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Anomaly-free discrete family symmetries

- arXiv:0805.1736 [JHEP 07 (2008) 085]
- arXiv:0807.1749 [PLB 670 (2009) 390]

Outline

- motivation
- basic mathematical concepts
- gauge origin & anomalies
- embedding finite into continuous groups
- discrete indices
- discrete anomaly conditions

Fermionic mass structure

quarks

$$m_u : m_c : m_t \sim \lambda_c^8 : \lambda_c^4 : 1$$

$$m_d : m_s : m_b \sim \lambda_c^4 : \lambda_c^2 : 1$$

$$\text{CKM} \sim \begin{pmatrix} 1 & \lambda_c & \lambda_c^3 \\ \lambda_c & 1 & \lambda_c^2 \\ \lambda_c^3 & \lambda_c^2 & 1 \end{pmatrix}$$

$\Rightarrow$ quark masses and mixing are hierarchical

leptons

$$m_e : m_\mu : m_\tau \sim \lambda_c^{4\,\text{or}\,5} : \lambda_c^2 : 1$$

$$m_{\nu_1} : m_{\nu_2} : m_{\nu_3} \sim \begin{cases} \lambda_c^{\geq 1} : \lambda_c : 1 \\ 1 : 1 : \lambda_c^{\geq 1} \\ 1 : 1 : 1 \end{cases}$$

$$\text{MNSP} \sim \begin{pmatrix} 0.8 & 0.5 & < 0.2 \\ 0.4 & 0.6 & 0.7 \\ 0.4 & 0.6 & 0.7 \end{pmatrix}$$

$\Rightarrow$ neutrino sector is different

Tri-bimaximal mixing

$$\mathrm{MNSP} \;\approx\; U_{\mathcal{TB}} \;\equiv\; \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$\Rightarrow$

MNSP-angles	tri-bimax.	3σ exp.
$\sin^2\theta_{12}$:	0.33	$0.24 - 0.40$
$\sin^2\theta_{23}$:	0.50	$0.34 - 0.68$
$\sin^2\theta_{13}$:	0	≤ 0.041

Effective neutrino mass matrix

- in basis where charged lepton mass matrix is diagonal

$$M_\nu \sim M_{\mathcal{TB}} = \begin{pmatrix} \alpha & \beta & \beta \\ \beta & \gamma & \alpha+\beta-\gamma \\ \beta & \alpha+\beta-\gamma & \gamma \end{pmatrix}$$

- How can we obtain such relations between entries of mass matrix?

  $\longrightarrow$ unify families into multiplets of a symmetry group $\mathcal{G}$ (assignment)

  $\longrightarrow$ break $\mathcal{G}$ spontaneously (vacuum alignment)

  $\longrightarrow$ construct invariants of $\mathcal{G}$ inserting vacuum structure

$$\boxed{\mathcal{G} \;=\; \text{non-Abelian finite group}}$$

Abelian finite groups: $\mathcal{Z}_N$

- $\mathcal{Z}_N = \{1 \; , \; e^{2\pi i \frac{1}{N}} \; , \; e^{2\pi i \frac{2}{N}} \; , \; ... \; , \; e^{2\pi i \frac{N-1}{N}}\}$
- N elements
- one generator $a \in \mathcal{Z}_N$ (e.g. $a = e^{2\pi i/N}$)
- group operation = multiplication of complex numbers
- only one-dimensional irreps

Non-Abelian finite groups, e.g. $\mathcal{S}_3$

group multiplication table:

	1	a_1	a_2	b_1	b_2	b_3
1	1	a_1	a_2	b_1	b_2	b_3
a_1	a_1	a_2	1	b_2	b_3	b_1
a_2	a_2	1	a_1	b_3	b_1	b_2
b_1	b_1	b_3	b_2	1	a_2	a_1
b_2	b_2	b_1	b_3	a_1	1	a_2
b_3	b_3	b_2	b_1	a_2	a_1	1

generators and the presentation:

choose generators $a \equiv a_1$ and $b \equiv b_1$ $\quad\Rightarrow\quad$ $a_2 = a^2,\ \ b_2 = ab,\ \ b_3 = ba$

$< a\,,\, b \mid a^3 \;=\; b^2 \;=\; 1\,,\, bab^{-1} \;=\; a^{-1} >$ defines the group uniquely

Irreducible representations

irreducible representations:

$$\mathbf{1}: \quad a = 1, \quad b = 1$$

$$\mathbf{1}': \quad a = 1, \quad b = -1$$

$$\mathbf{2}: \quad a = \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

conjugacy classes:

$$\begin{aligned} 1\,C_1(1) &= \{g\,1\,g^{-1} \,|\, g \in \mathcal{S}_3\} = \{1\} \\ 2\,C_2(a) &= \{g\,a\,g^{-1} \,|\, g \in \mathcal{S}_3\} = \{a,\, a^2\} \\ 3\,C_3(b) &= \{g\,b\,g^{-1} \,|\, g \in \mathcal{S}_3\} = \{b,\, ab,\, ba\} \end{aligned}$$

number of classes	=	number of irreps
number of elements	=	$\sum$ (dimension of irrep)^2

Kronecker products

general formula:

$$\mathbf{r} \otimes \mathbf{s} \;=\; d(\mathbf{r},\mathbf{s},\mathbf{t})\,\mathbf{t}\ , \qquad d(\mathbf{r},\mathbf{s},\mathbf{t}) \;=\; \tfrac{1}{N} \sum_i n_i\, \chi_i^{[\mathbf{r}]}\, \chi_i^{[\mathbf{s}]}\, \bar{\chi}_i^{[\mathbf{t}]}$$

- N = number of group elements
- n_i = number of elements in conjugacy class C_i
- character χ = trace of the matrix representation of element g

$\mathcal{S}_3$	$1C_1(1)$	$2C_2(a)$	$3C_3(b)$
$\chi^{[\mathbf{1}]}$	1	1	1
$\chi^{[\mathbf{1}']}$	1	1	-1
$\chi^{[\mathbf{2}]}$	2	-1	0

$$\mathbf{1}' \otimes \mathbf{1}' = \mathbf{1}$$
$$\mathbf{1}' \otimes \mathbf{2} = \mathbf{2}$$
$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} + \mathbf{1}' + \mathbf{2}$$

Possible discrete family symmetries $\mathcal{G}$

- symmetry group: SM $\times\, \mathcal{G}$
- three families
- $\mathcal{G}$ should have two- or three-dimensional irreps
- $\mathcal{G}$ is a finite subgroup of either
  - $SU(3)$ e.g. $\mathcal{PSL}_2(7),\ \mathcal{Z}_7 \rtimes \mathcal{Z}_3,\ \Delta(3n^2),\ \Delta(6n^2)$
  - $SO(3)$ e.g. $\mathcal{A}_4,\ \mathcal{S}_4,\ \mathcal{A}_5,\ \mathcal{D}_n$
  - $SU(2)$ e.g. $\mathcal{T}',\ \mathcal{Q}_{2n}$

example: $\mathcal{A}_4$ (irreps $\mathbf{1}$, $\mathbf{1}'$, $\overline{\mathbf{1}'}$, $\mathbf{3}$)

$$L \sim \mathbf{3}\,, \qquad E^c \sim \mathbf{1} + \mathbf{1}' + \overline{\mathbf{1}'}\,, \qquad N^c \sim \mathbf{3}$$

Gauging discrete symmetries

- discrete symmetries might be broken badly by quantum gravity effects
- as remnants of a gauge symmetry they are protected

$\longrightarrow$ "discrete gauge symmetry"

$$\boxed{G \supset \mathcal{G}}$$

gauge symmetry discrete symmetry

A N O M A L I E S

Possible anomalies

$$SU(3)_C \times SU(2)_W \times U(1)_Y \times G$$

- $G = U(1)$

$$SU(3)_C - SU(3)_C - U(1) \qquad U(1)_Y - U(1)_Y - U(1)$$

$$SU(2)_W - SU(2)_W - U(1) \qquad U(1)_Y - U(1) - U(1)$$

$$\text{Gravity} - \text{Gravity} - U(1) \qquad U(1) - U(1) - U(1)$$

$\longrightarrow$ constraints on possible $\mathcal{Z}_N \subset U(1)$ symmetries (Ibáñez & Ross)

- $G = SU(3)$

$$SU(3) - SU(3) - SU(3) \qquad SU(3) - SU(3) - U(1)_Y$$

$\longrightarrow$ What can we extract in this case ?

Recap: $G = U(1)$

- formulate anomaly equations for G with $U(1)$ charges z_i

$$\text{structure}: \qquad \sum_i z_i \; = \; 0$$

- insert $U(1)$ charges in terms of discrete charges q_i: $\boxed{z_i \; = \; q_i \bmod N}$

$$\sum_i q_i \; = \; 0 \bmod N$$

- separate light and heavy fermions ($\mathcal{Z}_N$ invariant mass term)

$$\sum_{i=\text{light}} q_i + \underbrace{\sum_{i=\text{heavy}} q_i}_{0 \bmod N \text{ or } \frac{N}{2}} \; = \; 0 \bmod N$$

$$\Longrightarrow \quad \sum_{i=\text{light}} q_i \; = \; 0 \bmod N \text{ or } \frac{N}{2}$$

Cubic anomaly for $G = SU(3)$

- formulate cubic anomaly equation for G with $SU(3)$ irreps $\boldsymbol{\rho}$

$$\sum_{\boldsymbol{\rho}} A(\boldsymbol{\rho}) = 0 \qquad\qquad A(\boldsymbol{\rho}) = \text{cubic Dynkin index}$$

- replace all $SU(3)$ parameters: $\boldsymbol{\rho} \rightarrow \sum_i \mathbf{r_i}$ and $\boxed{A(\boldsymbol{\rho}) = \sum_i \widetilde{A}(\mathbf{r_i}) \bmod N_A}$

$$\sum_i \widetilde{A}(\mathbf{r_i}) = 0 \bmod N_A \qquad\qquad \widetilde{A}(\mathbf{r_i}) = \text{discrete cubic index}$$

- separate light and heavy fermions ($\mathcal{G}$ invariant mass term)

$$\sum_{i=\text{light}} \widetilde{A}(\mathbf{r_i}) + \underbrace{\sum_{i=\text{heavy}} \widetilde{A}(\mathbf{r_i})}_{0 \bmod N_A'} = 0 \bmod N_A$$

$$\Longrightarrow \quad \sum_{i=\text{light}} \widetilde{A}(\mathbf{r_i}) = 0 \bmod N_A'$$

Mixed anomaly for $G = SU(3)$

- formulate mixed anomaly equation for G with $SU(3)$ irreps $\boldsymbol{\rho}$

$$\sum_{\boldsymbol{\rho}} \ell(\boldsymbol{\rho}) \cdot Y(\boldsymbol{\rho}) \;=\; 0 \qquad\qquad \ell(\boldsymbol{\rho}) = \text{quadratic Dynkin index}$$

- replace all $SU(3)$ parameters: $\boldsymbol{\rho} \to \sum_i \mathbf{r_i}$ and $\boxed{\ell(\boldsymbol{\rho}) = \sum_i \widetilde{\ell}(\mathbf{r_i}) \bmod N_\ell}$

$$\sum_i \widetilde{\ell}(\mathbf{r_i}) \cdot Y(\mathbf{r_i}) \;=\; 0 \bmod N_\ell \qquad \widetilde{\ell}(\mathbf{r_i}) = \text{discrete quadratic index}$$

- separate light and heavy fermions ($\mathcal{G}$ invariant mass term)

$$\sum_{i=\text{light}} \widetilde{\ell}(\mathbf{r_i}) \cdot Y(\mathbf{r_i}) + \underbrace{\sum_{i=\text{heavy}} \widetilde{\ell}(\mathbf{r_i}) \cdot Y(\mathbf{r_i})}_{0 \bmod N'_\ell} \;=\; 0 \bmod N_\ell$$

$$\Longrightarrow \quad \sum_{i=\text{light}} \widetilde{\ell}(\mathbf{r_i}) \cdot Y(\mathbf{r_i}) \;=\; 0 \bmod N'_\ell$$

In the following: $\mathcal{G} = \mathcal{Z}_7 \rtimes \mathcal{Z}_3 \quad (\mathcal{T}_7)$

Embedding $\mathcal{T}_7$ into $SU(3)$

irreps of $\mathcal{T}_7$:

$$\mathbf{3},\ \overline{\mathbf{3}},\ \mathbf{1},\ \mathbf{1}',\ \overline{\mathbf{1}'}$$

some $\mathcal{T}_7$ Kronecker products:

$$\mathbf{3} \otimes \mathbf{3} = (\mathbf{3} + \overline{\mathbf{3}})_s + \overline{\mathbf{3}}_a$$

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} + \mathbf{1}' + \overline{\mathbf{1}'} + \mathbf{3} + \overline{\mathbf{3}}$$

in $SU(3)$:

$$\mathbf{3} \otimes \mathbf{3} = \mathbf{6}_s + \overline{\mathbf{3}}_a$$

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} + \mathbf{8}$$

$SU(3) \supset \mathcal{T}_7$
$(10): \ \mathbf{3} = \mathbf{3}$
$(01): \ \overline{\mathbf{3}} = \overline{\mathbf{3}}$
$(20): \ \mathbf{6} = \mathbf{3} + \overline{\mathbf{3}}$
$(11): \ \mathbf{8} = \mathbf{1}' + \overline{\mathbf{1}'} + \mathbf{3} + \overline{\mathbf{3}}$
$(30): \ \mathbf{10} = \mathbf{1} + \mathbf{3} + 2 \cdot \overline{\mathbf{3}}$
$(21): \ \mathbf{15} = \mathbf{1} + \mathbf{1}' + \overline{\mathbf{1}'} + 2 \cdot (\mathbf{3} + \overline{\mathbf{3}})$
$(40): \ \mathbf{15}' = \mathbf{1} + \mathbf{1}' + \overline{\mathbf{1}'} + 2 \cdot (\mathbf{3} + \overline{\mathbf{3}})$
$(05): \ \mathbf{21} = \mathbf{1} + \mathbf{1}' + \overline{\mathbf{1}'} + 3 \cdot (\mathbf{3} + \overline{\mathbf{3}})$
$(13): \ \mathbf{24} = \mathbf{1} + \mathbf{1}' + \overline{\mathbf{1}'} + 3 \cdot \mathbf{3} + 4 \cdot \overline{\mathbf{3}}$
$(22): \ \mathbf{27} = \mathbf{1} + \mathbf{1}' + \overline{\mathbf{1}'} + 4 \cdot (\mathbf{3} + \overline{\mathbf{3}})$

Irreps $\boldsymbol{\rho}$ of $SU(3)$ and their indices

cubic index: $$\mathrm{Tr}\left(\left\{T_a^{[\boldsymbol{\rho}]}\,,\,T_b^{[\boldsymbol{\rho}]}\right\}\,T_c^{[\boldsymbol{\rho}]}\right) \;=\; A(\boldsymbol{\rho})\,\frac{d_{abc}}{2}$$

quadratic index: $$\mathrm{Tr}\left(\left\{T_a^{[\boldsymbol{\rho}]}\,,\,T_b^{[\boldsymbol{\rho}]}\right\}\right) \;=\; \ell(\boldsymbol{\rho})\,\delta_{ab}$$

Irreps $\boldsymbol{\rho}$ of $SU(3)$	$A(\boldsymbol{\rho})$	$\ell(\boldsymbol{\rho})$
$(10):\quad \mathbf{3}$	1	1
$(20):\quad \mathbf{6}$	7	5
$(11):\quad \mathbf{8}$	0	6
$(30):\quad \mathbf{10}$	27	15
$(21):\quad \mathbf{15}$	14	20

Irreps $\mathbf{r_i}$ of $\mathcal{T}_7$ and their indices

- $\mathbf{r_i}$ can originate from different irreps $\boldsymbol{\rho}$ of $SU(3)$

$$\boldsymbol{\rho} \longrightarrow \sum_i \mathbf{r_i}$$

- define discrete indices $\widetilde{A}(\mathbf{r_i})$, $\widetilde{\ell}(\mathbf{r_i})$ and N_A, N_ℓ such that:

$$\begin{aligned} A(\boldsymbol{\rho}) &= \sum_i \widetilde{A}(\mathbf{r_i}) \mod N_A \\ \ell(\boldsymbol{\rho}) &= \sum_i \widetilde{\ell}(\mathbf{r_i}) \mod N_\ell \end{aligned}$$

for all irreps $\boldsymbol{\rho}$ of $SU(3)$

Discrete indices of $\mathcal{T}_7$

definition:

Irreps $\mathbf{r_i}$ of $\mathcal{T}_7$	$\widetilde{A}(\mathbf{r_i})$ $(N_A = 7)$	$\widetilde{\ell}(\mathbf{r_i})$ $(N_\ell = 3)$
$\mathbf{1}'$	x	y
$\overline{\mathbf{1}'}$	$-x$	$1-y$
$\mathbf{3}$	1	1
$\overline{\mathbf{3}}$	-1	1

consistency:

$\boldsymbol{\rho}$	$\sum_i \mathbf{r_i}$	$A(\boldsymbol{\rho})$	$\sum_i \widetilde{A}(\mathbf{r_i})$	$\ell(\boldsymbol{\rho})$	$\sum_i \widetilde{\ell}(\mathbf{r_i})$
$\mathbf{3}$	$\mathbf{3}$	1	1	1	1
$\mathbf{6}$	$\mathbf{3}+\overline{\mathbf{3}}$	7	0	5	2
$\mathbf{8}$	$\mathbf{1}'+\overline{\mathbf{1}'}+\mathbf{3}+\overline{\mathbf{3}}$	0	0	6	3
$\mathbf{10}$	$\mathbf{1}+\mathbf{3}+2\cdot\overline{\mathbf{3}}$	27	-1	15	3
$\mathbf{15}$	$\mathbf{1}+\mathbf{1}'+\overline{\mathbf{1}'}+2\cdot(\mathbf{3}+\overline{\mathbf{3}})$	14	0	20	5

Proving the assignment of discrete indices

- assign discrete indices to irreps $\mathbf{r_i}$ using the decomposition of the smallest irreps $\boldsymbol{\rho}$ of $SU(3)$
- proof by induction that this assignment is consistent for all higher irreps $\boldsymbol{\rho}$
- make use of:

$$\boxed{I(\boldsymbol{\rho} \otimes \boldsymbol{\sigma}) \;=\; d(\boldsymbol{\rho})\, I(\boldsymbol{\sigma}) \;+\; I(\boldsymbol{\rho})\, d(\boldsymbol{\sigma})}$$

$$\begin{aligned} I &= \text{Dynkin index } A \text{ or } \ell \\ d &= \text{dimension of irrep} \end{aligned}$$

Discrete anomaly conditions

family symmetry $SU(3)$ particles live in irreps $\boldsymbol{\rho}$ [normalization: $Y(\boldsymbol{\rho}) \in \mathbb{Z}$]

$$\sum_{\boldsymbol{\rho}} A(\boldsymbol{\rho}) = 0 \qquad \sum_{\boldsymbol{\rho}} \ell(\boldsymbol{\rho}) \cdot Y(\boldsymbol{\rho}) = 0$$

$\downarrow$

family symmetry $\mathcal{T}_7$ particles live in irreps $\mathbf{r_i}$

- some acquire masses (heavy)
- some don't (light)

$$\sum_{i=\text{light}} \widetilde{A}_i + \sum_{i=\text{heavy}} \widetilde{A}_i = 0 \bmod N_A$$
$$\sum_{i=\text{light}} \widetilde{\ell}_i\, Y_i + \sum_{i=\text{heavy}} \widetilde{\ell}_i\, Y_i = 0 \bmod N_\ell$$

Effect of heavy fermions

- $\mathcal{T}_7$ invariant mass terms: $\mathbf{1}' \otimes \overline{\mathbf{1}'}$, $\mathbf{3} \otimes \overline{\mathbf{3}}$
- with $\mathcal{G} = \mathcal{T}_7$: only Dirac particles can be massive
- contribution of heavy fermions to discrete anomalies:

$$\mathbf{3} \otimes \overline{\mathbf{3}} : \quad \sum_{i=1}^{2} \widetilde{A}_i \; = \; 0 \qquad \sum_{i=1}^{2} \widetilde{\ell}_i \cdot Y_i \; = \; 0$$

$$\mathbf{1}' \otimes \overline{\mathbf{1}'} : \quad \sum_{i=1}^{2} \widetilde{A}_i \; = \; 0 \qquad \sum_{i=1}^{2} \widetilde{\ell}_i \cdot Y_i \; = \; (2y-1) \cdot Y(\mathbf{1}') \; \overset{!}{=} \; 0 \bmod 3$$

$\longrightarrow$ no contribution if $y = \frac{1}{2}$ or 2 (and integer hypercharge normalization)

$\mathcal{T}_7$ example: Luhn, Nasri, Ramond [PLB 652,27 (2007)]

– all hypercharges are integer

– hypercharge normalization $Y_Q = 1$

$$\left.\right\} \quad \begin{array}{lcl} \sum_{i=\text{light}} \widetilde{A}_i & = & 0 \bmod 7 \\ \sum_{i=\text{light}} \widetilde{\ell}_i\, Y_i & = & 0 \bmod 3 \end{array}$$

type of fermion	Q	U^c	D^c	L	E^c	N^c
$\mathcal{T}_7$ irrep	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$
# of fermions	6	3	3	2	1	1
hypercharge Y	1	-4	2	-3	6	0

$\widetilde{A}(\mathbf{3}) = 1$

$\widetilde{\ell}(\mathbf{3}) = 1$

$$\sum_{i=\text{light}} \widetilde{A}_i = 6+3+3+2+1+1 = 16 \neq 0 \bmod 7$$

$$\sum_{i=\text{light}} \widetilde{\ell}_i \cdot Y_i = 6\cdot 1 + 3\cdot(-4) + 3\cdot 2 + 2\cdot(-3) + 1\cdot 6 + 1\cdot 0 = 0$$

What does it mean?

discrete cubic anomaly is *not* satisfied

- light particle content of the model is *incomplete*
- some fermions which transform non-trivially under $\mathcal{G}$ have to be added
- these fermions must be massless at energies above $E_{\not{\mathcal{G}}}$
- if $E_{\not{\mathcal{G}}} \sim$ EWSB scale $\longrightarrow$ these fermions might show up in experiments

discrete mixed anomaly is *not* satisfied

- light particle content of the model is *incomplete* (as above)

or

- some heavy fermions carry *"fractional"* hypercharge, i.e. $\frac{Y_{\mathrm{heavy}}}{Y_Q} \notin \mathbb{Z}$
  $\longrightarrow$ electrically charged dark matter

Conclusion

- neutrino mixing motivates non-Abelian discrete family symmetry $\mathcal{G}$
- many candidates for $\mathcal{G}$ – many more models
- require gauge origin of $\mathcal{G} \subset G = SU(3), SO(3), SU(2)$
- discuss remnants of the high-energy anomaly conditions
- introduce discrete indices (individually for each group $\mathcal{G}$)
- discrete anomaly conditions
- some models might be incomplete in their light particle content

Different embeddings

- some groups $\mathcal{G}$ are subgroups of different continuous groups G
- e.g. $\mathcal{A}_4 \subset SU(3)$ or $\mathcal{A}_4 \subset SO(3)$
- Is it possible to define discrete indices independently from the embedding?

Embedding $\mathcal{A}_4$ into $SU(3)$

discrete indices of $\mathcal{A}_4$:

Irreps $\mathbf{r_i}$ of $\mathcal{A}_4$	$\widetilde{\ell}(\mathbf{r_i})$ $(N_\ell = 12)$	$\widetilde{A}(\mathbf{r_i})$ $(N_A = 2)$
$\mathbf{1}$	0	0
$\mathbf{1}'$	2	0
$\overline{\mathbf{1}'}$	2	0
$\mathbf{3}$	1	1

$\boldsymbol{\rho}$	$\sum_i \mathbf{r_i}$	$\ell(\boldsymbol{\rho})$	$\sum_i \widetilde{\ell}(\mathbf{r_i})$	$A(\boldsymbol{\rho})$	$\sum_i \widetilde{A}(\mathbf{r_i})$
$\mathbf{3}$	$\mathbf{3}$	1	1	1	1
$\mathbf{6}$	$\mathbf{1} + \mathbf{1}' + \overline{\mathbf{1}'} + \mathbf{3}$	5	5	7	1
$\mathbf{8}$	$\mathbf{1}' + \overline{\mathbf{1}'} + 2\cdot\mathbf{3}$	6	6	0	2
$\mathbf{10}$	$\mathbf{1} + 3\cdot\mathbf{3}$	15	3	27	3
$\mathbf{15}$	$\mathbf{1} + \mathbf{1}' + \overline{\mathbf{1}'} + 4\cdot\mathbf{3}$	20	8	14	4

Embedding $\mathcal{A}_4$ into $SO(3)$

discrete indices of $\mathcal{A}_4$:

Irreps $\mathbf{r_i}$ of $\mathcal{A}_4$	$\widetilde{\ell}(\mathbf{r_i})$ $(N_\ell = 12)$	
$\mathbf{1}$	0	
$\mathbf{1}'$	2	
$\overline{\mathbf{1}'}$	2	
$\mathbf{3}$	1	

$\boldsymbol{\rho}$	$\sum_i \mathbf{r_i}$	$\ell(\boldsymbol{\rho})$	$\sum_i \widetilde{\ell}(\mathbf{r_i})$		
$\mathbf{3}$	$\mathbf{3}$	1	1		
$\mathbf{5}$	$\mathbf{1}' + \overline{\mathbf{1}'} + \mathbf{3}$	5	5		
$\mathbf{7}$	$\mathbf{1} + 2\cdot\mathbf{3}$	14	2		
$\mathbf{9}$	$\mathbf{1} + \mathbf{1}' + \overline{\mathbf{1}'} + 2\cdot\mathbf{3}$	30	6		
$\mathbf{11}$	$\mathbf{1}' + \overline{\mathbf{1}'} + 3\cdot\mathbf{3}$	55	7		