Yangian symmetry of scattering amplitudes in $\mathcal{N} = 4$ SYM

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LAPTH Annecy

0807.1095 [JMD, Johannes Henn, Gregory Korchemsky, Emery Sokatchev]

0808.2475 [JMD, Johannes Henn]

0902.2987 [JMD, Johannes Henn, Jan Plefka]

Outline

- ✔ Tree-level amplitudes
- ✔ Superconformal and dual superconformal symmetry
- ✔ Superconformal + Dual superconformal $\Longrightarrow$ Yangian symmetry

$\mathcal{N} = 4$ Super-amplitudes

$\mathcal{N} = 4$ SYM is special because it is described by PCT self-conjugate supermultiplet:

Chiral representation:

$$\Phi(\eta) = G^{+} + \eta^{A}\Gamma_{A} + \frac{1}{2}\eta^{A}\eta^{B}S_{AB} + \frac{1}{3!}\eta^{A}\eta^{B}\eta^{C}\epsilon_{ABCD}\bar{\Gamma}^{D} + \frac{1}{4!}(\eta)^{4}G^{-}$$

$$p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}, \qquad q^{\alpha A} = \lambda^{\alpha}\eta^{A}, \qquad \bar{q}^{\dot{\alpha}}_{A} = \tilde{\lambda}^{\dot{\alpha}}\frac{\partial}{\partial\eta^{A}}.$$

Super-amplitudes:

$$\mathcal{A}(\Phi_1 \ldots \Phi_n) = (\eta_1)^4(\eta_2)^4\mathcal{A}(G_1^{-}G_2^{-}G_3^{+}\ldots G_4^{+}) + \ldots$$

$$p^{\alpha\dot{\alpha}} = \sum_i \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}, \qquad q^{\alpha A} = \sum_i \lambda_i^{\alpha}\eta_i^{A}, \qquad \bar{q}^{\dot{\alpha}}_{A} = \sum_i \tilde{\lambda}_i^{\dot{\alpha}}\frac{\partial}{\partial\eta_i^{A}}.$$

Symmetries:

$$h_i\mathcal{A}_n(\lambda_i, \tilde{\lambda}_i, \eta_i) = \mathcal{A}_n(\lambda_i, \tilde{\lambda}_i, \eta_i) \qquad h_i = -\frac{1}{2}\lambda_i^{\alpha}\frac{\partial}{\partial\lambda_i^{\alpha}} + \frac{1}{2}\tilde{\lambda}_i^{\dot{\alpha}}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\alpha}}} + \frac{1}{2}\eta_i^{A}\frac{\partial}{\partial\eta_i^{A}}$$

$$p\mathcal{A} = q\mathcal{A} = \bar{q}\mathcal{A} = 0 \implies \mathcal{A}(\Phi_1, \ldots, \Phi_n) = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle \ldots \langle n1\rangle}\mathcal{P}(\lambda, \tilde{\lambda}, \eta), \qquad \bar{q}\mathcal{P} = 0.$$

$$\mathcal{P} = \mathcal{P}^{\mathrm{MHV}} + \mathcal{P}^{\mathrm{NMHV}} + \ldots + \mathcal{P}^{\overline{\mathrm{MHV}}}.$$

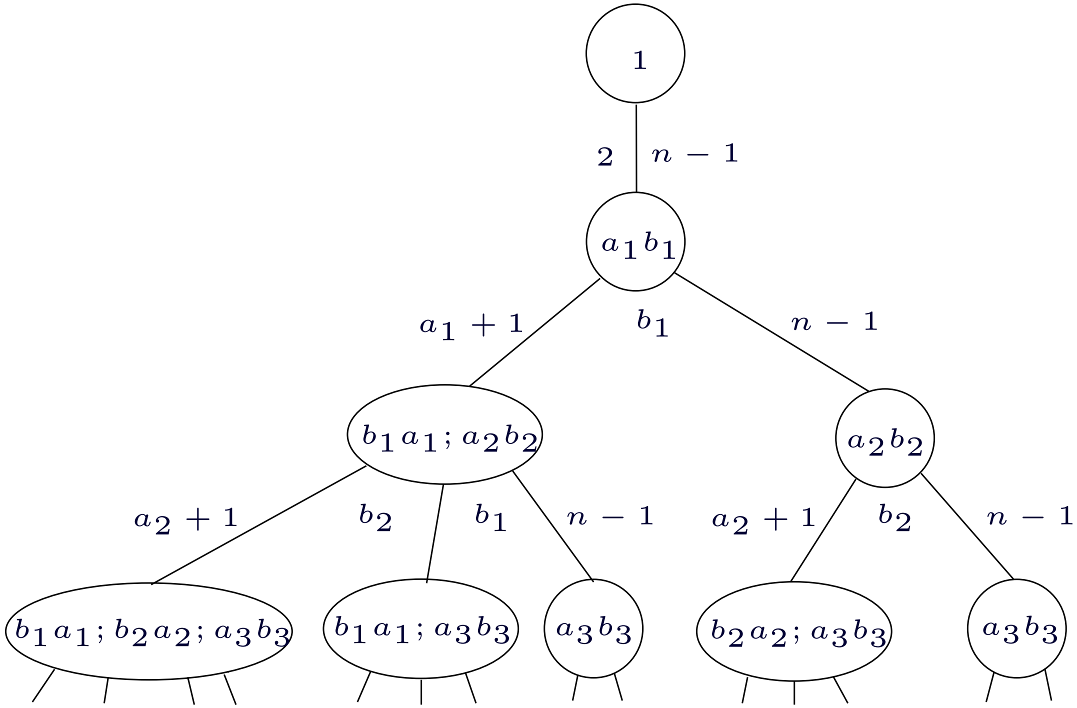
All tree-level amplitudes

The supersymmetric BCFW recursion relation
[Arkani-Hamed,Cachazo,Kaplan],[Brandhuber,Heslop,Travaglini],[Elvang,Freedman,Kiermaier]

$$\mathcal{A} = \mathcal{A}^{\mathrm{MHV}}\mathcal{P} = \sum_{P_i} \int d^4\eta_{P_i} \mathcal{A}_L(z_{P_i}) \frac{1}{P_i^2} \mathcal{A}_R(z_P)$$

admits a closed-form solution [JMD,Henn]:

$$\mathcal{P} = \sum \text{vertical paths in this picture} = 1 + \textstyle\sum_{a_1,b_1} R_{n;a_1 b_1} + \ldots$$



Superconformal symmetry

Since $\mathcal{N}=4$ SYM is a superconformal theory so we expect an action of the superconformal algebra on amplitudes [Witten].

$$p^{\dot\alpha\alpha}=\sum_i \tilde\lambda_i^{\dot\alpha}\lambda_i^{\alpha}, \qquad k_{\alpha\dot\alpha}=\sum_i \partial_{i\alpha}\partial_{i\dot\alpha},$$

$$\overline{m}_{\dot\alpha\dot\beta}=\sum_i \tilde\lambda_{i(\dot\alpha}\partial_{i\dot\beta)}, \qquad m_{\alpha\beta}=\sum_i \lambda_{i(\alpha}\partial_{i\beta)},$$

$$d=\sum_i[\tfrac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha}+\tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha}+1], \qquad r^{A}{}_{B}=\sum_i[-\eta_i^{A}\partial_{iB}+\tfrac{1}{4}\eta_i^{C}\partial_{iC}],$$

$$q^{\alpha A}=\sum_i \lambda_i^{\alpha}\eta_i^{A}, \qquad \bar q^{\dot\alpha}_{A}=\sum_i \tilde\lambda_i^{\dot\alpha}\partial_{iA},$$

$$s_{\alpha A}=\sum_i \partial_{i\alpha}\partial_{iA}, \qquad \bar s^{A}_{\dot\alpha}=\sum_i \eta_i^{A}\partial_{i\dot\alpha}.$$

$$c=\sum_i[1+\tfrac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha}-\tfrac{1}{2}\tilde\lambda^{\dot\alpha}\partial_{i\dot\alpha}-\tfrac{1}{2}\eta^{A}\partial_{iA}]$$

$$\partial_{i\alpha}=\frac{\partial}{\partial\lambda_i^{\alpha}}\,, \qquad \partial_{i\dot\alpha}=\frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}\,, \qquad \partial_{iA}=\frac{\partial}{\partial\eta_i^{A}}\,.$$

Some operators are zeroth order, some first order and some second order.
In twistor space they all become first order.

Dual conformal symmetry

Dual coordinates $x_i^\mu - x_{i+1}^\mu = p_i^\mu$.

Dual conformal symmetry: $K^\mu = \sum_i \left[x_i^\mu x_i \cdot \partial_i - \frac{1}{2} x_i^2 \partial_i^\mu\right]$

We need the action of the dual conformal generators on the spinors $\lambda, \tilde{\lambda}$.

The momenta satisfy two constraints:

$$\sum p_i^{\alpha\dot{\alpha}} = 0 \implies p_i^{\alpha\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}}$$

$$p_i^2 = 0 \implies p_i^{\alpha\dot{\alpha}} = \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}$$

Together these imply the constraints: $x_i - x_{i+1} - \lambda_i \tilde{\lambda}_i = 0$

Extend dual conformal generators so that they commute with the constraints up to constraints:

$$K_{\alpha\dot{\alpha}} = \sum_i [x_{i\alpha}{}^{\dot{\beta}} x_{i\dot{\alpha}}{}^{\beta} \partial_{i\beta\dot{\beta}} + x_{i\dot{\alpha}}{}^{\beta} \lambda_{i\alpha} \partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot{\beta}} \tilde{\lambda}_{i\dot{\alpha}} \partial_{i\dot{\beta}}]$$

Dual superconformal symmetry

[JMD,Henn,Korchemsky,Sokatchev]

Momentum conservation $\delta^4(p)$ suggests the introduction of the dual x_i:

$$x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}} - \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}} = 0.$$

Supersymmetry $\delta^8(q)$ suggests the introduction of dual θ_i:

$$\theta_i^{\alpha A} - \theta_{i+1}^{\alpha A} - \lambda_i^{\alpha}\eta_i^{A} = 0.$$

Now we can extend dual conformal symmetry to dual superconformal symmetry by extending the standard chiral representation so that all generators commute with the constraints up to constraints.

All generators of dual superconformal symmetry

$$P_{\alpha\dot\alpha} = \sum_i \partial_{i\alpha\dot\alpha},$$

$$Q_{\alpha A} = \sum_i \partial_{i\alpha A},$$

$$\overline{Q}^{A}_{\dot\alpha} = \sum_i [\theta_i^{\alpha A}\partial_{i\alpha\dot\alpha} + \eta_i^A \partial_{i\dot\alpha}],$$

$$D = \sum_i [-x_i^{\dot\alpha\alpha}\partial_{i\alpha\dot\alpha} - \tfrac{1}{2}\theta_i^{\alpha A}\partial_{i\alpha A} - \tfrac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} - \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha}],$$

$$C = \sum_i [-\tfrac{1}{2}\lambda_i^{\alpha}\partial_{i\alpha} + \tfrac{1}{2}\tilde\lambda_i^{\dot\alpha}\partial_{i\dot\alpha} + \tfrac{1}{2}\eta_i^A\partial_{iA}] = \sum_i h_i,$$

$$S^A_\alpha = \sum_i [-\theta^B_{i\alpha}\theta_i^{\beta A}\partial_{i\beta B} + x_{i\alpha}{}^{\dot\beta}\theta_i^{\beta A}\partial_{\beta\dot\beta} + \lambda_{i\alpha}\theta_i^{\gamma A}\partial_{i\gamma} + x_{i+1\,\alpha}{}^{\dot\beta}\eta_i^A\partial_{i\dot\beta} - \theta^B_{i+1\,\alpha}\eta_i^A\partial_{iB}],$$

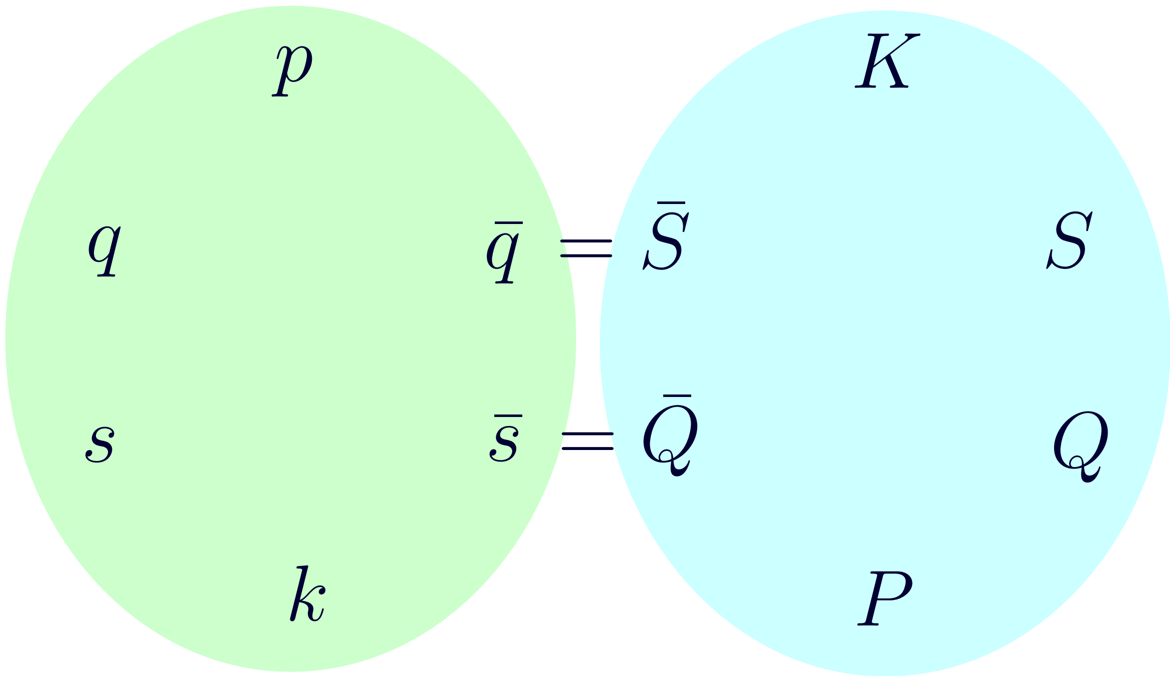
$$\overline{S}_{\dot\alpha A} = \sum_i [x_{i\dot\alpha}{}^{\beta}\partial_{i\beta A} + \tilde\lambda_{i\dot\alpha}\partial_{iA}],$$

$$K_{\alpha\dot\alpha} = \sum_i [x_{i\alpha}{}^{\dot\beta}x_{i\dot\alpha}{}^{\beta}\partial_{i\beta\dot\beta} + x_{i\dot\alpha}{}^{\beta}\theta^B_{i\alpha}\partial_{i\beta B} + x_{i\dot\alpha}{}^{\beta}\lambda_{i\alpha}\partial_{i\beta} + x_{i+1\,\alpha}{}^{\dot\beta}\tilde\lambda_{i\dot\alpha}\partial_{i\dot\beta} + \tilde\lambda_{i\dot\alpha}\theta^B_{i+1\,\alpha}\partial_{iB}].$$

$$\partial_{i\alpha\dot\alpha} = \frac{\partial}{\partial x_i^{\alpha\dot\alpha}}, \qquad \partial_{i\alpha A} = \frac{\partial}{\partial \theta_i^{\alpha A}}, \qquad \partial_{i\alpha} = \frac{\partial}{\partial \lambda_i^{\alpha}}, \qquad \partial_{i\dot\alpha} = \frac{\partial}{\partial \tilde\lambda_i^{\dot\alpha}}, \qquad \partial_{iA} = \frac{\partial}{\partial \eta_i^{A}}.$$

Conventional and dual superconformal symmetries

The generators of conventional and dual superconformal symmetry are not all independent:



Similar picture found by [Berkovits,Maldacena],[Beisert,Ricci,Tseytlin,Wolf] by combining bosonic T-duality with a fermionic one in the AdS sigma model.

How tree amplitudes behave under the symmetries

Invariance under the superconformal algebra.

$$J_a \mathcal{A}_n = 0$$

Covariance under the dual superconformal algebra:

$$\{P_{\alpha\dot\alpha}, Q_{\alpha A}, \bar Q^A_{\dot\alpha} = \bar s^A_{\dot\alpha}, \bar S_{\dot\alpha A} = \bar q_{\dot\alpha A}\}\mathcal{A}_n = 0$$

$$K^{\alpha\dot\alpha}\mathcal{A}_n = -\textstyle\sum_{i=1}^n x_i^{\alpha\dot\alpha}\mathcal{A}_n\,, \qquad S^A_\alpha \mathcal{A}_n = -\textstyle\sum_{i=1}^n \theta^A_{i\alpha}\mathcal{A}_n$$

[JMD,Henn,Korchemsky,Sokatchev],[Brandhuber,Heslop,Travaglini],[JMD,Henn].

Note: $$\bar s\mathcal{A} = 0 \implies s\mathcal{A} = 0 \implies k\mathcal{A} = 0$$

$$\mathcal{A}_{\rm tree} = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle \ldots \langle n1\rangle}\mathcal{P} = \frac{\delta^4(p)\delta^8(q)}{\langle 12\rangle \ldots \langle n1\rangle}\Big(1 + \sum_{s,t} R_{r;s,t} + \ldots\Big)$$

Here $R_{r;s,t}$ is a dual superconformal invariant (best to use x, θ, λ variables):

$$R_{r;s,t} = \frac{\langle s\ s-1\rangle\langle t\ t-1\rangle \delta^4(\langle r|x_{rs}x_{st}|\theta_{tr}\rangle + \langle r|x_{rt}x_{ts}|\theta_{sr}\rangle)}{x^2_{st}\langle r|x_{rs}x_{st}|t\rangle\langle r|x_{rs}x_{st}|t-1\rangle\langle r|x_{rt}x_{ts}|s\rangle\langle r|x_{rt}x_{ts}|s-1\rangle}$$

Commuting the two algebras

What algebraic structure combines both superconformal and dual superconformal algebras? [JMD,Henn,Plefka]

We want to commute charges coming from both algebras.

First we must reformulate dual superconformal symmetry as an invariance.

Subtract the weight terms:

$$\tilde{K}^{\alpha\dot{\alpha}} = K^{\alpha\dot{\alpha}} + \textstyle\sum_{i=1}^{n} x_i^{\alpha\dot{\alpha}} \qquad \text{and} \qquad \tilde{S}_{\alpha}^{A} = S_{\alpha}^{A} + \textstyle\sum_{i=1}^{n} \theta_{i\alpha}^{A}$$

So that: $\tilde{K}\mathcal{A} = 0$ and $\tilde{S}\mathcal{A} = 0$.

We want to remove all x and θ dependence.

Use $P_{\alpha\dot{\alpha}}$ and $Q_{\alpha A}$ to set $x_1 = 0$ and $\theta_1 = 0$. Eliminate all other x_i and θ_i in favour of $\lambda_i, \tilde{\lambda}_i, \eta_i$.

$$S'^{A}_{\alpha} = -\sum_{i=1}^{n}\left[\sum_{j=1}^{i-1}\lambda_j^{\gamma}\eta_j^{A}\lambda_{i\alpha}\frac{\partial}{\partial\lambda_i^{\gamma}} + \sum_{j=1}^{i}\lambda_{j\alpha}\tilde{\lambda}_j^{\dot{\beta}}\eta_i^{A}\frac{\partial}{\partial\tilde{\lambda}_i^{\dot{\beta}}} - \sum_{j=1}^{i}\lambda_{j\alpha}\eta_j^{B}\eta_i^{A}\frac{\partial}{\partial\eta_i^{B}} + \sum_{j=1}^{i-1}\lambda_{j\alpha}\eta_j^{A}\right]$$

Now on the same footing as the ordinary superconformal generators.

Yangians

Consider a Lie algebra: $[J_a, J_b] = f_{ab}{}^c J_c$

Can introduce some 'level one' generators

$$[J_a, J_b^{(1)}] = f_{ab}{}^c J_c^{(1)}$$

The Jacobi identity can be 'quantised' (Drinfeld):

$$[J_a^{(1)}, [J_b^{(1)}, J_c]] + \mathrm{cyc}(a,b,c) = h f_{ar}{}^l f_{bs}{}^m f_{ct}{}^n f^{rst} \{J_l, J_m, J_n\}$$

Then J and $J^{(1)}$ generate the Yangian.

On a chain the generators J can be given by sums of single site generators

$$J_a = \sum_i J_{ia}$$

Then $J_a^{(1)}$ can take the bilocal form [Dolan,Nappi,Witten]

$$J_a^{(1)} = f_a{}^{cb} \sum_{i<j} J_{ib} J_{jc}$$

if the representation $\mathcal{R}$ of J_i satisfies the condition that the adjoint appears only once in $\mathcal{R} \otimes \bar{\mathcal{R}}$.

From dual conformal symmetry to the Yangian

We want to identify two bilocal Yangian generators $J_a^{(1)}$ with the symmetries K' and S'

Inspecting the dimensions and Lorentz and $su(4)$ labels suggests the identification

$$p^{(1)}_{\alpha\dot{\alpha}} \sim K'_{\alpha\dot{\alpha}}, \qquad q^{(1)}{}_{\alpha}^{A} \sim S'^{A}_{\alpha}$$

Indeed we can add terms to S' which annihilate the amplitudes on their own

$$\Delta S^{A}_{\alpha} = \tfrac{1}{2}\left[-q^{A}_{\gamma} m^{\gamma}_{\alpha} + q^{A}_{\alpha}\tfrac{1}{2}d_{\lambda} + n q^{A}_{\alpha} + p^{\dot{\beta}}_{\alpha}\bar{s}^{A}_{\dot{\beta}} + q^{B}_{\alpha} r^{A}_{B} - q^{A}_{\alpha}\tfrac{1}{4}d_{\eta} + q^{A}_{\alpha}\right]$$

and we arrive at the bilocal formula

$$q^{(1)A}_{\alpha} := \sum_{i>j}\left[m^{\gamma}_{i\alpha} q^{A}_{j\gamma} - \tfrac{1}{2}(d_i + c_i) q^{A}_{j\alpha} + p^{\dot{\beta}}_{i\alpha}\bar{s}^{A}_{j\dot{\beta}} + q^{B}_{i\alpha} r^{A}_{jB} - (i \leftrightarrow j)\right].$$

The remaining generators in the level one multiplet come by acting with level zero generators.

The generator $p^{(1)}$ so obtained coincides with K' after similarly adding terms which annihilate the amplitude.

Cyclicity

Yangians are not normally consistent with the cyclicity of a closed chain.

Here this problem is avoided by a remarkable mechanism.

Consider

$$J_a^{(1)} = f_a{}^{cb} \sum_{1 \leq i < j \leq n} J_{ib} J_{jc}$$

$$\tilde{J}_a^{(1)} = f_a{}^{cb} \sum_{2 \leq i < j \leq n+1} J_{ib} J_{jc}$$

Then cyclicity implies $J_a^{(1)} - \tilde{J}_a^{(1)}$ should annihilate the amplitude.

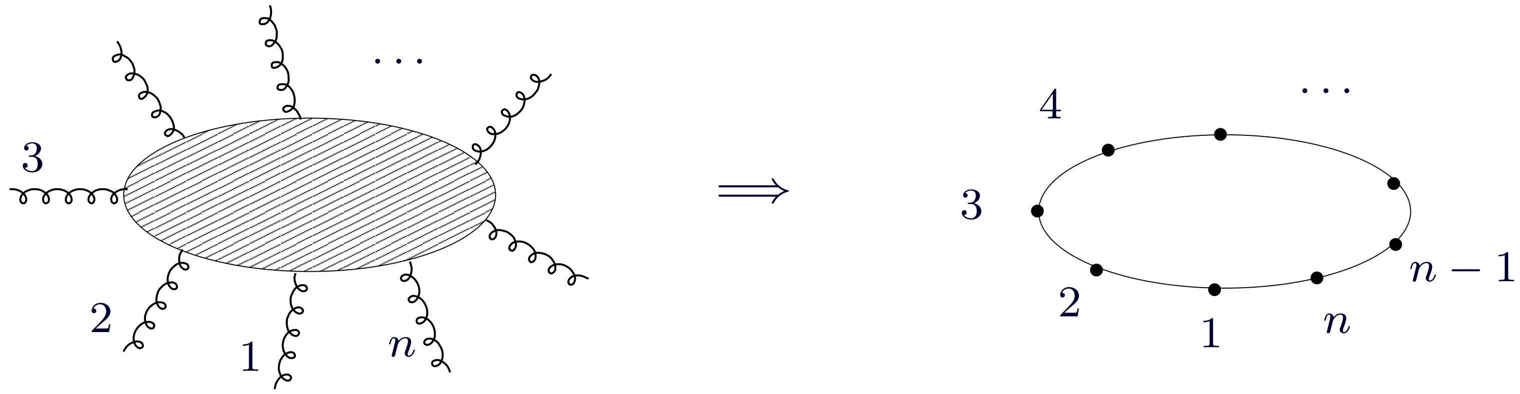
One finds the following term which, in general, does not annihilate the amplitude

$$f_a{}^{cb} f_{bc}{}^d J_{1d}$$

But for certain superalgebras this vanishes identically (those with vanishing Killing form):

$$psl(n|n), osp(2n+2|2), D(2,1;\alpha), P(n), Q(n)$$

As far as the algebraic representations are concerned amplitudes are identical to local operators.



Fields in a single trace operator can be written as (e.g.) $\Phi_{AB} = c^\dagger_A c^\dagger_B |0\rangle, \quad \Psi_{\alpha A} = a^\dagger_\alpha c^\dagger_A |0\rangle$

The tree-level get deformed by quantum corrections

They involve operators which can increase or decrease the number of sites

Yangian symmetry implies an infinite number of charges which commute with the spin chain Hamiltonian (anomalous dilatation generator)

Suggests that all the extremely powerful techniques applied to the spectral problem should have a version for amplitudes

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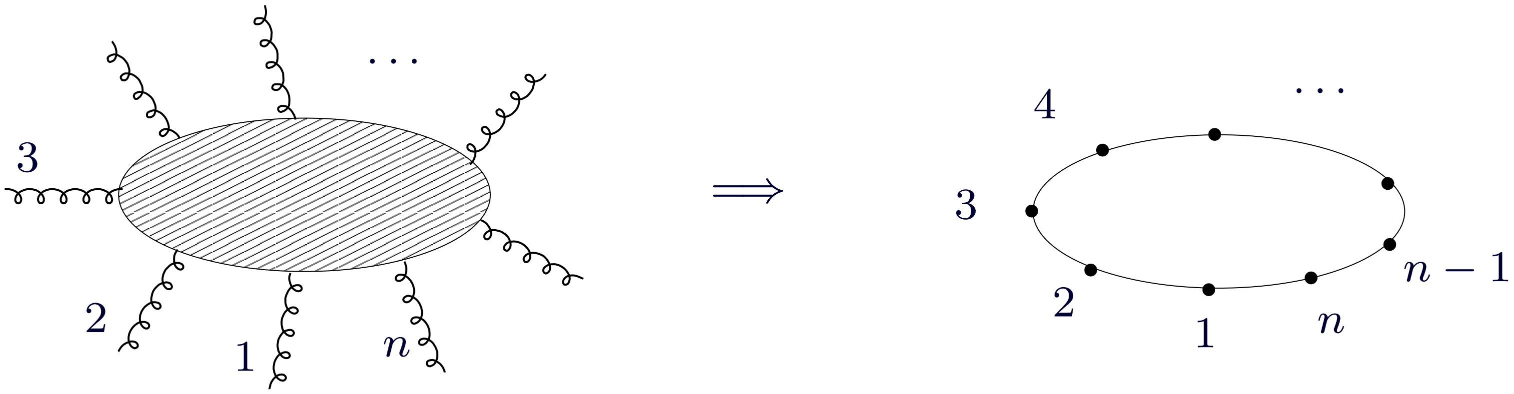

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The Algebraic S-matrix ?