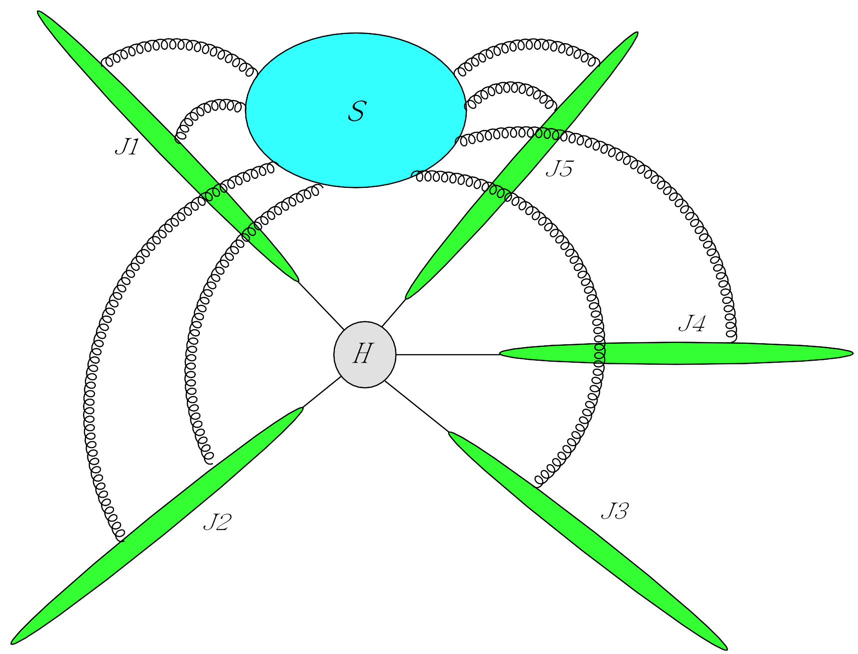
Infrared singularities in QCD amplitudes

Einan Gardi (Edinburgh)



Based on a recent paper with Lorenzo Magnea: *Factorization constraints for soft anomalous dimensions in QCD scattering amplitudes*, [arXiv:0901.1091] JHEP 0903:079, 2009.

Plan of the talk

- why study infrared singularties?
- factorization of multi-leg amplitudes
- partonic and eikonal jets
- the soft anomalous dimension
  - appetiser: surprise at two loops
  - rescaling, cusp anomaly and factorization constraints
  - sum-over-dipoles formula and possible corrections

Why study infrared singularties?

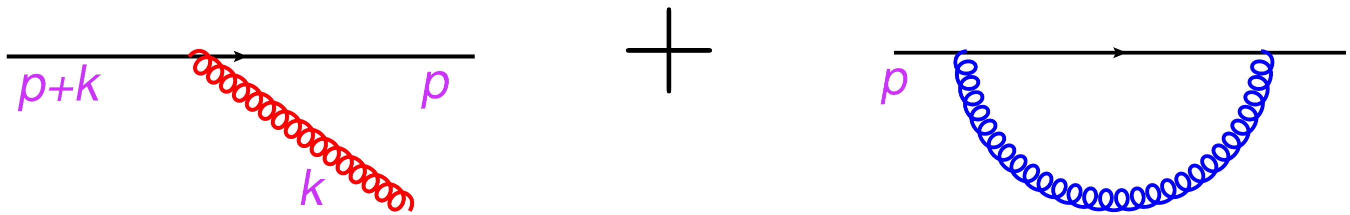
- Effective cancellation of infrared singularities: a major obstacle in multi-loop cross–section calculations.
- Infrared singularities are the key to resummation
  - parton showers
  - dedicated precision calculations
- Theory: Infrared singularities open a window into the all-order structure of perturbation theory and beyond

  Major progress in understanding $\mathcal{N} = 4$ SYM amplitudes at large N_c:
  - AdS-CFT allows strong coupling calculations (e.g. of the cusp anomalous dimension γ_K)
  - Integrability $\Longrightarrow$ Beisert-Eden-Staudacher equation for γ_K
  - BDS conjecture: iterative structure of amplitudes
  - Amplitude – Wilson loop duality

  But colour correlations only appear at finite N_c.

Why study infrared singularities?

Cross sections are finite only upon summing real and virtual contributions

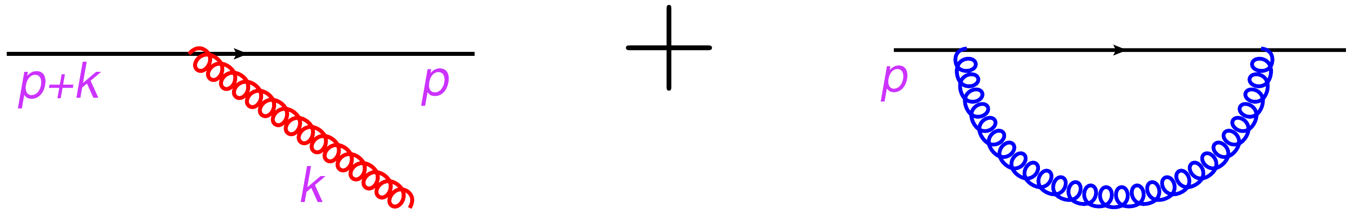


Dim. reg.: $$\frac{1}{\epsilon}\Bigg[\underbrace{\left(Q^2/m_j^2\right)^{\epsilon}}_{\text{real}}\underbrace{-\;1}_{\text{virtual}}\Bigg] \qquad \Longrightarrow \qquad \ln(Q^2/m_{\rm jet}^2)$$

- cross sections calculations:
  - For general kinematics (and cuts!) phase–space integration must be done numerically. $\Longrightarrow$ need to know the singularities before we start the calculation.
  - It is possible: the singular terms are universal.
  - At one loop we have general algorithms, allowing to determine and subtract the singularities for general kinematics. Example: dipole subtraction [Catani Seymour (1996)].
  - Such are needed in multi-loop calculations.

Why study infrared singularities? (II) resummation

Cross sections are finite only upon summing real and virtual contributions



Dim. reg.: $$\frac{1}{\epsilon}\left[\underbrace{\left(Q^2/m_j^2\right)^{\epsilon}}_{\text{real}}\ \underbrace{-\ 1}_{\text{virtual}}\right] \qquad \Longrightarrow \qquad \ln(Q^2/m_{\text{jet}}^2)$$

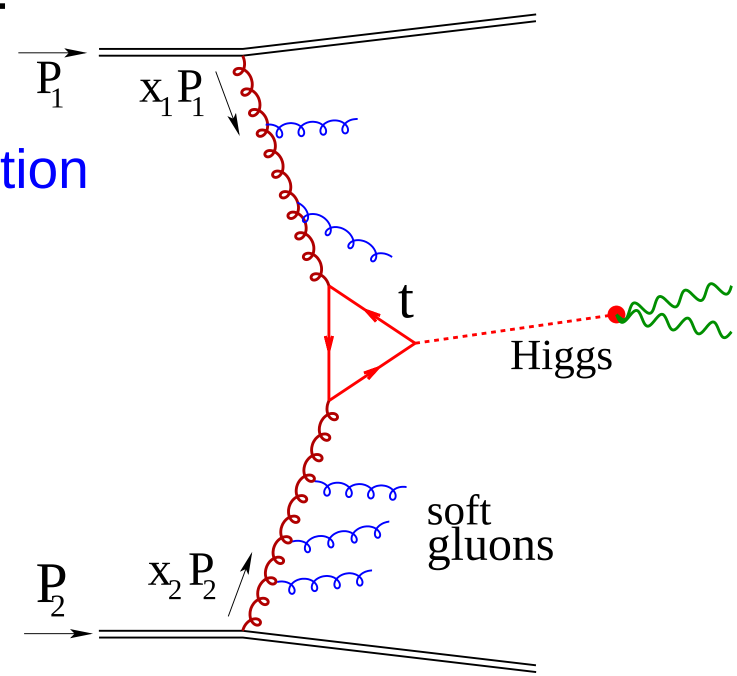
- cross sections calculations.
- Resummation: The logarithms are often large, and they spoil the convergence of the expansion in α_s: $\sum_{n,k} C_{n,k}\alpha_s^n \ln^k\left(Q^2/m_{\text{jet}}^2\right)$ But, knowing the singularity structure, they can be resummed to all orders — they exponentiate:
  - parton showers
  - dedicated precision calculations

Resummation: Example 1 — Higgs production at the LHC

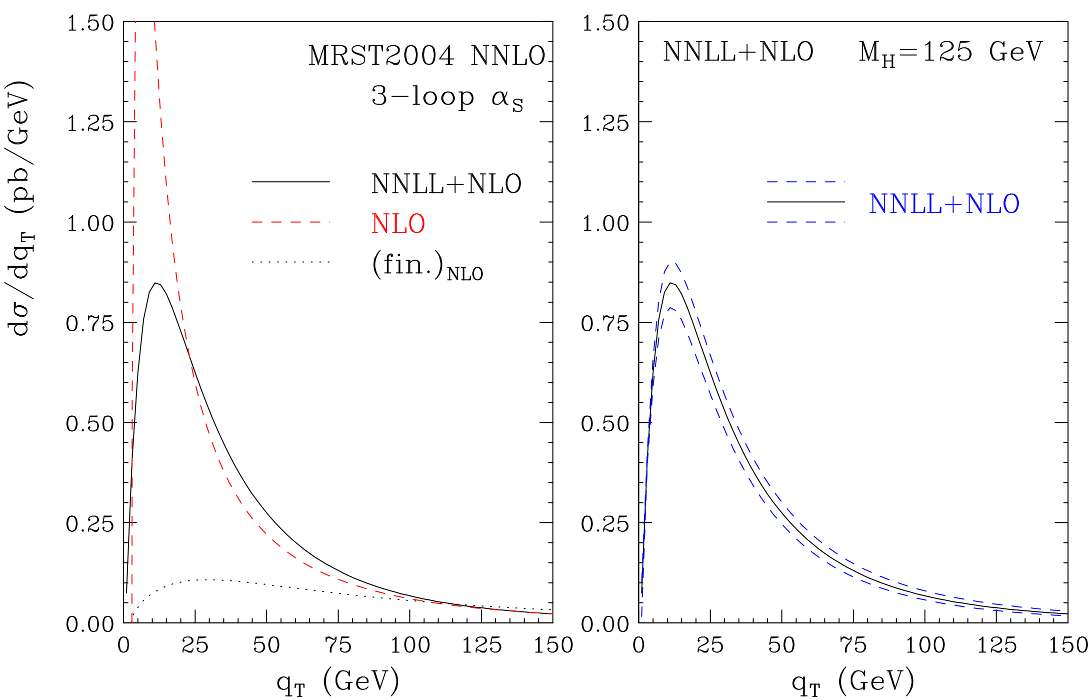
The main Higgs production channel: $gg \longrightarrow H + X$

gluon density $\Longrightarrow$ Higgs production occurs near partonic threshold:

- the total energy of gluons in the final state:
  $E_X = (\hat{s} - m_H^2)/2m_H \to 0$
- multiple soft gluon emission $\Longrightarrow$ resummation



Higgs P_T distribution [Bozzi et al. '05]

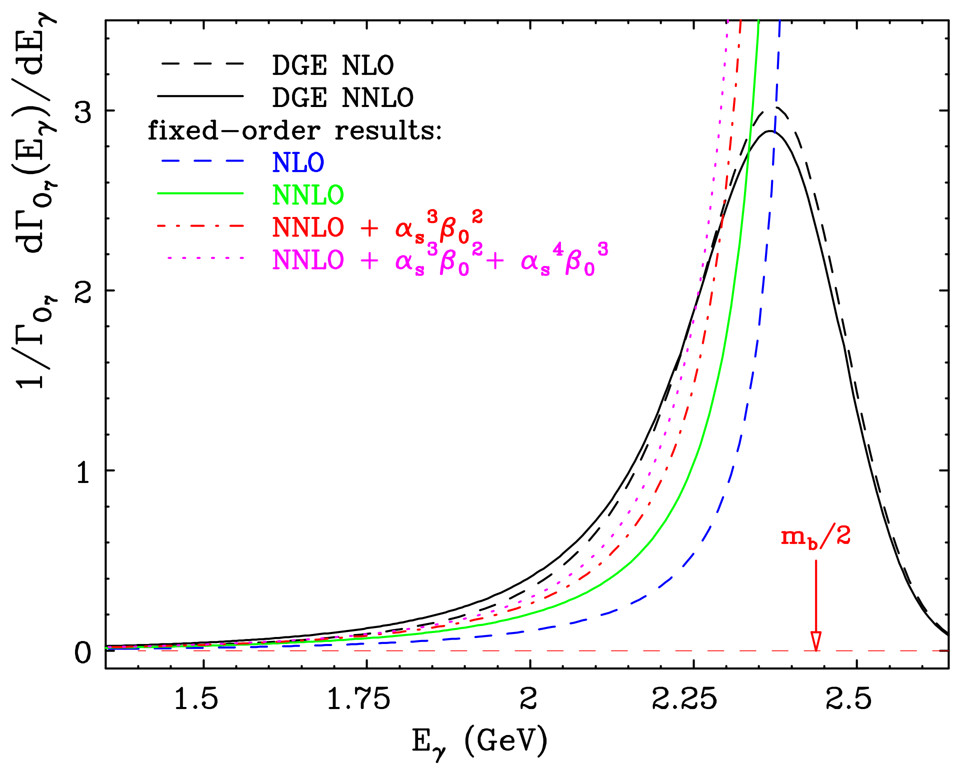


Resummation: Example 2 — precision flavour physics

Inclusive decay spectra in the B factories: $\bar{B} \to X_s \gamma$, $\bar{B} \to X_u l \nu$, ...

Resummation: Korchemsky Sterman; Bauer Fleming Pirjol Stewart (SCET); Lange Neubert Paz; Andersen Gardi, Aglietti et al., ...

$\bar{B} \to X_s \gamma$ spectrum [Andersen & Gardi '06]

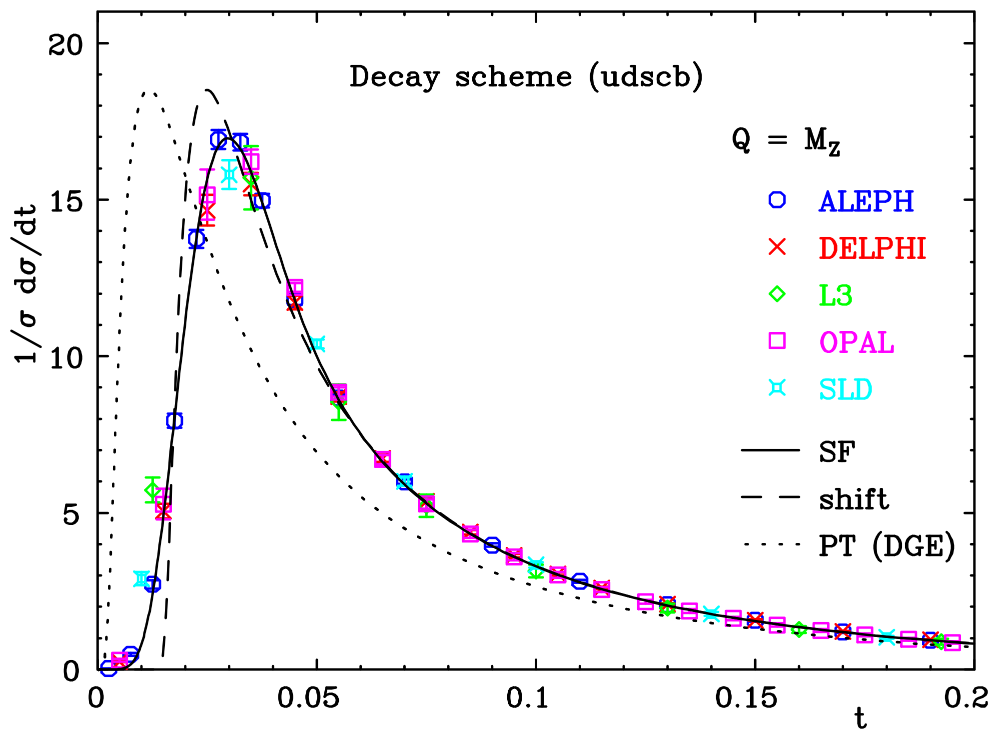


- $\bar{B} \to X_s \gamma$ branching fraction — bounds on new physics
- precise determination of $|V_{ub}|$!

Resummation: Example 3 — jet cross sections

Jets in $e^+e^- \to \mathrm{hadrons}$ — extensively studied at LEP
[Catani Trentadue Turnock Webber (92); Korchemsky Sterman (95);
Dokshitzer Webber (95), Dokshitzer Marchesini Webber (96), ...]

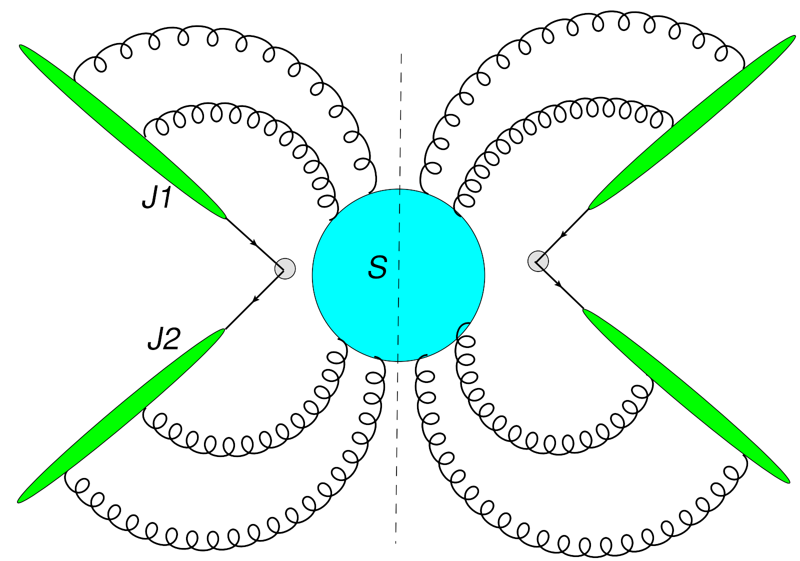
thrust distribution [Gardi & Rathsman '02]



- Determination of the strong coupling
- Quantitative understanding hadronization corrections

Factorization in inclusive cross sections

- Sudakov resummation in inclusive cross sections is well understood
- Soft and Jet sub-processes are incoherent $\Longrightarrow$ factorization
- Each sub-process is associated with
  - a single scale
  - a unique anomalous dimension — a function of the running coupling only: internal resummation
  - the overlap: the cusp anomalous dimension γ_K

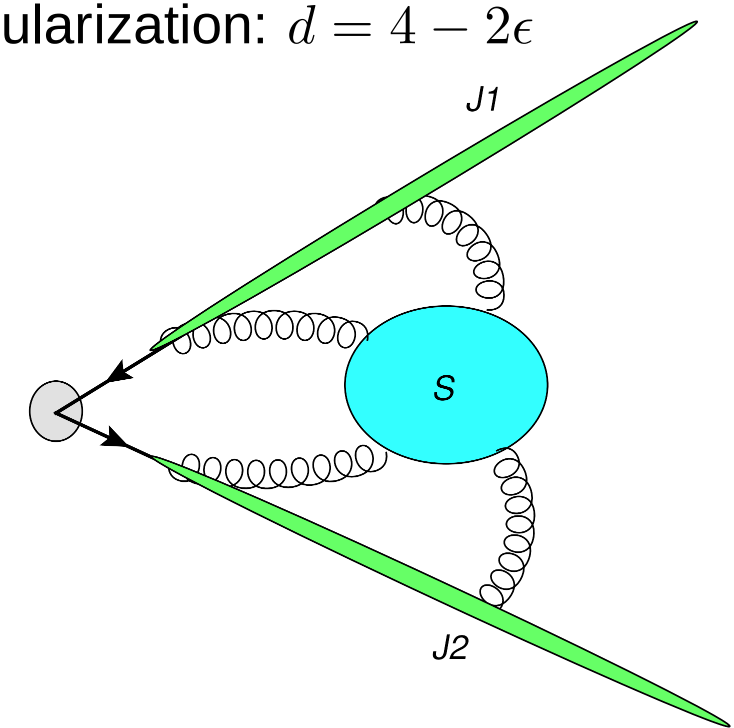


$$\mathrm{Sud}(m^2, N) = \exp\left\{ C_R \int_0^1 \frac{dr}{r} \Big[\underbrace{(1-r)^{N-1}}_{\text{real}} \underbrace{-1}_{\text{virtual}} \Big] R(m^2, r) \right\},$$

$$C_R \frac{R(m^2, r)}{r} = -\frac{1}{r} \left[\int_{r^2 m^2}^{r m^2} \frac{dk^2}{k^2} \gamma_K \left(\alpha_s(k^2)\right) + 2\mathcal{B}\left(\alpha_s(r m^2)\right) - 2\mathcal{D}\left(\alpha_s(r^2 m^2)\right) \right]$$

Factorization in a two-jet process: amplitude level

- Consider the amplitude (virtual corrections only). The logarithms we discussed before originate in infrared singularities of the amplitude:
  - gluons that are collinear to the hard external partons
  - soft gluons
- Need an infrared regulator (instead of the Mellin moment index N). Dimensional regularization: $d = 4 - 2\epsilon$

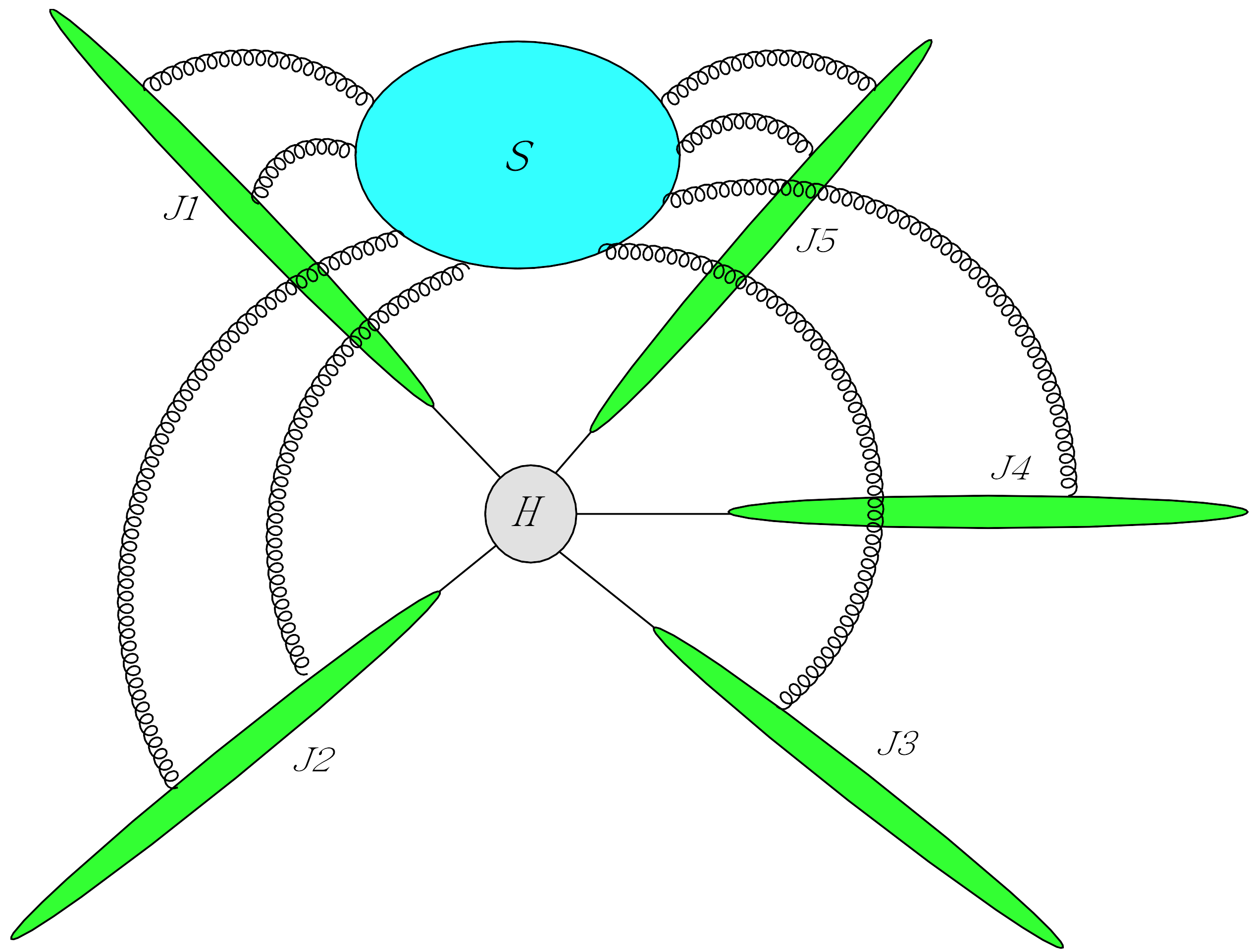


Sudakov form factor was extensively studied: Mueller (79), Collins (80), Sen (81), Korchemsky (89), Sterman and Magnea (90),...

Factorization of a multi-leg amplitude

Fixed–angle scattering amplitude in a massless gauge theory ($p_i^2 = 0$)

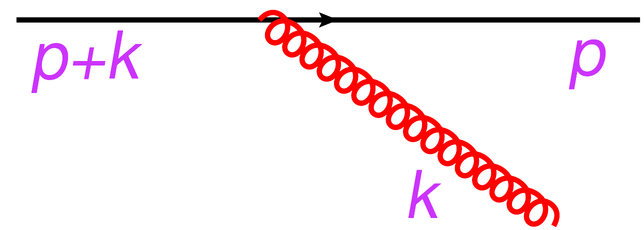
Mueller (81)
Sen (83)
Botts Sterman (89)
Kidonakis Oderda Sterman (98)
Catani (98)
Tejeda-Yeomans Sterman (02)
Kosower (03)
Aybat Dixon Sterman (06)
Becher Neubert (09)
Gardi Magnea (09)



Eikonal approximation

Eikonal Feynman rules

gluon emission in the limit $k \to 0$:



$$\bar{u}(p)\left(-\mathrm{i}g_s T^{(a)}\gamma^\mu\right)\frac{\mathrm{i}(\not{p}+\not{k}+m)}{(p+k)^2-m^2+\mathrm{i}\varepsilon} \longrightarrow \bar{u}(p)g_s T^{(a)}\frac{p^\mu}{p\cdot k+\mathrm{i}\varepsilon}$$

- Valid when all momentum components of k are small (not valid when k is collinear to p but hard)
- Only the direction and the colour charge of the emitter are important. Rescaling invariance: $\beta \propto p$

$$g_s T^{(a)}\frac{p^\mu}{p\cdot k+\mathrm{i}\varepsilon} = g_s T^{(a)}\frac{\beta^\mu}{\beta\cdot k+\mathrm{i}\varepsilon}$$

- Equivalent to radiation off a Wilson line along the quark trajectory:

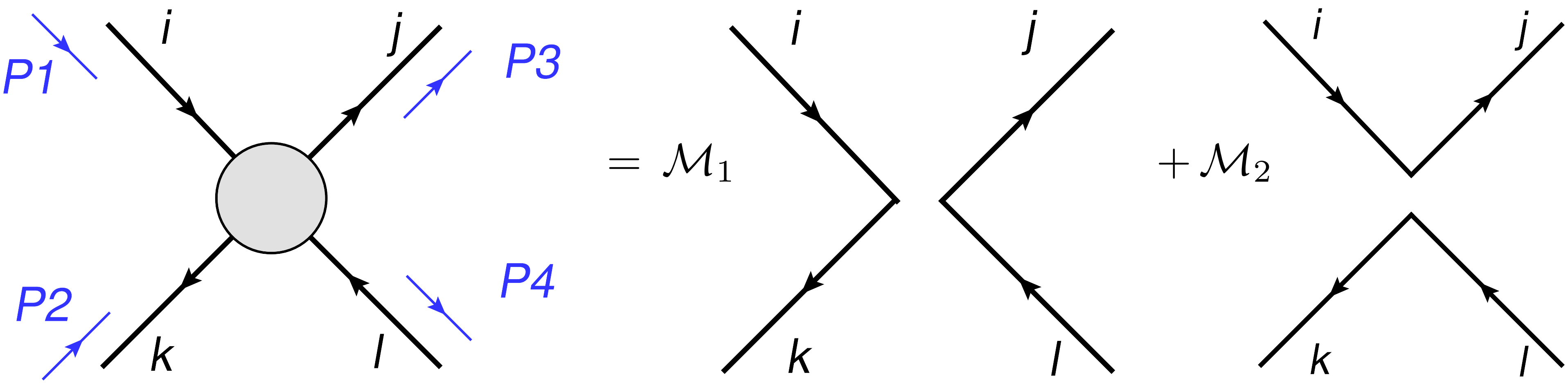
$$P\exp\left\{\mathrm{i}g_s\int_0^\infty d\lambda\beta\cdot A(\lambda\beta)\right\}$$

Colour flow

Decompose the amplitude in a colour basis (independent colour tensors with the index structure of the external partons):

Example:

$q(p_1)\bar{q}(p_2) \to q(p_3)\bar{q}(p_4)$ $\qquad c_1 = \delta_{ik}\delta_{jl}$ $\qquad c_2 = \delta_{ij}\delta_{kl}$



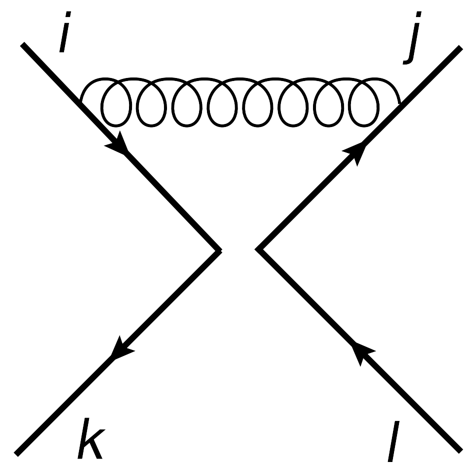
In general:

$$\mathcal{M}_{\{\alpha_i\}}\left(p_i/\mu, \epsilon\right) = \sum_{L=1}^{n_{\rm rep}} \mathcal{M}_L\left(p_i/\mu, \epsilon\right)\,\left(c_L\right)_{\{\alpha_i\}}$$

$n_{\rm rep}$ is the number of elements in the basis (number of irreducible representations that can be constructed with the given external particles).

Gluon exchange mixes between the two states

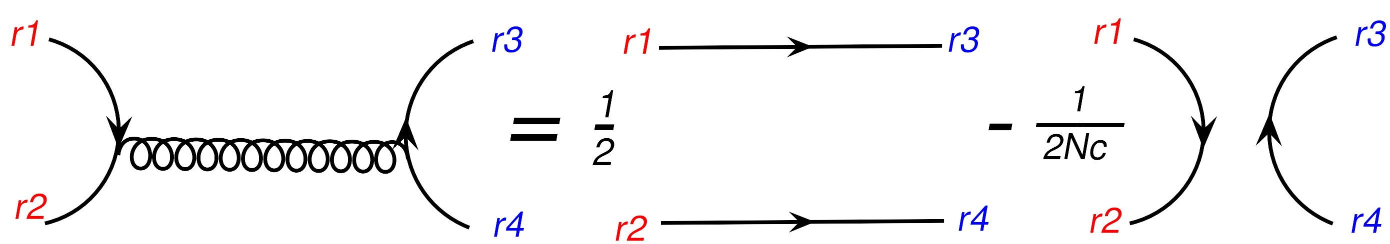
Dress c_1 by a soft gluon:



$$t^a_{ii'} t^a_{j'j} \overbrace{c_1}^{\delta_{i'k}\delta_{lj'}} = t^a_{ik} t^a_{lj} = \frac{1}{2}\delta_{ij}\delta_{kl} - \frac{1}{2N_c}\delta_{ik}\delta_{jl}$$

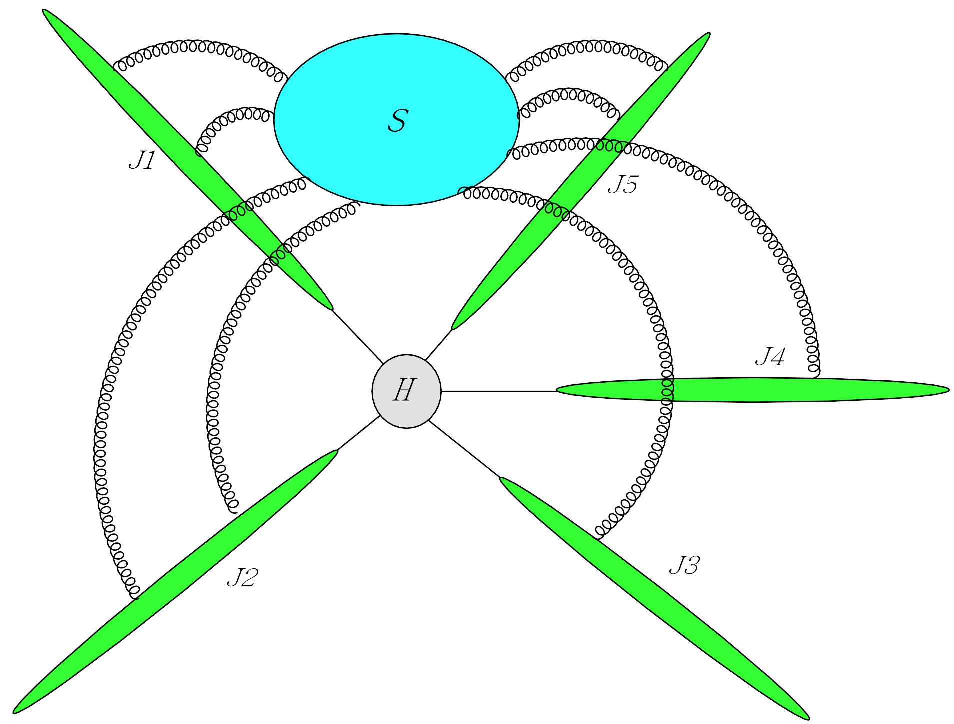
$$= \frac{1}{2}c_2 - \frac{1}{2N_c}c_1$$

Fierz Identity: $\sum_a t^a_{r_2 r_1} t^a_{r_3 r_4} = \frac{1}{2}\delta_{r_1 r_3}\delta_{r_2 r_4} - \frac{1}{2N_c}\delta_{r_2 r_1}\delta_{r_3 r_4}$



Factorization of a multi-leg amplitude

- All singularities are in $\mathcal{S}$, $J_i/\mathcal{J}_i$.
- colour:
  $\mathcal{S}$ is a matrix acting on H
- kinematics:
  $\mathcal{S}$ depends on all velocities;
  $J_i/\mathcal{J}_i$ depends on a single p_i

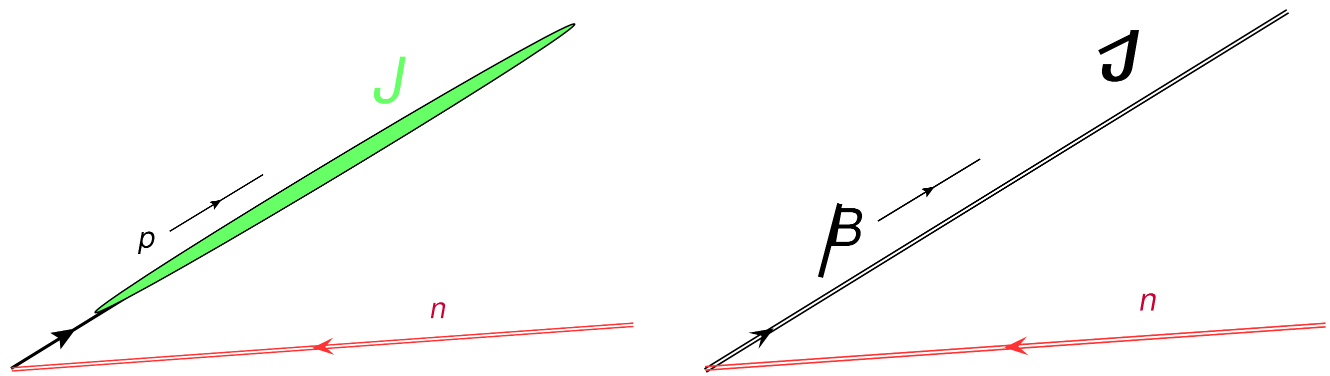


$$\mathcal{M}_N\left(p_i/\mu, \epsilon\right) = \sum_L \mathcal{S}_{NL}\left(\beta_i \cdot \beta_j, \epsilon\right)\, H_L\left(\frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}\right)$$
$$\times \prod_{i=1}^{n} J_i\left(\frac{(2p_i \cdot n_i)^2}{n_i^2\mu^2}, \epsilon\right) \Big/ \mathcal{J}_i\left(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \epsilon\right)$$

To avoid double counting of the soft-collinear region: $\mathcal{J}_i$ removes from J_i its eikonal part, which is already taken into account in $\mathcal{S}$.

The jet function: definition

- Introduce auxiliary vectors n_i ($n_i^2 \neq 0$) to separate collinear regions. Intuitive picture: jet i contains gluons (k) such that: $k \cdot p_i < n_i \cdot p_i$
- Define a gauge–invariant jet using a Wilson line along a ray n_i.

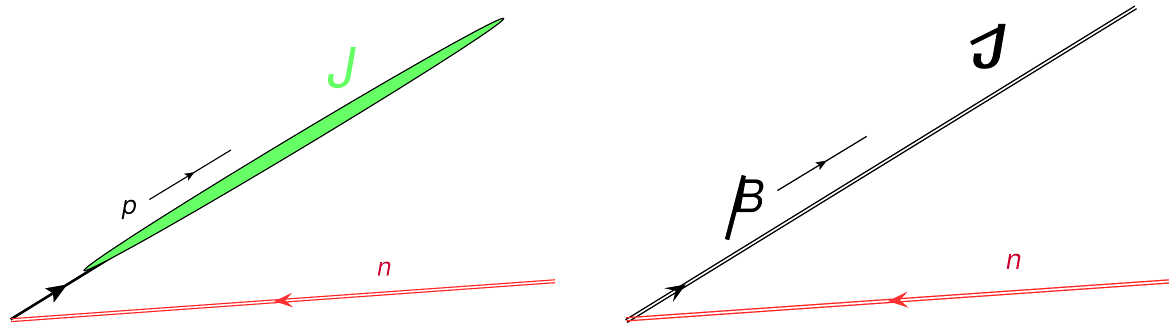


partonic jet: $$\overline{u}(p)\, J\left(\frac{(2p\cdot n)^2}{n^2\mu^2}, \epsilon\right) = \langle p \,|\, \overline{\psi}(0)\, \Phi_n(0,-\infty)\, |0\rangle$$

where $$\Phi_n(\lambda_2,\lambda_1) = P\exp\left[\mathrm{i}g\int_{\lambda_1}^{\lambda_2} d\lambda\, n\cdot A(\lambda n)\right]$$

eikonal jet: $$\mathcal{J}\left(\frac{2(\beta\cdot n)^2}{n^2}, \epsilon\right) = \langle 0|\, \Phi_\beta(\infty,0)\, \Phi_n(0,-\infty)\, |0\rangle$$

Jet functions: evolution equations



partonic jet: $$\overline{u}(p)\, J\left(\frac{(2p\cdot n)^2}{n^2\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \langle p \,|\, \overline{\psi}(0)\, \Phi_n(0,-\infty)\,|0\rangle$$

eikonal jet: $$\mathcal{J}\left(\frac{2(\beta\cdot n)^2}{n^2}, \alpha_s(\mu^2), \epsilon\right) = \langle 0 |\, \Phi_\beta(\infty,0)\, \Phi_n(0,-\infty)\,|0\rangle\,.$$

These operators are multiplicatively renormalizable $\Longrightarrow$ evolution equations:

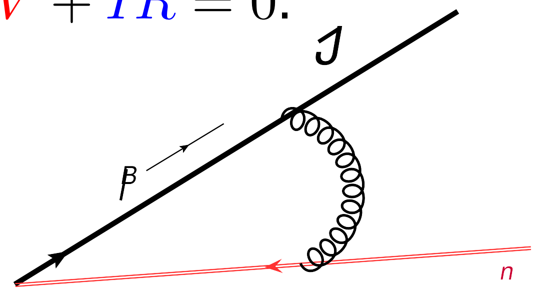
$$\mu\frac{d}{d\mu}\ln J_i\left(\frac{(2p\cdot n)^2}{n^2\mu^2}, \alpha_s(\mu^2), \epsilon\right) = -\,\gamma_{J_i}(\alpha_s(\mu^2))$$

$$\mu\frac{d}{d\mu}\ln \mathcal{J}_i\left(w_i, \alpha_s(\mu^2), \epsilon\right) \equiv -\gamma_{\mathcal{J}_i} = \frac{1}{2} G_{\mathcal{J}_i}\left(w_i, \alpha_s\right) - \underbrace{\frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\gamma_K^{(i)}\left(\alpha_s(\lambda^2,\epsilon)\right)}_{\mathcal{O}(1/\epsilon)}$$

$w_i \equiv 2(\beta_i\cdot n_i)^2/n_i^2$ [EG & Magnea (09), based on Sen (81), Korchemsky (89)...]

The eikonal jet and the cusp anomaly

- $\mathcal{J}$ doesn't depend on any kinematic scale; radiative corrections — only due to renormalization: In dimensional regularization (without dimensionful cutoff): $UV + IR = 0$.

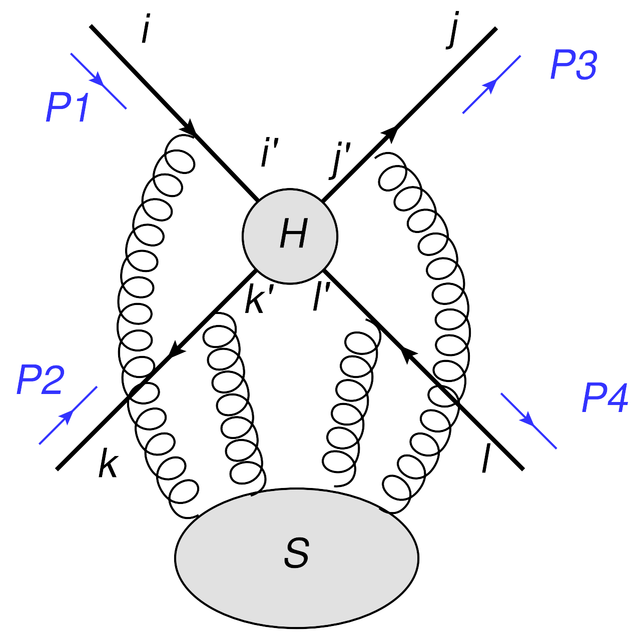


- for $\beta^2 \neq 0$: finite anom. dim., rescaling invariance: $(\beta \cdot n)^2/(\beta^2 n^2)$
- for $\beta^2 = 0$: overlapping ultraviolet and collinear singularities $\Longrightarrow$
  - double poles
  - single poles that carry $(\beta \cdot n)^2/n^2$ dependence, violating classical rescaling symmetry wrt β. This is the cusp anomaly!

$$\mathcal{J}_i\left(\frac{2(\beta\cdot n)^2}{n^2},\epsilon\right) = \exp\left\{\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left[\frac{1}{4}\delta_{\mathcal{J}_i}\Big(\alpha_s(\lambda^2,\epsilon)\Big) - \frac{1}{8}\gamma_K^{(i)}\Big(\alpha_s(\lambda^2,\epsilon)\Big)\,\ln\left(\frac{2(\beta\cdot n)^2\mu^2}{n^2\lambda^2}\right)\right]\right\}$$

The double poles as well as the entire kinematic dependence of the simple poles are governed by $\gamma_K^{(i)}$! [EG & Magnea (09)]

The soft function $\mathcal{S}$



Definition:

$$(c_N)_{ijkl}\,\mathcal{S}_{NL}\left(\beta_a\cdot\beta_b,\alpha_s(\mu^2),\epsilon\right) \;=$$

$$\sum_{i'j'k'l'} \langle 0|\,\Phi^{k,k'}_{-\beta_2}(0,\infty)\,\Phi^{i,i'}_{\beta_1}(\infty,0)\Phi^{j,j'}_{\beta_3}(0,\infty)\Phi^{l,l'}_{-\beta_4}(\infty,0)\;|0\rangle\;(c_L)_{i'j'k'l'}$$

multiplicatively renormalizable $\Longrightarrow$ matrix evolution equation:

$$\mu\frac{d}{d\mu}\mathcal{S}_{JL}\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon\right) =$$

$$-\sum_N \left[\mathbf{\Gamma}_{\mathcal{S}}\right]_{JN}\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon\right)\;\mathcal{S}_{NL}\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon\right)$$

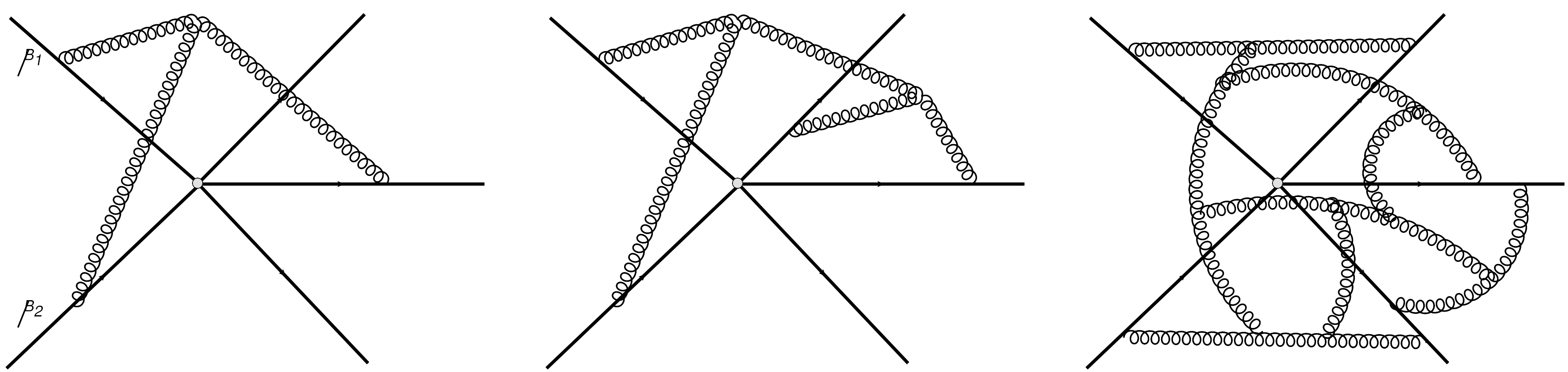
The soft function $\mathcal{S}$

Evolution $\Longrightarrow$ Exponentiation:

$$\mathcal{S}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) = P \exp\left\{-\frac{1}{2}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \mathbf{\Gamma}_{\mathcal{S}}\left(\beta_i \cdot \beta_j, \alpha_s(\lambda^2, \epsilon), \epsilon\right)\right\}$$

$\mathbf{\Gamma}_{\mathcal{S}}$ is a matrix of anomalous dimensions.

- A priori, $\mathbf{\Gamma}_{\mathcal{S}}$ can be very complicated: at each order in α_s it may contain new colour structures and kinematic dependence corresponding to sums of webs:



- In fact $\mathbf{\Gamma}^{\mathcal{S}}$ is (much?) simpler.

The soft anomalous dimension $\mathbf{\Gamma}_{\mathcal{S}}$ at two loops

Remarkable discovery: [Aybat Dixon Sterman (06)]

For any multi-leg amplitude:

$$\mathbf{\Gamma}_{\mathcal{S}}^{(2)} = \frac{K}{2}\mathbf{\Gamma}_{\mathcal{S}}^{(1)}$$

where $\mathbf{\Gamma}_{\mathcal{S}} = \sum_{n=1}^{\infty} \mathbf{\Gamma}_{\mathcal{S}}^{(n)} \left(\frac{\alpha_s(\mu)}{\pi}\right)^n$ and $K = \left(\frac{67}{18} - \zeta(2)\right) C_A - \frac{10}{9} T_F N_f$.

so at two loops: no new colour matrices, no new kinematic dependence...

- why?
- where is K coming from?

  This is the famous coefficient of the cusp anomalous dimension $\gamma_K^{(i)}$ [Korchemsky Radyushkin (87), Kodaira Trentadue (82),...] :

$$\gamma_K^{(i)} = 2C_i \frac{\alpha_s}{\pi} + K C_i \left(\frac{\alpha_s}{\pi}\right)^2 + \cdots$$

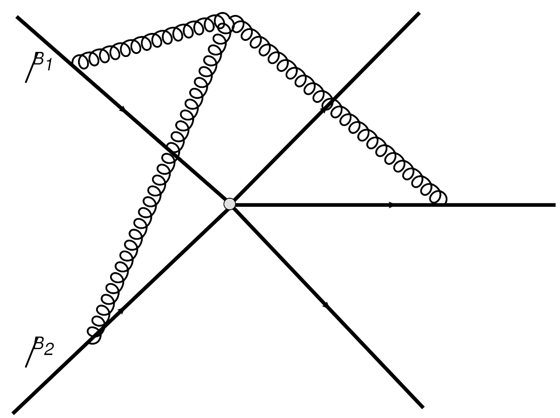
very suggestive... does this extend to higher orders?

The soft anomalous dimension $\Gamma_{\mathcal{S}}$ at two loops

One of the crucial elements in proving

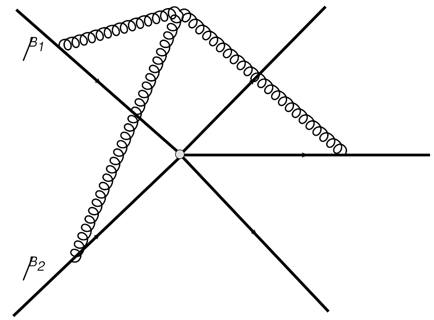
$$\mathbf{\Gamma}_{\mathcal{S}}^{(2)} = \frac{K}{2}\mathbf{\Gamma}_{\mathcal{S}}^{(1)}$$

is the vanishing of the diagram



We can prove it as follows [EG (09)]

Γ_S at two loops: vanishing of F_{3g}



$$F_{3g}(\beta_1,\beta_2,\beta_3) = \int_0^\infty dt_1 i g_s \beta_1^{\hat\mu} \int_0^\infty dt_2 i g_s \beta_2^{\hat\nu} \int_0^\infty dt_3 i g_s \beta_3^{\hat\sigma} \underbrace{\theta(t_1\, t_2\, t_3 < T)}_{\text{IR cutoff}}$$

$$\int d^d z \Gamma^{\mu\nu\sigma} D_{\mu\hat\mu}(t_1\beta_1 - z) D_{\nu\hat\nu}(t_2\beta_2 - z) D_{\sigma\hat\sigma}(t_3\beta_3 - z)$$

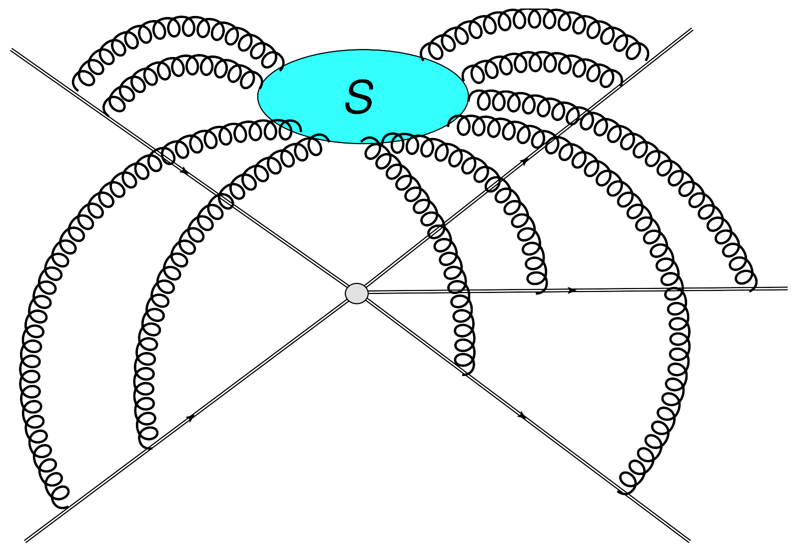
- Antisymmetry of the three gluon vertex $\Gamma^{\mu\nu\sigma}$ under any replacement (Bose symmetry) which implies $F_{3g}(\beta_1,\beta_2,\beta_3) = -F_{3g}(\beta_2,\beta_1,\beta_3)$
- Rescaling invariance of the velocities (without affecting the IR cutoff),
- If β_1 and β_2 are lightlike, then:

$$F_{3g}(\beta_1,\beta_2,\beta_3) = f(\beta_1\cdot\beta_3, \beta_2\cdot\beta_3, \beta_1\cdot\beta_2, \beta_3^2) = f(\kappa\beta_1\cdot\beta_3, \beta_2\cdot\beta_3/\kappa, \beta_1\cdot\beta_2, \beta_3^2)$$

$$= f\left(\beta_2\cdot\beta_3, \beta_1\cdot\beta_3, \beta_1\cdot\beta_2, \beta_3^2\right)$$

$$= -f\left(\beta_1\cdot\beta_3, \beta_2\cdot\beta_3, \beta_1\cdot\beta_2, \beta_3^2\right)$$

The soft function $\mathcal{S}$



$$\mu \frac{d}{d\mu} \mathcal{S}_{JL} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) =$$

$$- \sum_N \left[\mathbf{\Gamma}_{\mathcal{S}} \right]_{JN} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) \, \mathcal{S}_{NL} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right)$$

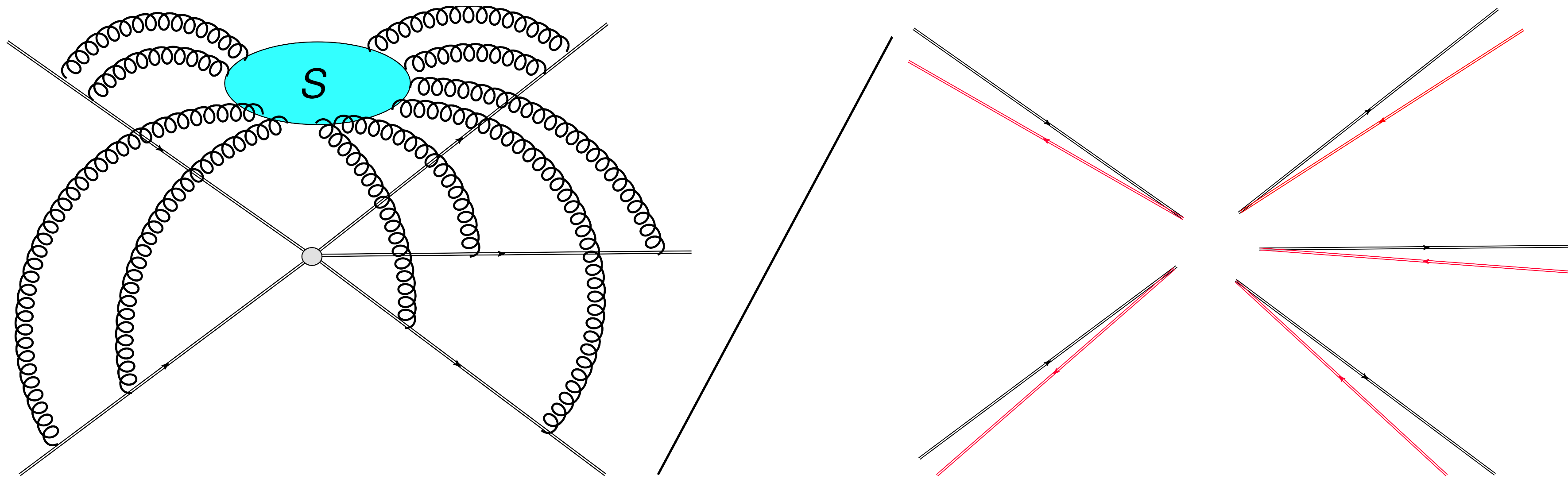
$\mathbf{\Gamma}_{\mathcal{S}}$ has cusp singularities, and therefore, similarly to $\gamma_{\mathcal{J}}$

- it has poles in ϵ ($\mathcal{S}$ itself has double poles).
- it is not invariant with respect to $\beta_i \longrightarrow \kappa_i \beta_i$

Both these issues can be ‘fixed’ by dividing by appropriate eikonal jets...

The reduced soft function $\overline{\mathcal{S}}$

$\overline{\mathcal{S}} \equiv$



$$\overline{\mathcal{S}}_{JL}\left(\rho_{ij}, \epsilon\right) = \frac{\mathcal{S}_{JL}\left(\beta_i \cdot \beta_j, \epsilon\right)}{\displaystyle\prod_{i=1}^{n} \mathcal{J}_i\left(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \epsilon\right)}$$

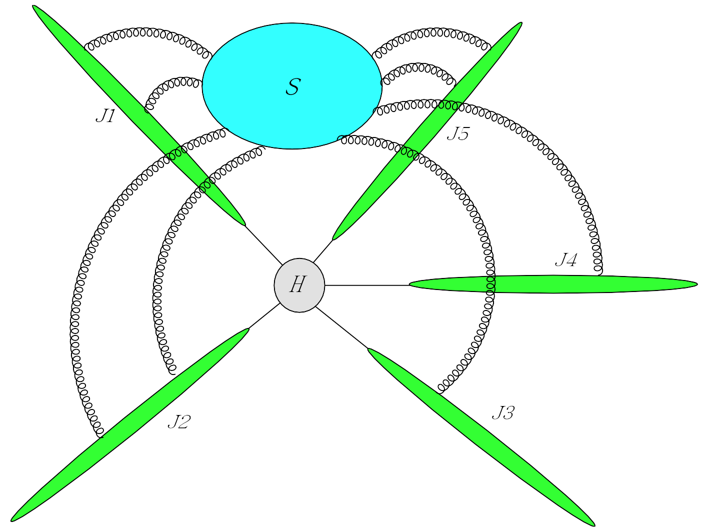
Having removed the collinear regions, $\overline{\mathcal{S}}$ does not suffer from the cusp anomaly, and must therefore respect rescaling $\beta_i \longrightarrow \kappa_i \beta_i$:

$\Longrightarrow$ $\overline{\mathcal{S}}$ depends only on

$$\rho_{ij} \equiv \frac{(\beta_i \cdot \beta_j)^2}{\left[2(\beta_i \cdot n_i)^2/n_i^2\right]\left[2(\beta_j \cdot n_j)^2/n_j^2\right]}$$

Factorization in terms of the reduced soft function $\overline{\mathcal{S}}$

$$\mathcal{M}_N\left(p_i/\mu, \epsilon\right) =$$



$$=$$

$$= \sum_L \mathcal{S}_{NL}\left(\beta_i \cdot \beta_j, \epsilon\right) H_L \prod_{i=1}^{n} \frac{J_i\left(\frac{(2p_i \cdot n_i)^2}{n_i^2\mu^2}, \epsilon\right)}{\mathcal{J}_i\left(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \epsilon\right)}$$

$$= \sum_L \overline{\mathcal{S}}_{NL}\left(\rho_{ij}, \epsilon\right) H_L \prod_{i=1}^{n} J_i\left(\frac{(2p_i \cdot n_i)^2}{n_i^2\mu^2}, \epsilon\right)$$

- $\overline{\mathcal{S}}$ has only single poles due to large-angle soft gluons.
- $\overline{\mathcal{S}}$, like $\mathcal{M}$, cannot depend on the normalization of the velocities!

Consequences of factorization + rescaling invariance of $\overline{\mathcal{S}}$

$$\mu \frac{d}{d\mu} \overline{\mathcal{S}}_{IK}\left(\rho_{ij}, \alpha_s(\mu^2), \epsilon\right) = -\sum_J \Gamma^{\overline{\mathcal{S}}}_{IJ}\left(\rho_{ij}, \alpha_s(\mu^2)\right) \overline{\mathcal{S}}_{JK}\left(\rho_{ij}, \alpha_s(\mu^2), \epsilon\right)$$

$\Gamma^{\overline{\mathcal{S}}}_{IJ}$ — in contrast to $\Gamma^{\mathcal{S}}_{IJ}$ and $\gamma_{\mathcal{J}}$ — is free of singularities.

$$\overline{\mathcal{S}}_{IK}\left(\rho_{ij}, \alpha_s(\mu^2), \epsilon\right) \equiv \frac{\mathcal{S}_{IK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)}{\prod_{i=1}^{n} \mathcal{J}_i\left(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)}$$

$$\begin{aligned}
\Gamma^{\overline{\mathcal{S}}}_{IJ}\left(\rho_{ij}, \alpha_s\right) &= \Gamma^{\mathcal{S}}_{IJ}\left(\beta_i \cdot \beta_j, \alpha_s, \epsilon\right) - \delta_{IJ} \sum_{k=1}^{n} \gamma_{\mathcal{J}_k}\left(\frac{2(\beta_k \cdot n_k)^2}{n_k^2}, \alpha_s, \epsilon\right) \\
&= \Gamma^{\mathcal{S}}_{IJ}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2, \epsilon), \epsilon\right) - \delta_{IJ} \sum_{k=1}^{n} \Bigg[-\frac{1}{2} \delta_{\mathcal{J}_k}\left(\alpha_s(\mu^2, \epsilon)\right) \\
&\quad + \frac{1}{4} \gamma_K^{(k)}\left(\alpha_s(\mu^2, \epsilon)\right) \ln\left(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}\right) + \frac{1}{4} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \gamma_K^{(k)}\left(\alpha_s(\xi^2, \epsilon)\right) \Bigg]
\end{aligned}$$

Consequences of factorization $+$ rescaling invariance of $\overline{\mathcal{S}}$

$$\Gamma^{\overline{\mathcal{S}}}_{IJ}\left(\rho_{ij},\alpha_s\right) = \Gamma^{\mathcal{S}}_{IJ}\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2,\epsilon),\epsilon\right) - \delta_{IJ}\sum_{k=1}^{n}\Bigg[-\frac{1}{2}\delta_{\mathcal{J}_k}\left(\alpha_s(\mu^2,\epsilon)\right)$$
$$+\frac{1}{4}\gamma_K^{(k)}\left(\alpha_s(\mu^2,\epsilon)\right)\,\ln\left(\frac{2(\beta_k\cdot n_k)^2}{n_k^2}\right) + \frac{1}{4}\int_0^{\mu^2}\frac{d\xi^2}{\xi^2}\gamma_K^{(k)}\Big(\alpha_s(\xi^2,\epsilon)\Big)\Bigg]$$

- off diagonal terms in $\Gamma^{\mathcal{S}}$ are finite and must depend only on conformal cross ratios

$$\rho_{ijkl} \equiv \frac{(\beta_i\cdot\beta_j)(\beta_k\cdot\beta_l)}{(\beta_i\cdot\beta_k)(\beta_j\cdot\beta_l)} = \left(\frac{\rho_{ij}\,\rho_{kl}}{\rho_{ik}\,\rho_{jl}}\right)^{1/2}$$

- diagonal terms in $\Gamma^{\mathcal{S}}$ have the following singularity

$$\Gamma^{\mathcal{S}}_{IJ}\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2,\epsilon),\epsilon\right) = \delta_{IJ}\sum_{k=1}^{n}\frac{1}{4}\int_0^{\mu^2}\frac{d\xi^2}{\xi^2}\gamma_K^{(k)}\Big(\alpha_s(\xi^2,\epsilon)\Big) + \mathcal{O}(\epsilon^0)$$

and must also contain finite terms with specific dependence on $\beta_i\cdot\beta_j$ so as to combine with the $(\beta_i\cdot n_i)^2/n_i^2$ to generate ρ_{ij}.

Consequences of factorization $+$ rescaling invariance of $\overline{\mathcal{S}}$

$$\Gamma^{\overline{\mathcal{S}}}_{IJ}\left(\rho_{ij},\alpha_s\right) = \Gamma^{\mathcal{S}}_{IJ}\left(\beta_i\cdot\beta_j,\alpha_s(\mu^2,\epsilon),\epsilon\right) - \delta_{IJ}\sum_{k=1}^{n}\Bigg[-\frac{1}{2}\delta_{\mathcal{J}_k}\left(\alpha_s(\mu^2,\epsilon)\right) + \frac{1}{4}\gamma_K^{(k)}\left(\alpha_s(\mu^2,\epsilon)\right)\,\ln\left(\frac{2(\beta_k\cdot n_k)^2}{n_k^2}\right) + \frac{1}{4}\int_0^{\mu^2}\frac{d\xi^2}{\xi^2}\gamma_K^{(k)}\Big(\alpha_s(\xi^2,\epsilon)\Big)\Bigg]$$

Taking a derivative with respect to $(\beta_i\cdot n_i)^2/n_i^2$ we get:

$$\sum_{j\neq i}\frac{\partial}{\partial\ln(\rho_{ij})}\,\Gamma^{\overline{\mathcal{S}}}_{IJ}\left(\rho_{ij},\alpha_s\right) = \frac{1}{4}\,\gamma_K^{(i)}\left(\alpha_s\right)\,\delta_{IJ}\,, \qquad \forall i,\ I,\ J$$

On the l.h.s. we used the definition $\rho_{ij}\equiv\dfrac{(\beta_i\cdot\beta_j)^2}{4\,\left[(\beta_i\cdot n_i)^2/n_i^2\right]\,\left[(\beta_j\cdot n_j)^2/n_j^2\right]}$

with the chain rule: $\dfrac{\partial}{\partial\ln(\beta_i\cdot n_i)^2/n_i^2}F\left(\rho_{ij}\right) = -\sum_{j\neq i}\dfrac{\partial}{\partial\ln\rho_{ij}}F\left(\rho_{ij}\right)$

The equations for $\Gamma^{\overline{\mathcal{S}}}$

Factorization $+$ rescaling invariance imply:
$\Gamma^{\overline{\mathcal{S}}}$ for any multi-leg amplitude, in any colour basis, obeys:

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \mathbf{\Gamma}^{\overline{\mathcal{S}}} \left(\rho_{ij}, \alpha_s\right) = \frac{1}{4} \gamma_K^{(i)} \left(\alpha_s\right) \ , \qquad \forall i$$

[Gardi Magnea (09)]

This is true to all orders, as well as at strong coupling.

- We have related the soft anomalous dimension of a general multi-leg amplitude to the cusp anomalous dimension.
- Intriguing relation between kinematics and colour.

Solving for $\Gamma^{\overline{\mathcal{S}}}$

$$\sum_{j\neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \mathbf{\Gamma}^{\overline{\mathcal{S}}} (\rho_{ij}, \alpha_s) = \frac{1}{4} \gamma_K^{(i)} (\alpha_s) \, , \qquad \forall i$$

Does this set of differential equations have a unique solution?

- For two or three legs - yes! Then $\Gamma^{\overline{\mathcal{S}}}$ can be written in terms of γ_K, with explicitly determined kinematic dependence.
- For four or more legs - no: functions of conformal cross ratios

$$\rho_{ijkl} \equiv \frac{(\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)} = \left(\frac{\rho_{ij}\, \rho_{kl}}{\rho_{ik}\, \rho_{jl}} \right)^{1/2}$$

satisfy the homogeneous equation.

Yet, it has a simple all-order solution (minimal solution)

The sum-over-dipoles formula

$\gamma_K^{(i)}$ admits quadratic Casimir scaling ($C_i \equiv \mathrm{T}_i \cdot \mathrm{T}_i$) (at least to 3 loops):

$$\gamma_K^{(i)} = 2C_i \frac{\alpha_s}{\pi} + K C_i \left(\frac{\alpha_s}{\pi}\right)^2 + K^{(2)} C_i \left(\frac{\alpha_s}{\pi}\right)^3 + \cdots = C_i \, \widehat{\gamma}_K(\alpha_s) + \underbrace{\widetilde{\gamma}_K^{(i)}(\alpha_s)}_{\text{Higher Casimirs}}$$

The equations:

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}_{\text{Q.C.}}(\rho_{ij}, \alpha_s) = \frac{1}{4} \mathrm{T}_i \cdot \mathrm{T}_i \, \widehat{\gamma}_K(\alpha_s) \,, \qquad \forall i$$

are solved by the sum-over-dipoles formula [Gardi Magnea (09)]:

$$\Gamma^{\overline{\mathcal{S}}}_{\text{Q.C.}}(\rho_{ij}, \alpha_s) = -\frac{1}{8} \widehat{\gamma}_K(\alpha_s) \sum_{i \neq j} \ln(\rho_{ij}) \, \mathrm{T}_i \cdot \mathrm{T}_j + \frac{1}{2} \widehat{\delta}_{\overline{\mathcal{S}}}(\alpha_s) \sum_{i=1}^{n} \mathrm{T}_i \cdot \mathrm{T}_i \,,$$

- Generalises the two loop result to all orders (minimal solution!)
- Kinematics and colour are directly correlated.

The same formula was simultaneously proposed by Becher and Neubert.

The sum-over-dipoles formula: a solution to the constraints

$$\Gamma^{\overline{S}}_{\text{Q.C.}}\left(\rho_{ij},\alpha_s\right)\Big|_{\rm ansatz} = -\frac{1}{8}\,\widehat{\gamma}_K\left(\alpha_s\right)\sum_{i\neq j}\ln(\rho_{ij})\,\sum_a \mathrm{T}_i^{(a)}\mathrm{T}_j^{(a)} + \frac{1}{2}\,\widehat{\delta}_{\overline{\mathcal{S}}}(\alpha_s)\sum_{i=1}^{n}\sum_a \mathrm{T}_i^{(a)}\mathrm{T}_i^{(a)}\,, \tag{1}$$

Proof: take a derivative of (1) with respect to ρ_{ij} (for fixed i and j),

$$\frac{\partial\Gamma^{\overline{S}}\left(\rho_{ij},\alpha_s\right)}{\partial\ln(\rho_{ij})} = -\frac{1}{4}\,\widehat{\gamma}_K\left(\alpha_s\right)\,\sum_a \mathrm{T}_i^{(a)}\mathrm{T}_j^{(a)}$$

then sum over j (all external partons, excluding i) to get:

$$\sum_{j,\,j\neq i}\frac{\partial\Gamma^{\overline{S}}\left(\rho_{ij},\alpha_s\right)}{\partial\ln(\rho_{ij})} = -\frac{1}{4}\widehat{\gamma}_K\left(\alpha_s\right)\,\sum_{j,\,j\neq i}\sum_a \mathrm{T}_i^{(a)}\mathrm{T}_j^{(a)}$$

$$= -\frac{1}{4}\widehat{\gamma}_K\left(\alpha_s\right)\,\sum_a \mathrm{T}_i^{(a)}\left(-\mathrm{T}_i^{(a)}\right)$$

where colour conservation was used $\sum_{i=1}^{n}\mathrm{T}_i^{(a)} = 0$.

Beyond the minimal solution

Corrections to the sum-over-dipoles formula are of two kinds

- terms that are induced by higher Casimir contributions to γ_K — they may appear starting at four loops and must satisfy the equations

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}}_{\text{H.C.}} \left(\rho_{ij}, \alpha_s\right) = \frac{1}{4} \, \widetilde{\gamma}_K^{(i)} \left(\alpha_s\right) \, , \qquad \forall i,$$

- solutions of the homogeneous equations

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\overline{\mathcal{S}}} \left(\rho_{ij}, \alpha_s\right) = 0 \qquad \forall i$$

  namely, functions of conformal cross ratios. These may appear starting at three loops, four legs.

Absence of $\hat{\mathbf{H}}^{(2)}_{[\text{f}]} = \sum_{j,k,l} \sum_{a,b,c} \text{i} \, f_{abc} \, \text{T}^a_j \text{T}^b_k \text{T}^c_l \, \ln\left(\rho_{ijkl}\right) \, \ln\left(\rho_{iklj}\right) \, \ln\left(\rho_{iljk}\right)$

at the two-loops $\Gamma^{\overline{\mathcal{S}}}$ supports the minimal solution!

Conclusions

- Detailed understanding of infrared singularities in QCD amplitudes is needed for cross section calculations and for resummation.
- Recent progress:
  - Remarkable simplicity at two loops — now better understood.
  - A completely general constraint was derived based on factorization and rescaling symmetry.
    It relates soft singularities in any amplitude, and any loop order, to the cusp anomalous dimension.
  - An all-loop sum-over-dipoles formula naturally emerges as a minimal solution.
- Several research avenues have opened up. The full beauty of gauge theory amplitudes is not yet revealed...