On gluon scattering amplitudes

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based on

arXiv:0807.1095 [hep-th] Drummond, J.H., Korchemsky, Soktachev

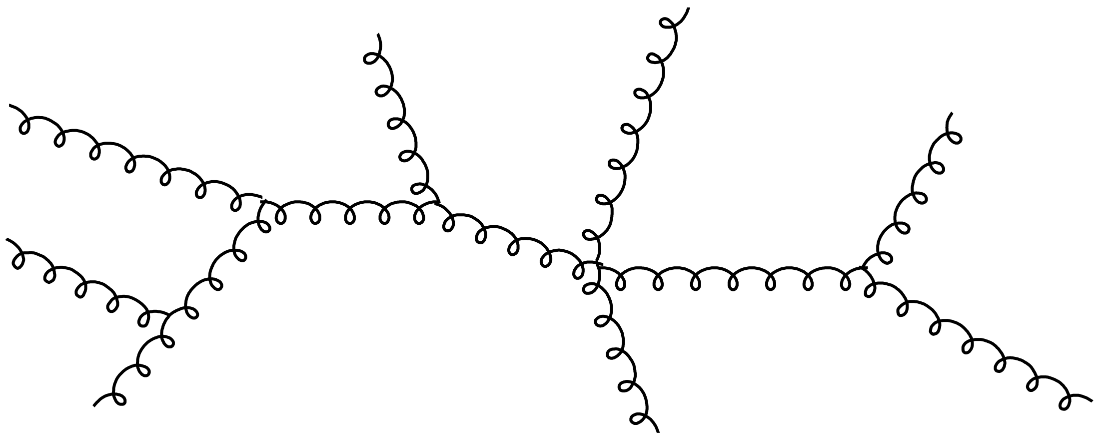
arXiv:0808.0491 [hep-th] Drummond, J.H., Korchemsky, Soktachev

arXiv:0808.2475 [hep-th] Drummond, J.H.

arXiv:0902.2987 [hep-th] Drummond, J.H., Plefka

Motivation and outline

- ✔ gluon scattering amplitudes in Yang-Mills theory

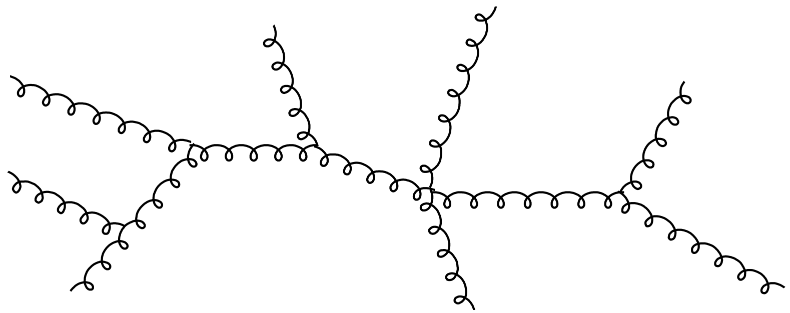


number of external gluons	4	5	6	7	8	9	10
number of diagrams	4	25	220	2485	34300	559405	10525900

Questions we want to ask:

- ✔ can we compute tree-level amplitudes for an arbitrary number of gluons?
- ✔ what are the symmetry properties of the amplitudes?

Tree-level gluon scattering amplitudes



- ✔ how to compute them efficiently?
  review e.g. [Dixon, hep-ph/9601359]
  - ✗ depend on $\{p_i^\mu, h_i = \pm 1, a_i\}$
  - ✗ colour decomposition

$$A^{\text{tree}}(\{p_i, h_i, a_i\}) = \sum_\sigma \text{Tr}\,(t^{a_{\sigma(1)}} \dots t^{a_{\sigma(n)}}) A^{\text{tree}}(\sigma(1), \dots \sigma(n))$$

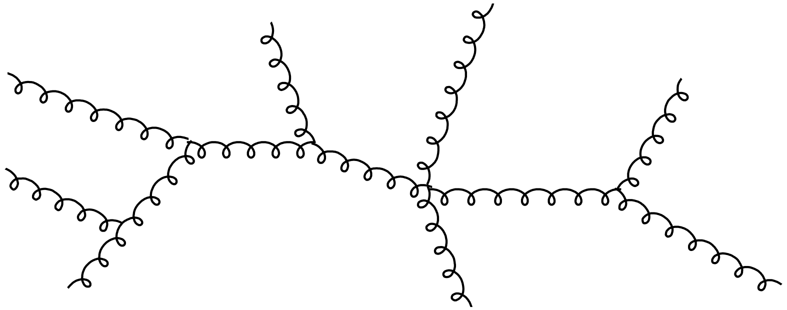
  - ✗ helicity classification

    supersymmetry constraints: $A(+,+,\dots,+) = 0\,, \qquad A(-,+,\dots,+) = 0$

    maximally helicity violating amplitudes (MHV): $A(-,+,\dots,+,-,+,\dots,+)$, etc.

    more ‘flipped’ (negative) helicities: non-MHV amplitudes

Tree-level gluon scattering amplitudes



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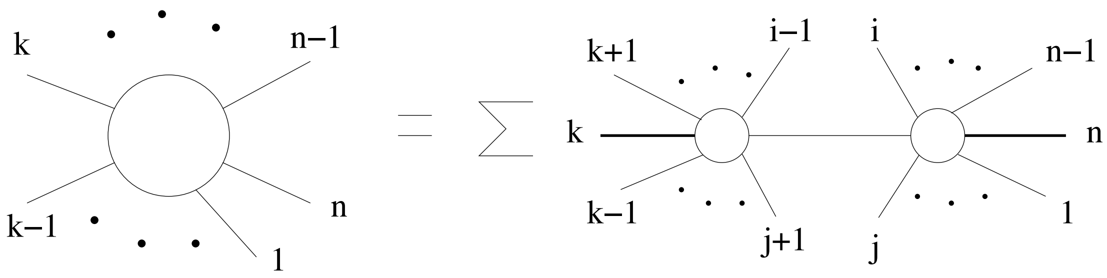
    maximally helicity violating amplitudes (MHV): $A(-,+,\ldots,+,-,+,\ldots,+)$, etc.
    more 'flipped' (negative) helicities: non-MHV amplitudes
- ✔ example: MHV case: [Parke, Taylor 1986],[Berends, Giele 1988]

$$A(-,+,\ldots,+,-_k,+,\ldots,+) = \frac{\langle 1\, k\rangle^4}{\langle 1\, 2\rangle \ldots \langle n\, 1\rangle}\, \delta^{(4)}(\sum_i p_i)$$

  on-shell momenta $p_i^{\alpha\dot\alpha} = \lambda_i^\alpha \tilde\lambda_i^{\dot\alpha}$, useful notation $\langle i\, j\rangle = \lambda_i^\alpha \lambda_{j\,\alpha}$
- ✔ non-MHV case much more complicated
  e.g. next-to-MHV amplitudes [Kosower, hep-th/0406175], split-helicity amplitudes [Britto et al, hep-th/0503198]
- ✔ can we compute all tree-level amplitudes?

On-shell recursion relations

- ✔ recursion relations from analytical behaviour of the amplitudes [Britto, Cachazo, Feng + Witten, 2004]



- ✔ n-point amplitudes are obtained recursively from lower-point amplitudes
- ✔ all amplitudes are on-shell
- ✔ special cases can be solved analytically
  e.g. split-helicity amplitudes $A(-,\dots,-,+,\dots+)$
- ✔ does not require arbitrary reference spinor(s) as in MHV vertex approach

Scattering amplitudes in $N = 4$ super Yang-Mills

- at tree-level: gluon amplitudes in Yang-Mills = gluon amplitudes in super Yang-Mills
- why $\mathcal{N} = 4$ SYM? maximal supersymmetry, AdS/CFT correspondence, expected integrability
- particle content: gauge field, 4 fermions, 6 scalars
- superwavefunction

$$\Phi(p,\eta) = G^{+}(p) + \eta^{A}\Gamma_{A}(p) + 1/2\eta^{A}\eta^{B}S_{AB} + \ldots + 1/4!\epsilon_{ABCD}\eta^{A}\eta^{B}\eta^{C}\eta^{D}G^{-}(p)$$

  Grassmann variable η^{A} for bookkeeping, $A = 1,2,3,4$

- superamplitudes $\mathcal{A}_n = < \Phi_1 \ldots \Phi_n >$
- supersymmetry $p = \sum_{i=1}^{n} \lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}}\,, \qquad q = \sum_{i=1}^{n} \lambda_i^{\alpha}\eta_i^{A}\,, \qquad \bar{q} = \sum_{i=1}^{n} \tilde{\lambda}_i^{\dot{\alpha}}\frac{\partial}{\partial\eta_i^{A}}$

  implies $\mathcal{A}_n = \delta^{(4)}(p)\delta^{(8)}(q)\frac{\mathcal{P}_n}{\langle 1\,2\rangle\ldots\langle n\,1\rangle}$

  $\mathcal{P}_n$ is a polynomial in the $\{\eta_i\}$, MHV: η^0, NMHV: η^4, NNMHV: η^8 etc.
- MHV case: [Nair 1988]

$$\mathcal{P}_n^{\mathrm{MHV}} = 1\,, \qquad \mathcal{A}_n^{\mathrm{MHV}} = \frac{\delta^{(4)}(p)\delta^{(8)}(q)}{\langle 1\,2\rangle\ldots\langle n\,1\rangle}$$

Use of on-shell superspace for scattering amplitudes

[Nair 1988]

[Witten 2003]
[Georgiou, Glover, Khoze 2004][Brandhuber, Spence, Travaglini 2004]
[Huang 2005]

[Arkani-Hamed, Bianchi, Brandhuber, Cachazo, Drummond, Elvang, Freedman, J.H., Heslop, Kaplan, Kiermaier, Korchemsky, Sokatchev, Travaglini ... 2008]

- ✔ MHV vertex approach
- ✔ on-shell recursion relations
- ✔ (for loops:) generalised unitarity

Important common feature:

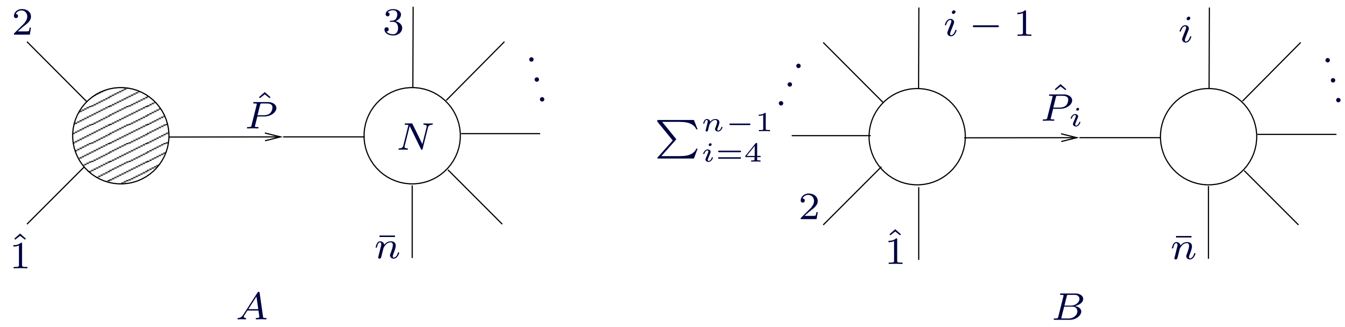
intermediate state sums are replaced by Grassmann integral

$$\sum_{\text{states}} A_L \frac{1}{P^2} A_R \quad \Rightarrow \quad \int d^4\eta \mathcal{A}_L \frac{1}{P^2} \mathcal{A}_R$$

Grassmann integral can be easily carried out thanks to supersymmetry

$$\mathcal{A} \propto \delta^{(8)} \left(\sum_i \lambda_i^\alpha \eta_i^A \right)$$

[Drummond, J.H. 2008]



- ✔ white circles: MHV superamplitude $\mathcal{A}_n^{\rm MHV} = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}$
- ✔ inhomogeneous term

$$B = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}\sum_{i=4}^{n-1} R_{n;2\,i}$$

- ✔ solution

$$\mathcal{A}_n^{\rm NMHV} = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\langle 1\,2\rangle\langle 2\,3\rangle\ldots\langle n\,1\rangle}\sum_{2\le s<t\le n-1} R_{n;st}$$

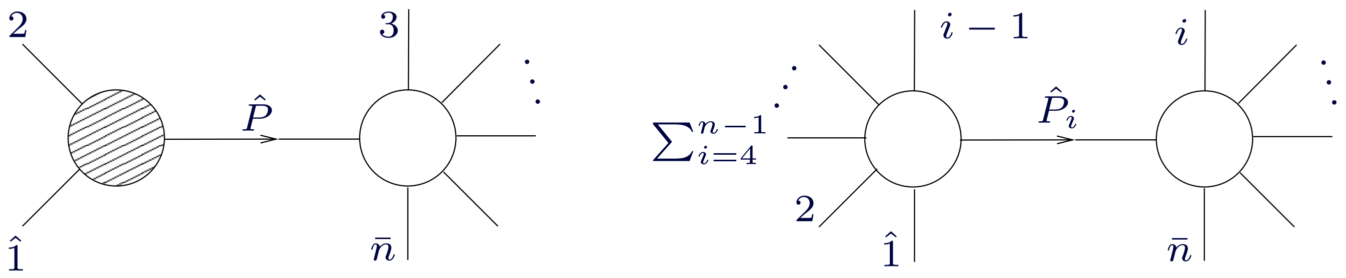
- ✔ proof by induction

$$A = \frac{\delta^{(4)}(p)\,\delta^{(8)}(q)}{\prod_{j=1}^{n}\langle j\,j+1\rangle}\sum_{3\le s<t\le n-1} R_{n;st} \qquad \Rightarrow \qquad A + B = \mathcal{A}_n^{\rm NMHV}$$

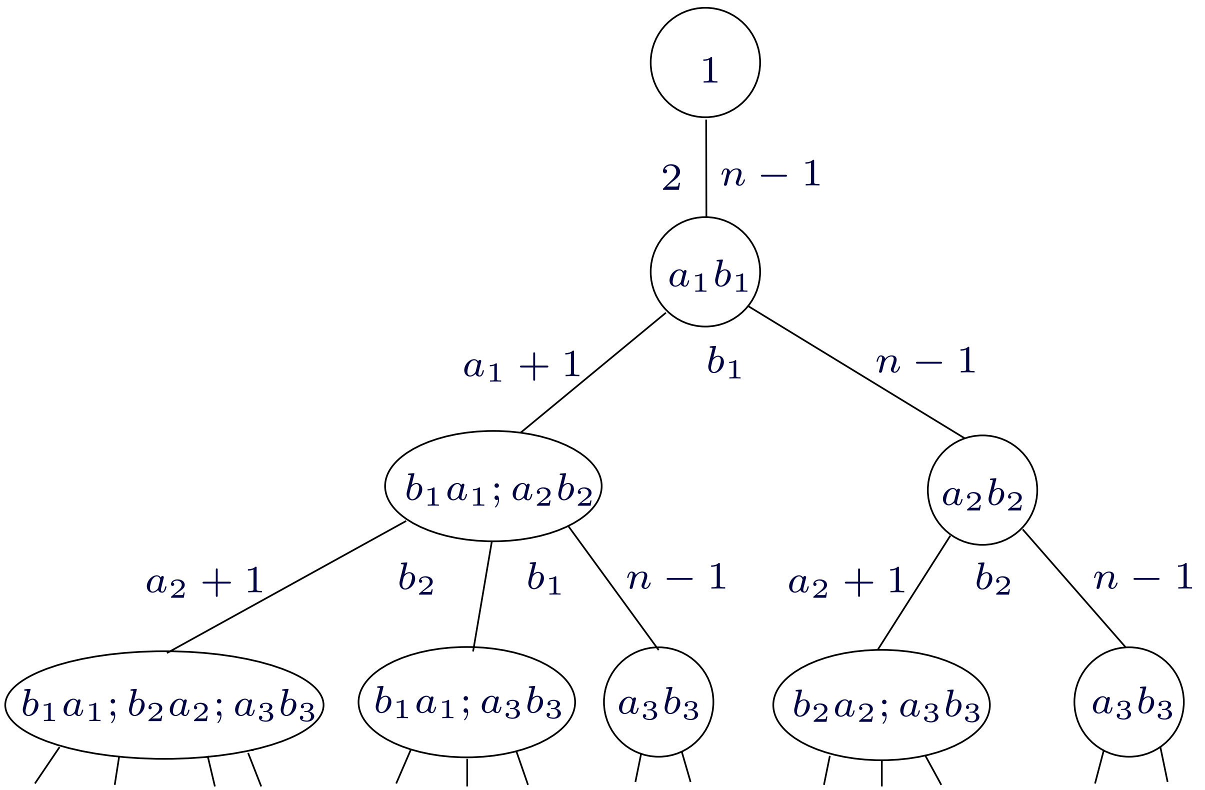
All tree-level scattering amplitudes in $\mathcal{N}=4$ super Yang-Mills

[Drummond and J.H. 2008]

- recursion relation for full superamplitude



- solution



All tree-level scattering amplitudes in $\mathcal{N}=4$ super Yang-Mills

- ✔ reminder: supersymmetry implies $\mathcal{A}_n = \delta^{(4)}(p)\delta^{(8)}(q)\dfrac{\mathcal{P}_n}{\langle 1\,2\rangle \ldots \langle n\,1\rangle}$

- ✔ first cases [Drummond, J.H., Korchemsky, Sokatchev 2008]

$$\mathcal{P}_n^{\mathrm{MHV}} = 1\,, \qquad \mathcal{P}_n^{\mathrm{NMHV}} = \sum_{2\leq i,j\leq n-1} R_{n;i,j}$$

with

$$R_{n;i,j} = \frac{< i\; i-1 >< j\; j-1 > \delta^{(4)}(\Xi_{n;ij})}{x_{ij}^2 < n|x_{ni}x_{ij}|j >< n|x_{ni}x_{ij}|j-1 >< n|x_{nj}x_{ji}|i >< n|x_{nj}x_{ji}|i-1 >}$$

where

$$\Xi_{n;ij} =< n|x_{ni}x_{ij}|\theta_{jn} > + < n|x_{nj}x_{ji}|\theta_{in} >$$

and where x and θ are dual variables defined by

$$\lambda_i^{\alpha}\tilde{\lambda}_i^{\dot{\alpha}} = x_i^{\alpha\dot{\alpha}} - x_{i+1}^{\alpha\dot{\alpha}}\,, \qquad \lambda_i^{\alpha}\eta_i^{A} = \theta_i^{\alpha A} - \theta_{i+1}^{\alpha A}$$

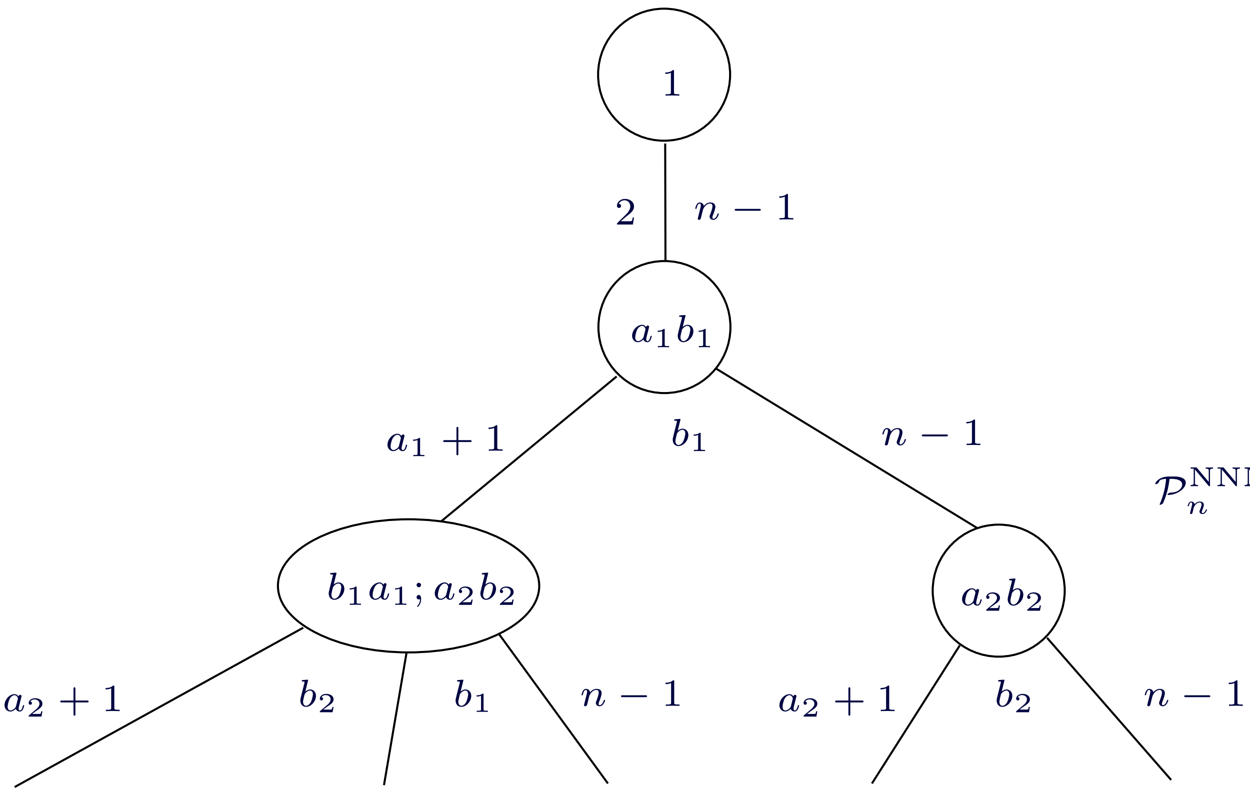
All tree-level scattering amplitudes in $\mathcal{N}=4$ super Yang-Mills

- reminder: supersymmetry implies $\mathcal{A}_n = \delta^{(4)}(p)\delta^{(8)}(q)\dfrac{\mathcal{P}_n}{\langle 1\,2\rangle \ldots \langle n\,1\rangle}$

- first cases [Drummond, J.H., Korchemsky, Sokatchev 2008]

$$\mathcal{P}_n^{\mathrm{MHV}} = 1\,, \qquad \mathcal{P}_n^{\mathrm{NMHV}} = \sum_{2\le i,j\le n-1} R_{n;i,j}$$

- solve supersymmetrised recursion relations [Drummond and J.H. 2008]



$$\mathcal{P}^{\mathrm{MHV}} = 1$$

$$\mathcal{P}_n^{\mathrm{NMHV}} = \textstyle\sum_{2\le a_1,b_1\le n-1} R_{n;a_1b_1}$$

$$\mathcal{P}_n^{\mathrm{NNMHV}} = \sum R_{n;a_1b_1}\Big[\sum R^{0;a_1b_1}_{n;b_1a_1;a_2b_2} + \\ + R^{a_1b_1;0}_{n;a_2b_2}\Big]$$

$$\ldots$$

How to extract gluon amplitudes

- reminder: superwavefunction

$$\Phi(p,\eta) = G^{+}(p) + \eta^{A}\Gamma_{A}(p) + \ldots + 1/4!\epsilon_{ABCD}\eta^{A}\eta^{B}\eta^{C}\eta^{D}G^{-}(p)$$

negative helicity gluon state at point i $\Longleftrightarrow$ $1/4!\epsilon_{ABCD}\eta_i^A\eta_i^B\eta_i^C\eta_i^D = (\eta_i)^4$

positve helicity gluon state at point j $\Longleftrightarrow$ $\eta_j^A = 0$

- gluon amplitudes correspond to $(\eta_i)^4$ components of the superamplitudes

  example:

$$\mathcal{A}_n = (\eta_1)^4(\eta_2)^4(\eta_4)^4 A(- - + - + \ldots +) + \ldots$$

  equivalently,

$$A(- - + - + \ldots +) = \int d^4\eta_1\, d^4\eta_2\, d^4\eta_4\, \mathcal{A}_n$$

- structure of $\mathcal{A}_n$: product of Grassmann delta functions $\Longrightarrow$ Grassmann integrals can be evaluated
- for explicit examples see [Drummond and J.H. 2008]
- conclusion:

  extracting gluon amplitudes from the superamplitudes $=$ (simple) linear algebra

Generalised unitarity in on-shell superspace

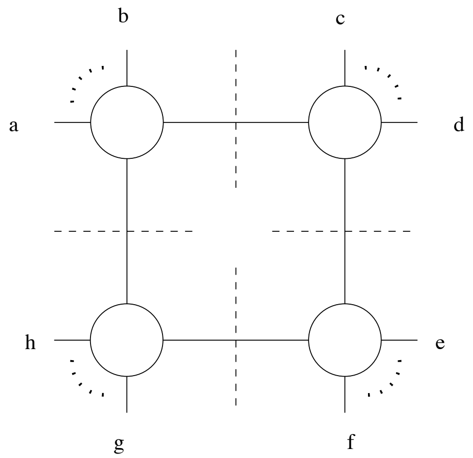
- one loop amplitudes in $\mathcal{N} = 4$ SYM given by box integrals

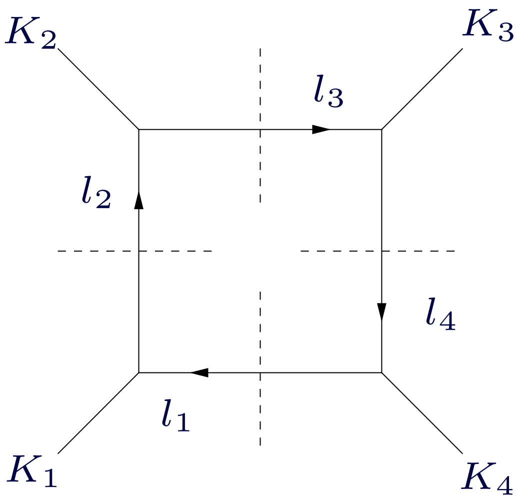
$$A_{n;1} = \sum (c^{4\mathrm{m}} I^{4\mathrm{m}} + c^{3\mathrm{m}} I^{3\mathrm{m}} + c^{2\mathrm{mh}} I^{2\mathrm{mh}} + c^{2\mathrm{me}} I^{2\mathrm{me}} + c^{1\mathrm{m}} I^{1\mathrm{m}})$$

with

$$I(K_1, K_2, K_3, K_4) = -i(4\pi)^{2-\epsilon} \int \frac{d^{4-2\epsilon} l}{(2\pi)^{4-2\epsilon}} \frac{1}{l^2 (l+K_1)^2 (l+K_1+K_2)^2 (l-K_4)^2}$$

- box coefficients can be obtained from generalised unitarity [Britto, Cachazo, Feng 2005]

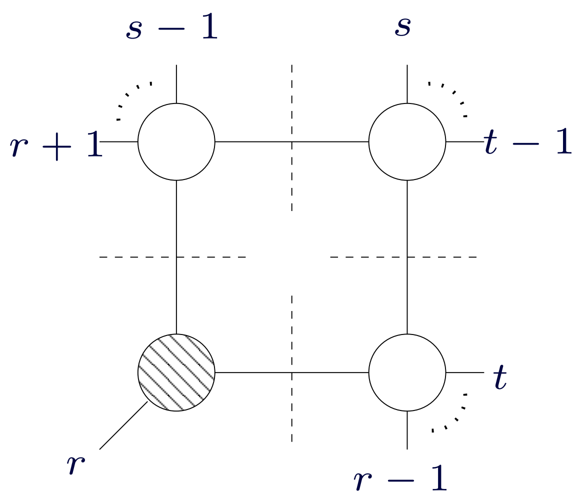




Example: NMHV superamplitudes at one loop

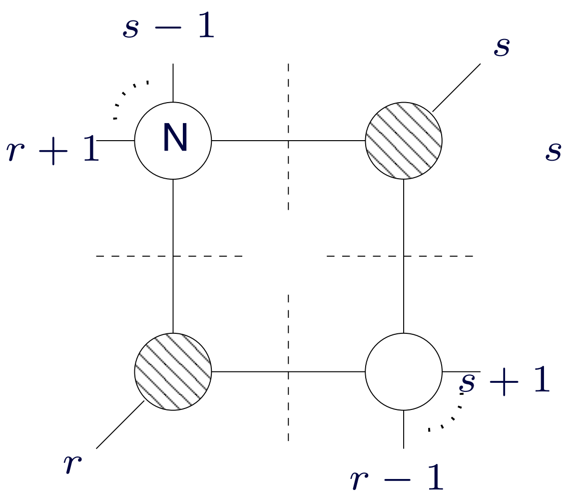
[Drummond, J.H., Korchemsky, Sokatchev 2008]

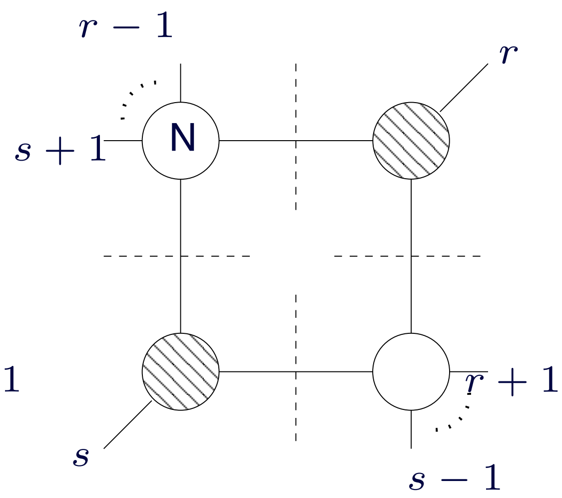
✔ three mass box coefficients



$$\sum_{r,s,t} C^{3\mathrm{m}}_{r,r+1,s,t} I_{r,r+1,s,t} = \frac{\delta^{(8)}(q)}{\prod_1^n \langle i\, i{+}1\rangle} \sum_{r,s,t} R_{rst}\, F_{r,r+1,s,t}$$

✔ 'two mass easy' box coefficients

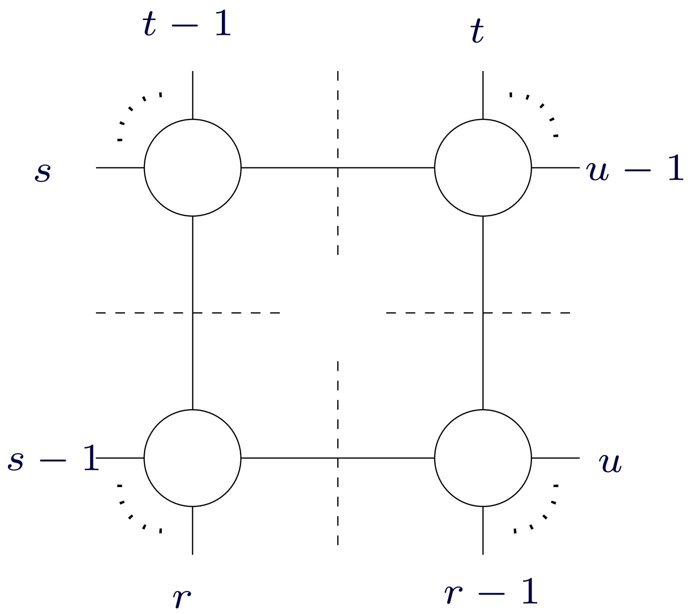




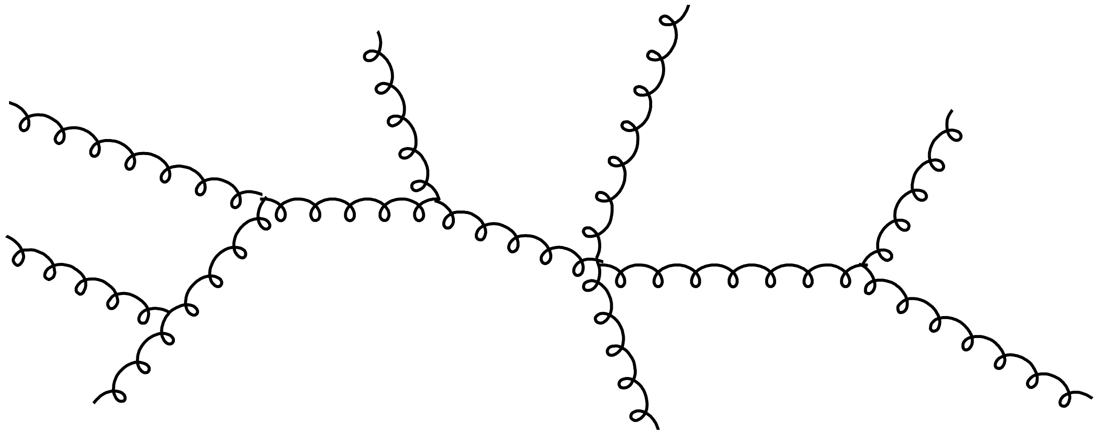
Beyond NMHV amplitudes at one loop

[Drummond, J.H., Korchemsky, Sokatchev 2008]

- four-mass box integrals appear; coefficient (e.g. for NNMHV superamplitudes):



Summary



- ✔ formula for all tree-level gluon scattering amplitudes
  - ✗ explicit analytical expression
  - ✗ does not contain arbitrary reference spinor(s) as in the MHV vertex approach
  - ✗ simplicity and compactness of $\mathcal{N}=4$ formula related to symmetries
- ✔ generalised unitarity in $\mathcal{N}=4$ on-shell superspace
  - ✗ intermediate helicity sums turn into simple Grassmann integrals
  - ✗ may be useful in higher loop calculations

Symmetries of scattering amplitudes in $N = 4$ super Yang-Mills

- superconformal symmetry $psu(2,2|4)$ — cf. [Witten 2003]

  $psu(2,2|4)$ algebra:
  $$[J_a, J_b\} = f_{ab}{}^{c} J_c\,, \qquad J_a = \sum_{i=1}^{n} J_{ia}$$

  for example
  $$p^{\alpha\dot\alpha} = \sum_{i=1}^{n} \lambda_i^{\alpha} \tilde\lambda_i^{\dot\alpha}\,, \qquad \bar q_A^{\dot\alpha} = \sum_{i=1}^{n} \tilde\lambda_i^{\dot\alpha} \frac{\partial}{\partial \eta_i^A}\,, \qquad k_{\alpha\dot\alpha} = \sum_{i=1}^{n} \frac{\partial}{\partial\lambda_i^{\alpha}} \frac{\partial}{\partial\tilde\lambda_i^{\dot\alpha}}$$

- 'dual' superconformal symmetry — [Drummond, J.H., Korchemsky, Sokatchev 2008]

- closure of algebra give Yangian $Y(psu(2,2|4))$ — [Drummond, J.H., Plefka 2009]

  level-one Yangian generators:
  $$Q_a = f_a{}^{cb} \sum_{1\le i<j\le n} J_{ib} J_{jc}$$

- spin chain analogy