A Numerical study of the Schwinger-Dyson Equations of QCD

David Wilson

Friday 19^{th} June 2009

Work in progress with M.R. Pennington

Strongly-coupled QCD

$$\mathcal{L} = \bar{\psi}^a \left(i \not{D}_{ab} - m \right) \psi^b - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a - \frac{1}{2\xi} \left(\partial_\mu A^\mu_a \right)^2 + \left(\partial^\mu \bar{c}_a \right) D^{ab}_\mu c_b$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

$$D^{ab}_\mu = \delta^{ab} \partial_\mu - g f^{abc} A^c_\mu$$

Confinement and Hadronisation

- How is confinement realised in nature?
- How do the fields of the lagrangian become hadrons?

Nonperturbative Mass corrections

- What is the mechanism behind dynamical chiral symmetry breaking?
  $m(p^2) = m_0 \left(1 + \alpha_s(\mu^2) c_1 \mathrm{Log} \left(p^2/\mu^2 \right) + \mathcal{O}(\alpha_s^2) \right)$
- Is there mass generation in QCD even if $m_0 \to 0$?

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Schwinger-Dyson Equations

- Field equations of a QFT – can expand in g to give perturbation theory.
- Valid non-perturbatively. Need to understand NP physics to understand how the particles of the Lagrangian become hadrons.
- Infinite tower of equations - 2 point Greens function depends on 3 and 4 point Greens functions. The 3 and 4 point Greens functions also depend on higher Greens functions, and so on...

- Always need to truncate the tower somewhere.
- Essential that the truncation respects gauge invariance to be physically meaningful.
- Neglect higher contributions or model them in a sensible way.
- Ward-Slavnov-Taylor identities help impose Gauge constraints.
- Multiplicative Renormalisability gives additional constraints.

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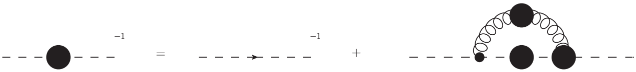
The Schwinger-Dyson Equations of QCD



Heavy dots correspond to fully dressed vertices and propagators, including all loop effects.
Light dots and normal propagators are those that we use in perturbative QCD.

The Ghost SDE

- Ghost SDE is particularly simple.
- All recent SDE studies use covariant gauges – Ghosts are present.
- This is a nonlinear coupled integral equation and can be formally derived using functional methods.
- Can also be found by summing an infinite series of the all of the 1PI polarisation diagrams.
- One vertex is always bare in these types of diagrams to avoid double counting.

"Feynman Rules"

For the propagators we just have to include a dressing function, in a general covariant gauge we can use the following.

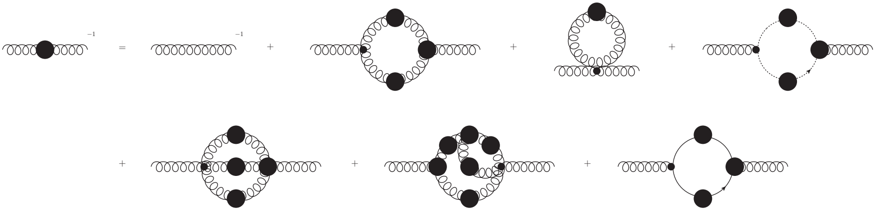
$$D_{\mu\nu}(p) = \frac{G\ell(p^2)}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \xi\frac{p_\mu p_\nu}{p^4} \tag{1}$$

$$D^{-1}_{\mu\nu}(p) = \frac{p^2}{G\ell(p^2)}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \frac{p_\mu p_\nu}{\xi} \tag{2}$$

$$D(p) = -\frac{Gh(p^2)}{p^2} \tag{3}$$

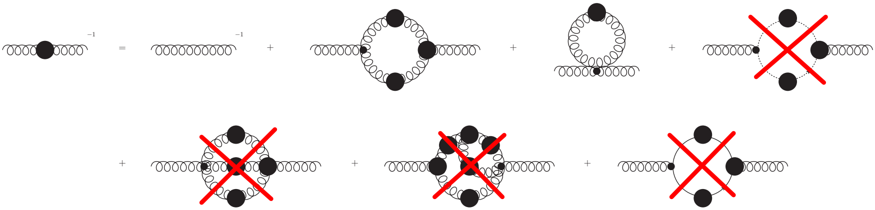
The new functions $G\ell\left(p^2\right)$ and $Gh\left(p^2\right)$ contain the non-perturbative effects. Setting these to 1 recovers the usual propagators.

The Gluon SDE



- Couples to fermions and ghosts simultaneously.
- Contains (superficially) quadratically divergent terms.
- Several attempts have been made to solve these in different approximations.
- The dressed two-loop integrals are very tricky numerically – have never been solved exactly, only in certain limits.

Neglecting Ghosts



- Quenched approximation – $N_f \to 0$.
- Also assume ghosts are negligible, $Gh\left(p^2\right) \to 1$.
- Ward-Slavnov-Taylor Identity constrains longitudinal part of the triple gluon vertex to be dressed like $\Gamma_L \sim \Gamma^0/G\ell$.
- Tadpole diagram is independent of external momentum and never contributes.

Various Refs: Pagels, Mandelstam, Bar-Gadda

Numerical Method

- Use Landau gauge with $\partial_\mu A^\mu = 0$ and $\xi \to 0$.
- We Wick rotate to Euclidean space – this allows use of squared momenta.
- Loop integrals may then be performed without approximation via,

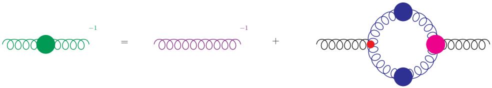
$$\int \frac{d^4k}{(2\pi)^4} \mathcal{F}(p^2, k^2, \psi) = \frac{1}{(2\pi)^3} \int_0^{\kappa^2} k^2 dk^2 \int_0^{\pi} d\psi \; \sin^2\psi \; \mathcal{F}(p^2, k^2, \psi)$$

- We must also contract the tensor structure in the Gluon equation, this is usually done with a "projector" of the form

$$\mathcal{P}_{\mu\nu}(p, \zeta) = g_{\mu\nu} - \zeta \frac{p_\mu p_\nu}{p^2}$$

where p is the external momenta and ζ is a free parameter that can select different tensor structures.

Numerical Method



$$D^{-1}_{\mu\nu}(p) = D^{(0),-1}_{\mu\nu}(p) + \int \frac{d^4k}{(2\pi)^4} \Gamma^0_{\mu\rho\sigma}\, D^{\rho\eta}(k_+)\, \Gamma_{\eta\phi\nu}\, D^{\phi\sigma}(k_-)$$

Applying $\mathcal{P}_{\mu\nu}$, using a vertex dressing proportional to the bare, and renormalising (momentum subtraction scheme),

$$G\ell(p^2)^{-1} = Z_3 + \frac{g^2 N_c}{3(2\pi)^3} Z_1 \int_0^{\kappa^2} k^2 dk^2 \int_0^{\pi} d\theta \sin^2\theta\; Q(p^2, k^2, \theta)\, G\ell(k_+^2)\, G\ell(k_-^2)\, \Gamma(p, k)$$

Integrals evaluated using gaussian quadrature rules. A starting guess is iterated using a Newton-Raphson technique until convergence.

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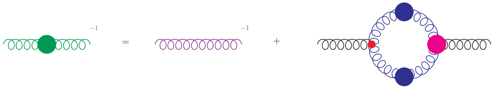

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A Closer Look at the Gluon Equation

Usually cancellations between different diagrams ensures our result is gauge invariant and free from quadratic divergences. In Landau gauge the gluon is also transverse. The projector $\mathcal{P}_{\mu\nu}(p,\zeta)$ picks out different tensor structures, depending on which value of ζ we choose, if the inverse gluon propagator is,

$$D^{-1}_{\mu\nu}(p) = A(p^2)p^2 g_{\mu\nu} + B(p^2)p_\mu p_\nu,$$

then

$$\mathcal{P}^{\mu\nu}(p,\zeta)D^{-1}_{\mu\nu}(p) = (d-\zeta)p^2A(p^2) + (1-\zeta)p^2B(p^2).$$

Quadratic divergences

- The quadratic divergences found in individual diagrams are always proportional to $g_{\mu\nu}$, so setting $\zeta \to 4$ in 4 dimensions, we are guaranteed to be free from these.
- The function $B(p^2)$ then determines $G\ell(p^2)$.

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Results neglecting Ghosts

First Numerical Studies

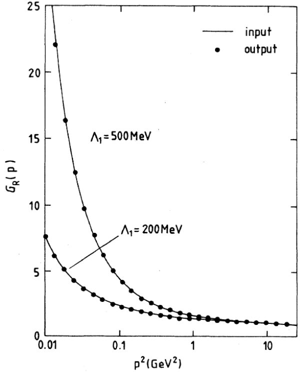
- Gluon dressing is singular in the IR
- Dressing function $G\ell(p^2) \sim p^{-2}$ in the IR.

$\longrightarrow$ Dressed propagator singular as p^{-4}.

$\longrightarrow$ A fourier transform of this propagator leads to a linearly rising confining potential for large distances.

- Similar to those found to work in potential model studies of heavy mesons.

Caveat: In Landau gauge, the Gluon propagator is neither gauge invariant nor an experimental observable.



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Is this how confinement works??

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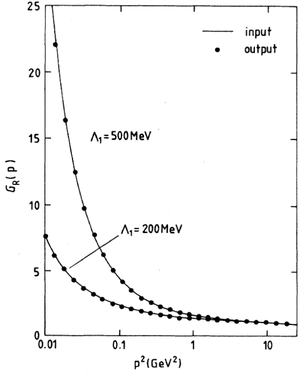

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Confinement Scenarios → *IR Ghost enhancement?*

Kugo-Ojima Confinement Criterion

Unbroken BRST symmetry → Ghost more singular than a simple pole. Confinement restricts the physically allowed state space of QCD. Must separate physical and unphysical states.

1. If BRST symmetry is an unbroken symmetry of gauge fixed NP QCD it can be used to define a physical part of the state space.
2. If the global charge is well-defined everywhere, they show that the physical part of the state space contains only colour singlets.
3. Cluster decomposition has to be violated somewhere in the total state space, but not the physical part.

Condition (2.) Leads to the requirement on $D(p^2) = -\frac{Gh(p^2)}{p^2} = -\frac{1}{p^2}\frac{1}{1+u(p^2)}$ that $u(0) = -1$. This tells us that, provided the above are true, the ghost propagator has to be more singular than the bare propagator.

+Gribov!

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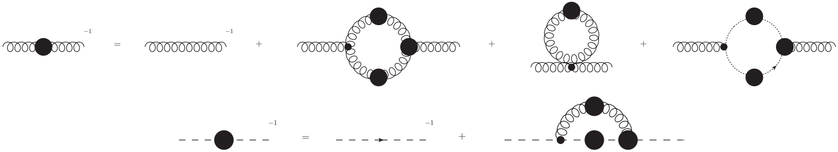
Including Ghosts



- In the UV we ensure our functions match onto resummed 1-loop PT.

$\longrightarrow$ Calculate up to cutoff κ, and ensure smooth matching over a wide region in the UV between numerical and perturbative results.

$\longrightarrow$ Essential to preserve gauge invariance for a consistent truncation.

- Slavnov-Taylor Identities help dress the triple-gluon vertex and protect gauge invariance. In the truncation we consider the relation is not exact, but a simplified form.
- In Landau gauge the ghost-gluon vertex does not get renormalised ($\tilde{Z}_1 = 1$) and reduces to its bare form for symmetric momenta $\rightarrow$ bare vertex may not be a bad approximation to the full.

The Tübingen-Graz Solutions (TG)

IR scaling solutions

Lerche and von Smekal find scaling solutions in the deep IR matching the confinement scenarios.
$G\ell(p^2) \to (p^2)^{2\kappa}$, $Gh(p^2) \to (p^2)^{-\kappa}$, with $\kappa \simeq 0.595$. Several studies have used this relation using different values of κ.

Full momentum range

- Alkofer, Fischer *et al* find numerical solutions connecting the IR scaling to 1-loop PT in the UV.
- They use the $\zeta \to 1$ projector, and they subtract the quadratic divergences by hand.
- Uses a Ghost-Antighost symmetric extension of landau gauge.
- Widely used NP Running coupling definition, $\alpha_s(p^2) = \frac{g^2}{4\pi} Gh^2(p^2) G\ell(p^2)$.

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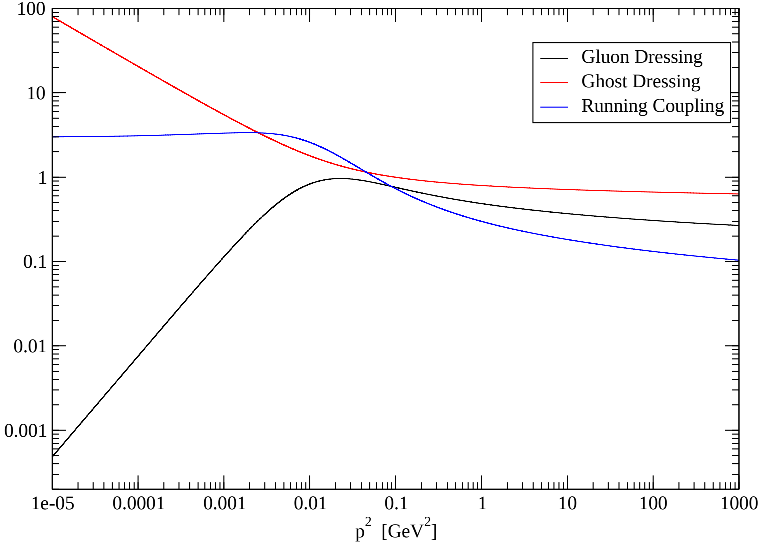

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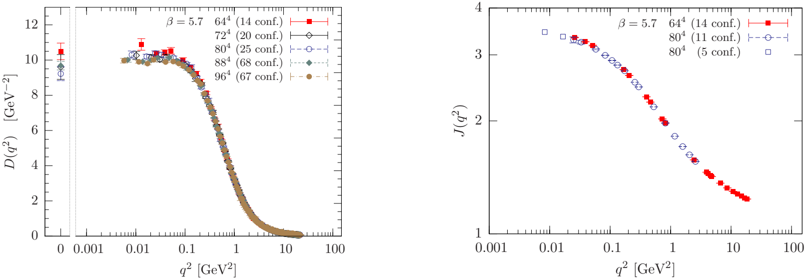
Contradiction with the Lattice studies



Left: The Gluon $D(p^2) = G\ell(p^2)/p^2$ appears to go to a constant.
Dressing is $G\ell(p^2) = \text{const} \times p^2$ in the IR.

Right: $J(p^2) = Gh(p^2)$. Ghost dressing function weakly increasing - weaker than $(p^2)^{-0.595}$.

- These solutions should be directly comparable. They are performed using the quenched approximation and in Landau gauge.
- Lattice solutions may suffer finite volume effects and finite lattice spacing effects (Always have $V < \infty$ and $a > 0$).
- Are there other solutions in this truncation that better match this new lattice data?

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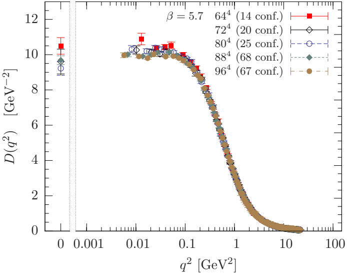

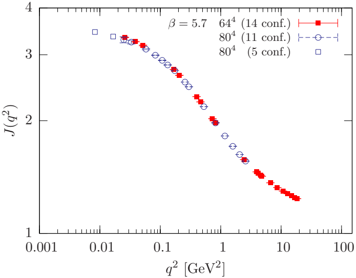

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What happens in this truncation if $\zeta = 4$?

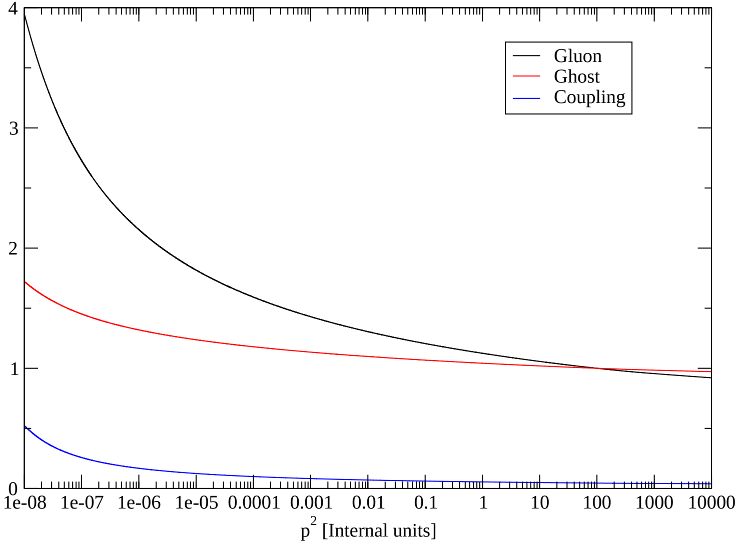
This truncation has some nice features and is simple enough to quickly make changes to its various terms. $\zeta = 4$ is a more natural way of removing the quadratic divergences and ensuring transversality without prior knowledge of the quadratic term, so this will be investigated.

What is known so far...

- TG IR analysis does not hold for this projector, and this is dependent upon the subtraction of the quadratic divergence in the Gluon loop.

  (Ref: J.C.R. Bloch, hep-ph/0303125).
- TG are unable to find solutions over the whole momentum region.
- Bloch has found solutions in a related truncation by including effective dressed two-loop diagrams

Solutions with small coupling



One-loop pertubation theory works!

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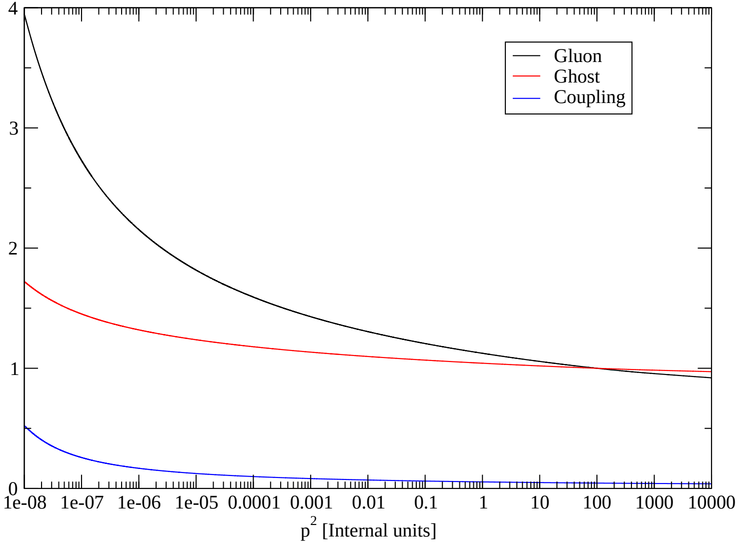

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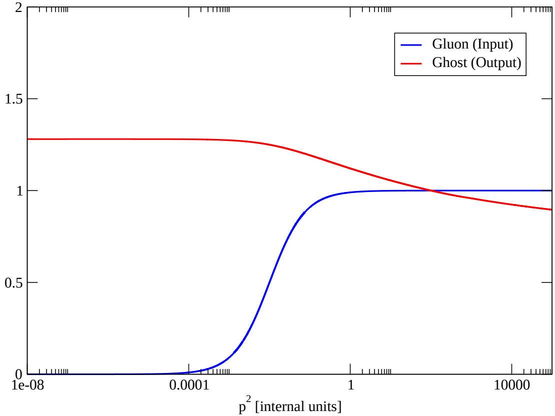
Solutions with IR behaviour similar to the lattice

- The non–linear integral equations that we wish to solve have no gauranteed number of solutions – Solutions found may be dependent upon the starting conditions.
- Solving the ghost equation shows us that for a ghost that is constant in the IR, we require a gluon dressing $\sim p^2$ in the IR.
- Using as input $G\ell(p^2) = \frac{p^2}{p^2+\lambda}$ the Ghost output looks similar to the lattice studies.
- No fully self-consistent solutions found.



What next?

- Essential to have self–consistent solutions with a strong coupling.
- If lattice is correct $\rightarrow$ Ghost equation looks sensible $\rightarrow$ keep bare ghost-gluon-ghost vertex.

$\rightarrow$ The place to concentrate on is then the gluon loop in the gluon equation.

$\rightarrow$ Triple-Gluon vertex ansatz could be too simple – does not satify WSTI.

- A similar study uses fully dressed *effective* two-loop diagrams to modify the behaviour in the intermediate regime.
- Finds solutions only with the dressed two loop diagrams included. Matches on to power laws in the IR. Ref: J.C.R. Bloch, hep-ph/0303125.

- For a more interesting phenomenology, introduction of quarks would be necessary.
- The Fermion equation is similar to ghost equation, with a more interesting vertex structure.

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Conclusions

- The strongly coupled gauge sector of QCD has been studied via its Schwinger-Dyson equations.
- Current studies using this method produce results that are in disagreement with those found on the lattice.

- Possible problems have been found in the existing methods but in correcting these we have not yet produced any strongly–coupled self–consistent solutions.
- We have written the numerical machinery and it has been heavily tested which has been a time–consuming process, but should yield results quickly for slight modifications of the current equations.