamma[\text{ep}] \Gamma[-\text{z1}] \Gamma[1 + 2 \text{ ep} + \text{z1}] + \Gamma[-\text{ep}] \right. \right. \\ \left. \left. \left. \Gamma[\text{ep} - \text{z1}] \Gamma[1 + \text{ep} + \text{z1}] (2 \text{ EulerGamma} + \text{Log}[\text{sg}] + \text{PolyGamma}[0, -2 \text{ ep}] + \right. \right. \\ \left. \left. \left. \text{PolyGamma}[0, -\text{ep}] - \text{PolyGamma}[0, \text{ep} - \text{z1}] + \text{PolyGamma}[0, 2 + \text{ep} + \text{z1}]) \right) \right) \right) / \\ \left(\Gamma[-3 \text{ ep}] \Gamma[2 + \text{ep} + \text{z1}] \Gamma[2 + 2 \text{ ep} + \text{z1}] \right), \\ \left\{ \left\{ \text{ep} \rightarrow 0 \right\}, \right. \\ \left. \left\{ \left\{ \text{z1} \rightarrow -\frac{1}{4} \right\} \right\} \right]$$

(* a piece of this: *)

In[15]:= **V20 =**

$$\left(2 \text{ sg}^{\text{z1}} \Gamma[-2 \text{ ep}] \Gamma[-\text{ep}]^2 \Gamma[\text{ep}] \Gamma[-\text{z1}] \Gamma[1 + \text{ep} + \text{z1}] \Gamma[1 + 2 \text{ ep} + \text{z1}] \right) / \\ \left(\Gamma[-3 \text{ ep}] \Gamma[2 + \text{ep} + \text{z1}] \right);$$

(* no 1/ep^4 poles here??? *)

In[16]:= **Series[V20 E^(2 EulerGamma ep), {ep, 0, -4}]**

$$\text{Out[16]} = \frac{1}{\text{O}[\text{ep}]^3}$$

In[17]:= **V20 /. sg^z1 -> E^(I Pi z1)**

$$\text{Out[17]} = \left(2 e^{i \pi \text{z1}} \Gamma[-2 \text{ ep}] \Gamma[-\text{ep}]^2 \Gamma[\text{ep}] \Gamma[-\text{z1}] \Gamma[1 + \text{ep} + \text{z1}] \Gamma[1 + 2 \text{ ep} + \text{z1}] \right) / \\ \left(\Gamma[-3 \text{ ep}] \Gamma[2 + \text{ep} + \text{z1}] \right)$$

In[18]:= **% /. Gamma[2 + ep + z1] -> Gamma[1 + ep + z1] (1 + ep + z1)**

$$\text{Out[18]} = \frac{2 e^{i \pi \text{z1}} \Gamma[-2 \text{ ep}] \Gamma[-\text{ep}]^2 \Gamma[\text{ep}] \Gamma[-\text{z1}] \Gamma[1 + 2 \text{ ep} + \text{z1}]}{(1 + \text{ep} + \text{z1}) \Gamma[-3 \text{ ep}]}$$

In[19]:= **(% /. Gamma[-z1] -> (-1)^n/n!) /. z1 -> n**

$$\text{Out[19]} = \frac{2 (-1)^n e^{i n \pi} \Gamma[-2 \text{ ep}] \Gamma[-\text{ep}]^2 \Gamma[\text{ep}] \Gamma[1 + 2 \text{ ep} + n]}{(1 + \text{ep} + n) n! \Gamma[-3 \text{ ep}]}$$

In[20]:= **% /. e^{i n \pi} -> (-1)^n**

$$\text{Out[20]} = \frac{2 (-1)^{2n} \Gamma[-2 \text{ ep}] \Gamma[-\text{ep}]^2 \Gamma[\text{ep}] \Gamma[1 + 2 \text{ ep} + n]}{(1 + \text{ep} + n) n! \Gamma[-3 \text{ ep}]}$$

In[21]:= **% /. (-1)^{2n} -> 1**

$$\text{Out[21]} = \frac{2 \Gamma[-2 \text{ ep}] \Gamma[-\text{ep}]^2 \Gamma[\text{ep}] \Gamma[1 + 2 \text{ ep} + n]}{(1 + \text{ep} + n) n! \Gamma[-3 \text{ ep}]}$$

In[22]:= **Sum[%, {n, 0, Infinity}]**

$$\text{Out[22]} = -\frac{2 \pi \text{Csc}[2 \text{ ep} \pi] \Gamma[-2 \text{ ep}] \Gamma[-\text{ep}]^2 \Gamma[\text{ep}] \Gamma[1 + \text{ep}]}{\Gamma[1 - \text{ep}] \Gamma[-3 \text{ ep}]}$$

```
In[23]:= Simplify[Normal[Series[% E ^ (2 EulerGamma ep), {ep, 0, -4}]]]
```

$$\text{Out[23]} = -\frac{3}{2 \text{ep}^4}$$

(* a singularity in epsilon arises when integrating over large values of z *)

(* do not use the loop-by-loop strategy of
deriving MB representations for nonplanar diagrams *)

$$\frac{1}{2\pi i} \int_C \frac{\Gamma(1 + 2\epsilon + z)\Gamma(-z)}{1 + \epsilon + z} (-1)^z \mathbf{d}z$$

$$\frac{1}{2\pi i} \int_C \frac{\Gamma(1 + 2\epsilon + z)\Gamma(-z)}{1 + \epsilon + z} (-1)^z \mathbf{d}z$$

$z = x + iy$; ϵ **real**

$$\frac{1}{2\pi i} \int_C \frac{\Gamma(1 + 2\epsilon + z)\Gamma(-z)}{1 + \epsilon + z} (-1)^z \mathbf{d}z$$

$z = x + iy$; ϵ **real**

$$= \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} \frac{\Gamma(1 + 2\epsilon + x + iy)\Gamma(-x - iy)}{1 + \epsilon + x + iy} e^{-i\pi x + \pi y} \mathbf{d}y$$

$$\frac{1}{2\pi i} \int_C \frac{\Gamma(1 + 2\epsilon + z)\Gamma(-z)}{1 + \epsilon + z} (-1)^z \mathbf{d}z$$

$z = x + iy$; ϵ **real**

$$= \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} \frac{\Gamma(1 + 2\epsilon + x + iy)\Gamma(-x - iy)}{1 + \epsilon + x + iy} e^{-i\pi x + \pi y} \mathbf{d}y$$

$$\Gamma(x \pm iy) \sim \sqrt{2\pi} e^{\pm i\frac{\pi}{4}(2x-1)} e^{\pm iy(\ln y - 1)} e^{-\frac{\pi}{2}y}$$

when $y \rightarrow +\infty$

$$\frac{1}{2\pi i} \int_C \frac{\Gamma(1 + 2\epsilon + z)\Gamma(-z)}{1 + \epsilon + z} (-1)^z \mathbf{d}z$$

$z = x + iy$; ϵ **real**

$$= \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} \frac{\Gamma(1 + 2\epsilon + x + iy)\Gamma(-x - iy)}{1 + \epsilon + x + iy} e^{-i\pi x + \pi y} \mathbf{d}y$$

$$\Gamma(x \pm iy) \sim \sqrt{2\pi} e^{\pm i\frac{\pi}{4}(2x-1)} e^{\pm iy(\ln y - 1)} e^{-\frac{\pi}{2}y}$$

when $y \rightarrow +\infty$

The integrand behaves like

$$2\pi \frac{1}{y^{1-2\epsilon}}$$

In[17]:= % / 4

Out[17]= $\left\{ \frac{17}{4}, 0, 1, 0, -1 \right\}$

In[18]:= $\frac{17}{4} \text{Zeta}[4] + \text{Zeta}[2]$

Out[18]= $\frac{\pi^2}{6} + \frac{17 \pi^4}{360}$

In[19]:= % - x0

Out[19]= $0. \times 10^{-100}$

(* For example,
this is the list of constants up to level 6 where Log[2] is involved *)

```
x[1] = N[Zeta[5], 500] ;
x[2] = N[Zeta[3] Zeta[2], 500] ;
x[3] = N[Log[2] Zeta[4], 500] ;
x[4] = N[PolyLog[5, 1/2], 500] ;
x[5] = N[Log[2] PolyLog[4, 1/2], 500] ;
x[6] = N[Log[2]^5, 500] ;
x[7] = N[Log[2]^2 Zeta[3], 500] ;
x[8] = N[Log[2]^3 Zeta[2], 500] ;
x[9] = N[Zeta[4], 500] ;
x[10] = N[PolyLog[4, 1/2], 500] ;
x[11] = N[Log[2]^4, 500] ;
x[12] = N[Log[2]^2 Zeta[2], 500] ;
x[13] = N[Zeta[3] Log[2], 500] ;
x[14] = N[Zeta[3], 500] ;
x[15] = N[Log[2] Zeta[2], 500] ;
x[16] = N[Log[2]^3, 500] ;
x[17] = N[Zeta[2], 500] ;
x[18] = N[Log[2]^2, 500] ;
x[19] = N[Log[2], 500] ;
x[20] = N[Zeta[6], 500] ;
x[21] = N[Zeta[5] Log[2], 500] ;
x[22] = N[Zeta[4] Log[2]^2, 500] ;
x[23] = N[Zeta[3] Log[2]^3, 500] ;
x[24] = N[Zeta[3]^2, 500] ;
x[25] = N[Zeta[3] Zeta[2] Log[2], 500] ;
x[26] = N[Log[2]^6, 500] ;
x[27] = N[Zeta[2] Log[2]^4, 500] ;
x[28] = N[PolyLog[6, 1/2], 500] ;
x[29] = N[PolyLog[5, 1/2] Log[2], 500] ;
x[30] = N[PolyLog[4, 1/2] Log[2]^2, 500] ;
x[31] = N[PolyLog[4, 1/2] Zeta[2], 500] ;
x[32] = N[1, 500] ;
x[33] = N[appHPLs6, 500] ;
```

appHPLs6 =

```
0.98744142640329971377165000804182021413602714891426845727515905154561369911878895355 \
863956404477558915203056255385484634405674146808430963398029349014388073675782238914 \
431072868245446242746979584265585272725425183010859167628333328685038039329144638374 \
255888866551887721158448693857022393231698312530376047305702516583202905810931661490 \
473936355942820783847449825120107884979332566716367272547925735281416166238736093320 \
673713594250387141809300098954218750377349286950447367387499265350018206576197727 ;
```

PSLQ: an integer relation algorithm

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x_0 is an unknown constant.

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PSLQ looks for integers c_0, c_1, \dots such that

$$c_0x_0 + \sum_{i=1}^n c_i x[i] = 0$$

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We suspect that it is a linear combination of known constants $x[1], x[2], \dots, x[n]$

PSLQ looks for integers c_0, c_1, \dots such that

$$c_0 x_0 + \sum_{i=1}^n c_i x[i] = 0$$

Then

$$x_0 = -c_0^{-1} \sum_{i=1}^n c_i x[i]$$

1

π^2

$\zeta(3)$

π^4

$\zeta(5), \zeta(2)\zeta(3)$

$\pi^6, \zeta(3)^2$

1 π^2 $\zeta(3)$ π^4 $\zeta(5), \zeta(2)\zeta(3)$ $\pi^6, \zeta(3)^2$ $\ln 2, \ln^2 2, \dots, \mathbf{Li}_4(1/2), \dots,$ $s_6 = \zeta(-5, -1) + \zeta(6), \dots$

1

π^2

$\zeta(3)$

π^4

$\zeta(5), \zeta(2)\zeta(3)$

$\pi^6, \zeta(3)^2$

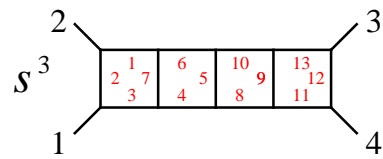
$\ln 2, \ln^2 2, \dots, \mathbf{Li}_4(1/2), \dots,$

$s_6 = \zeta(-5, -1) + \zeta(6), \dots$

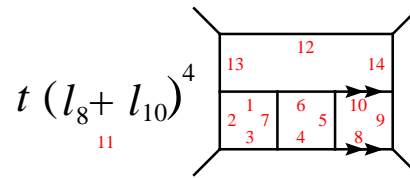
uniform transcendentalty

The four-loop cusp (soft) anomalous dimension

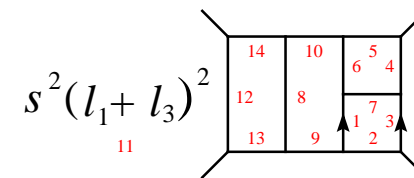
[Z. Bern, M. Czakon, L. Dixon, D.A. Kosower, & V.A. Smirnov'06]



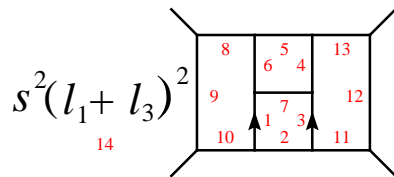
(a)



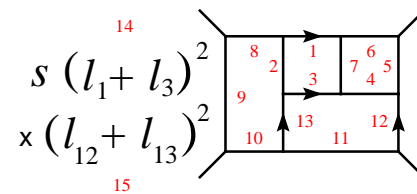
(b)



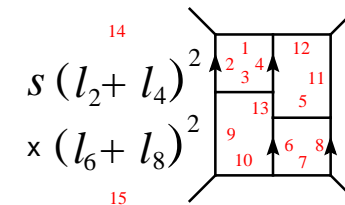
(c)



(d)



(e)



(f)

```
<< MB/MB.m ";
<<MB/MBresolve.m";
```

MB 1.1

by Michal Czakon

more info in hep-ph/0511200

last modified 06 Mar 08

MBresolve 1.0

by Alexander Smirnov

last modified 22 Oct 08

```
F1 = - (S^{-5-4 ep-z7} T^{z7} Gamma[1+z1] Gamma[-1-ep-z1-z2] Gamma[-z2] Gamma[-1-ep-z1-z3]
Gamma[-z3] Gamma[1+z1+z2+z3] Gamma[2+ep+z1+z2+z3] Gamma[z10-z4]
Gamma[-z1+z4] Gamma[-ep+z1+z2-z4-z5] Gamma[-z5] Gamma[-ep+z1+z3-z4-z6]
Gamma[-z6] Gamma[1+z4+z5+z6] Gamma[1+ep-z1-z2-z3+z4+z5+z6]
Gamma[-z7] Gamma[1+z7] Gamma[-z10+z7] Gamma[-ep-z10+z4+z5-z8]
Gamma[-z8] Gamma[-ep+z10-z7+z8] Gamma[-ep-z10+z4+z6-z9]
Gamma[1+ep-z10+z7-z8-z9] Gamma[-z9] Gamma[-ep+z10-z7+z9]
Gamma[1+z10+z8+z9] Gamma[1+ep+z10-z4-z5-z6+z8+z9]) /
(Gamma[-2 ep] Gamma[1-z2] Gamma[1-z3] Gamma[1-2 ep+z1+z2+z3]
Gamma[1-z5] Gamma[1-z6] Gamma[1-2 ep+z4+z5+z6]
Gamma[1-z8] Gamma[1-z9] Gamma[1-2 ep+z10+z8+z9]);
```

```
F1cont = MBresolve[F1, ep]
```

CREATING RESIDUES LIST.....654.3281 seconds

EVALUATING RESIDUES.....16.9375 seconds

A very large output was generated. Here is a sample of it:

$$\left\{ \text{MBint} \left[\frac{\text{EulerGamma}^3 T^{-1-4 \text{ep}} \text{Gamma}[-2 \text{ep}]^2 \text{Gamma}[-\text{ep}]^4 \text{Gamma}[1+\text{ep}]^2 \text{Gamma}[1+4 \text{ep}]}{S^4 \text{Gamma}[1-\text{ep}]^2 \text{Gamma}[-4 \text{ep}]} + \ll 180 \gg + \right. \right.$$

$$\left. \frac{T^{-1-4 \text{ep}} \text{Gamma}[-2 \text{ep}]^2 \text{Gamma}[-\text{ep}]^4 \ll 5 \gg \ll 1 \gg \ll 1 \gg^2 \text{Gamma}[1+4 \text{ep}] \text{PolyGamma}[2, -2 \text{ep}]}{2 S^4 \text{Gamma}[1-\text{ep}]^2 \text{Gamma}[-4 \text{ep}]} , \{ \{ \text{ep} \rightarrow 0 \}, \{ \} \} \right\} ,$$

$$\ll 654 \gg , \text{MBint} \left[- \frac{S^{-5-\ll 1 \gg} \ll 1 \gg^{z7} \ll 27 \gg \ll 1 \gg}{\text{Gamma}[-2 \text{ep}] \ll 8 \gg \ll 1 \gg} , \{ \{ \ll 1 \gg \}, \ll 1 \gg \} \right] \left. \right\}$$

Show Less

Show More

Show Full Output

Set Size Limit...

Length[F1cont]

656

```
Flxep = MBexpand[F1select, E^(4 EulerGamma ep), {ep, 0, 0}] // Timing
```

A very large output was generated. Here is a sample of it:

$$\left\{ 2337.28, \left\{ \text{MBint} \left[-\frac{19}{16 \text{ep}^8 \text{S}^4 \text{T}} - \frac{11 \pi^2}{24 \text{ep}^6 \text{S}^4 \text{T}} + \frac{167 \pi^4}{720 \text{ep}^4 \text{S}^4 \text{T}} + \frac{14 213 \pi^6}{7560 \text{ep}^2 \text{S}^4 \text{T}} + \frac{975 257 \pi^8}{226 800 \text{S}^4 \text{T}} + \right. \right.$$

$$\left. \left. \ll 117 \gg + \frac{974 \text{Log[S]} \text{PolyGamma}[6,1]}{315 \text{S}^4 \text{T}} + \frac{54 983 \text{Log[T]} \text{PolyGamma}[6,1]}{2520 \text{S}^4 \text{T}}, \{\{\text{ep} \rightarrow 0\}, \{\}\}\right\}, \right.$$

$$\left. \left. \ll 553 \gg, \text{MBint} \left[-\frac{2 \ll 21 \gg \text{Gamma}[1-z4-z5-z6+z7+z8]}{\text{Gamma}[1-z2] \text{Gamma}[1-z3] \ll 1 \gg \text{Gamma}[1-z6]}, \{\{\text{ep} \rightarrow 0\}, \{\ll 1 \gg\}\}\right\] \right\}$$

[Show Less](#)
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```
Flrules = MBOptimizedRules[F1, ep → 0, {}, {ep}]
```

```
Flrules = MBcorrectContours[MBOptimizedRules[F1, ep → 0, {}, {ep}], 10 000]
```

```
Mlcont = MBcontinue[F1, ep → 0, Flrules]
```

```
2200" integral(s) found"
```

'Inverse Feynman parameters'

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \rightarrow \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

'Inverse Feynman parameters'

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \rightarrow \int_0^1 t^{a-1}(1-t)^{b-1} \mathbf{d}t$$

Integrating over a MB variable (not by a Barnes lemma)

$$\begin{aligned} & \frac{1}{2\pi i} \int_C \Gamma(a+z)\Gamma(b+z)\Gamma(c-z)\Gamma(-z)x^z \mathbf{d}z \\ &= \frac{\Gamma(a)\Gamma(b)\Gamma(a+c)\Gamma(b+c)}{\Gamma(a+b+c)} {}_2F_1(a; b; a+b+c; 1-x) \\ &= \Gamma(a)\Gamma(b+c) \int_0^1 t^{b-1}(1-t)^{a+c-1}(1-t+tx)^{-a} \mathbf{d}t \end{aligned}$$

MB tools at <http://projects.hepforge.org/mbtools/>:

MB tools at <http://projects.hepforge.org/mbtools/>:

[MB.m](#) (updated)

MB tools at <http://projects.hepforge.org/mbtools/>:

MB.m (updated)

MBresolve.m

MB tools at <http://projects.hepforge.org/mbtools/>:

MB.m (updated)

MBresolve.m

MBasymptotics.m [M. Czakon]

MB tools at <http://projects.hepforge.org/mbtools/>:

[MB.m](#) (updated)

[MBresolve.m](#)

[MBasymptotics.m](#) [M. Czakon]

[Barnes.m](#), . . . [D. Kosower]

(applying Barnes lemmas automatically)

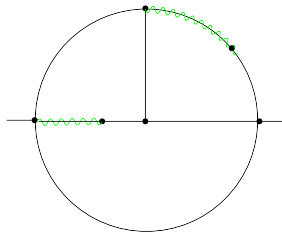
Evaluating the three-loop quark static potential (fermionic contributions)

[A.V. Smirnov, V.A. Smirnov, and M. Steinhauser'08]

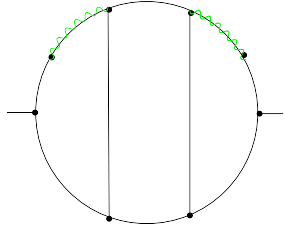
Reduction by FIRE.

Evaluating master integrals by MB, with checks by FIESTA.

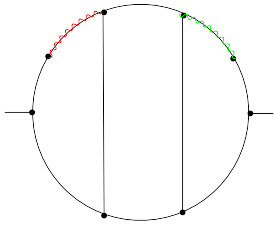
For example,



$$\frac{(i\pi^{d/2})^3}{q^2 v^2} \left[\frac{32\pi^4}{135\epsilon} - \frac{128\pi^4}{135} + \frac{88\pi^2\zeta(3)}{9} + \frac{188\zeta(5)}{3} + O(\epsilon) \right]$$

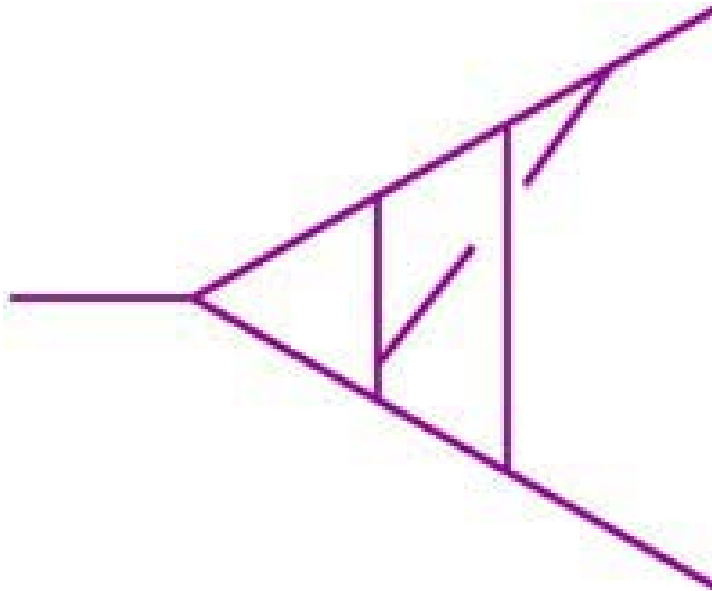


$$\frac{(i\pi^{d/2})^3}{(q^2)^3 v^2} \left[\frac{56\pi^4}{135\epsilon} + \frac{112\pi^4}{135} + \frac{16\pi^2\zeta(3)}{9} + \frac{8\zeta(5)}{3} + O(\epsilon) \right]$$



$$\frac{(i\pi^{d/2})^3}{(q^2)^3 v^2} \left[-\frac{64\pi^4}{135\epsilon} - \frac{128\pi^4}{135} - \frac{32\pi^2\zeta(3)}{9} + \frac{8\zeta(5)}{3} + O(\epsilon) \right]$$

An example of unsolved problem:
three-loop quark form-factor
A92



to be continued