On Double-Cuts of One-Loop Amplitudes

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Outline

- Multiple-Cuts & Analytic Calculations
- 2-point coefficients from Double-Cuts
  - Simplified Phase-Space Integration
- Contour Integrals of Rational Functions
  - Hermite Polynomial Reduction
- An example with S@M: A(g,g,g,Z,Z)
- Summary

Process-Independent Strategy

✱ Properties of the S-Matrix

- a general mathematical property: **Analyticity** of Scattering-Amplitudes
  ▷ *Scattering Amplitudes are determined by their singularities*

- a general physical property: **Unitarity** of Scattering-Amplitudes
  ▷ *The residues at singular points are products of simpler amplitudes, with lower number of particles and/or less loops*

Motivations

- Integration addicted (blame Remiddi for it!)
- Impressive results from Numerical Unitarity — Maitre & Zanderighi's talk
- many 2 => 3, and some 2 =>4 process can be computed analytically (virtual cnt'n)
- Improving Numerical results with new Analytic insights
  - The Rational-Term of One-Loop Amplitudes
  - D-dim Double-Cuts are sensitive to the Rational Terms — Kunszt's talk

One-Loop Scattering Amplitudes

- n-particle Scattering: $1+2 \rightarrow 3+4+\ldots+n$

- Reduction in D-shifted Basis

Passarino-Veltman; Tarasov;
Bern, Chalmers, Dixon, Dunbar, Kosower, Morgan;
Binoth, Guillet, Heinrich;
Giele, Kunszt, Melnikov

$$A_n^{(D)} = \sum_{r=0} e_{(r)} I_5^{(D+2r)} + \sum_{r=0} d_{(r)} I_4^{(D+2r)} + \sum_{r=0} c_{(r)} I_3^{(D+2r)} + + \sum_{r=0} b_{(r)} I_2^{(D+2r)} + \sum_{r=0} a_{(r)} I_1^{(D+2r)}$$

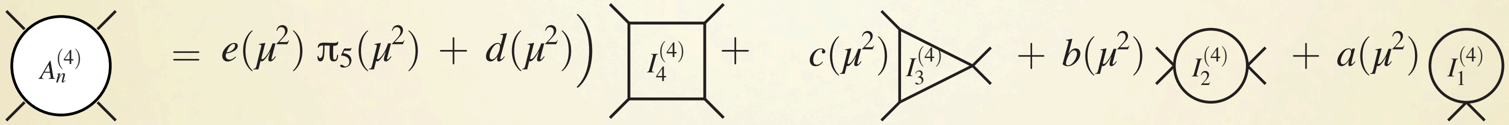
$$\overset{\varepsilon \to 0}{=} \quad \text{PolyLogarithms} \quad + \quad \text{Rational}$$

- $a, b, c, d, e, \ldots, f, g$ are the unknowns: they are known to be rational functions of kinematic invariants, and D-independent in this basis.

One-Loop Scattering Amplitudes (cont'd)

- Loop Splitting: $L_{(D)} = \ell_{(4)} + \mu_{(-2\varepsilon)} \quad \Rightarrow \quad \int d^{4-2\varepsilon}L = \int d^{-2\varepsilon}\mu \int d^4\ell$

- 4-dim Kernel



▷ $e(\mu^2), d(\mu^2), \ldots, a(\mu^2)$ are polynomials of in μ^2

Ossola, Papadopoulos, Pittau
Ellis, Giele, Kunszt, Melnikov
Britto, Feng, Yang
Britto, Feng & P.M.
Badger

- The polynomial structure of $e(\mu^2), d(\mu^2), \ldots, a(\mu^2)$

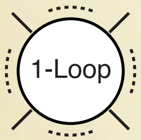
  is responsible for the D-shifted integrals:

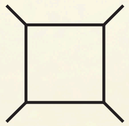
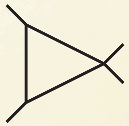
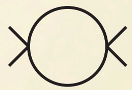
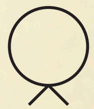
$$\int \frac{d^{-2\varepsilon}\mu}{(2\pi)^{-2\varepsilon}} (\mu^2)^r f(\mu^2) = -\varepsilon(1-\varepsilon)(2-\varepsilon)\cdots(r-1-\varepsilon)(4\pi)^r \int \frac{d^{2r-2\varepsilon}\mu}{(2\pi)^{2r-2\varepsilon}} f(\mu^2)$$

▷ The reconstruction of the 4-dim kernel of any one-loop amplitude contains all the information for the complete reconstruction of the amplitude in D-dimensions.

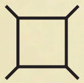
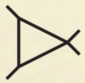
One-Loop Scattering Amplitudes (cont'd)

- n-particle Scattering: $1+2 \to 3+4+\ldots+n$
- Reduction to a Scalar-Integral Basis Passarino-Veltman



$$= \sum_{10^2-10^3} \int d^D\ell \, \frac{\ell^\mu \ell^\nu \ell^\rho \ldots}{D_1 D_2 \ldots D_n} = c_4$$  $$+ c_3$$  $$+ c_2$$  $$+ c_1$$ 

- Known: Master Integrals Ellis, Zanderighi + FF + LoopTools

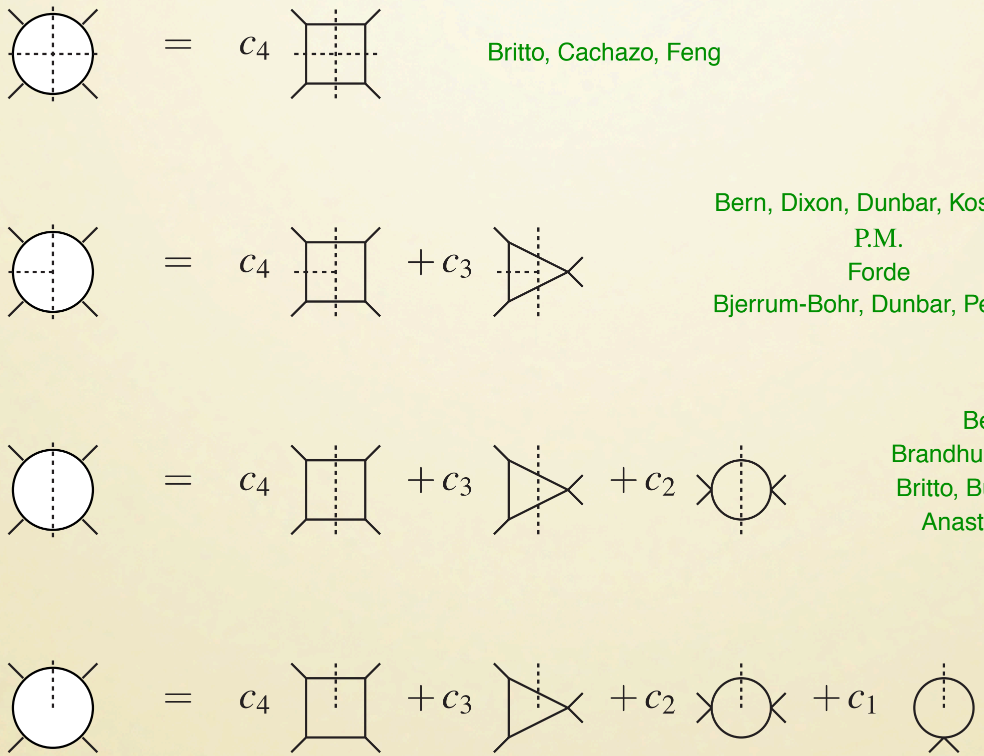
 $$= \int d^D\ell \, \frac{1}{D_1 D_2 D_3 D_4} \, ,$$  $$= \int d^D\ell \, \frac{1}{D_1 D_2 D_3} \, ,$$  $$= \int d^D\ell \, \frac{1}{D_1 D_2} \, ,$$  $$= \int d^D\ell \, \frac{1}{D_1}$$

- Unknowns: c_i are rational functions of external kinematic invariants

Generalised Unitarity: Analytic Techniques

- Multiple-cuts as *optical filters*

Replacing the original amplitude with simpler integrals fulfilling the same algebraic decomposition



Britto, Cachazo, Feng

Bern, Dixon, Dunbar, Kosower
P.M.
Forde
Bjerrum-Bohr, Dunbar, Perkins

Bern, Dixon, Dunbar, Kosower
Brandhuber, McNamara, Spence, Travaglini
Britto, Buchbinder, Cachazo, Feng, ⊕ P.M.
Anastasiou, Britto, Feng, Kunszt, P.M.
Forde; Badger

Glover, Williams
Britto, Feng

Cut-Conditions

- under Multiple On-shellness Conditions :

- the loop-momentum becomes complex ;
- some of its components (if not all) are frozen;
- the left over free components are *integration*-variable

$$q^2 = p^2 = 0 \,, \qquad \ell_\mu = x_1 \, p_\mu + x_2 \, q_\mu + x_3 \, \frac{\langle q|\gamma_\mu|p]}{2} + x_4 \, \frac{\langle p|\gamma_\mu|q]}{2}$$

- Closer look at the Integrand Structure

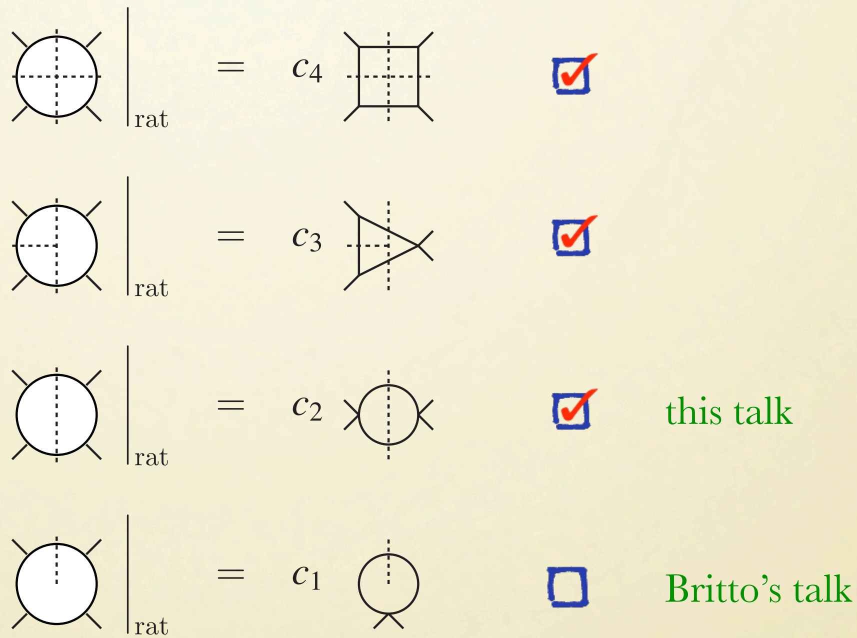
Numerator and denominator of the n-particle cut-integrand are mutivariate-polynomials in $(4-n)$ complex-variables:

$$\text{Cut}_n = \oint dx_1 \ldots dx_{4-n} \;\; \frac{P(x_1, \ldots, x_{4-n})}{Q(x_1, \ldots, x_{4-n})}$$

▷ Contour Integrals of Rational Functions $\sim$ Integrals by *partial fractioning*

Out of Cuts

- Any n-particle cut, after integration, might contain a *Rational Term* and a *Logarithmic Term*
- Extract the n-point coefficient from the *Rational Term* of the n-particle cut.

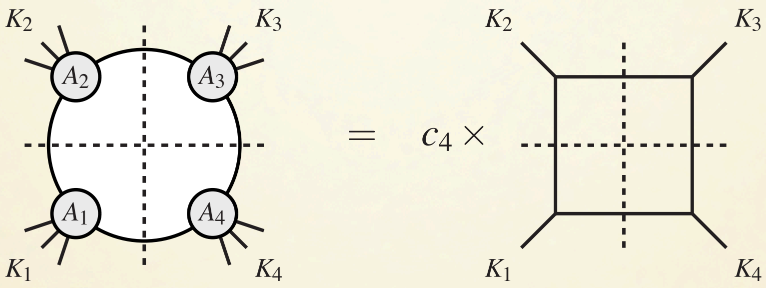


Let's try to avoid PV-Tensor Reduction

Quadruple-Cuts

Britto, Cachazo, Feng (2004)

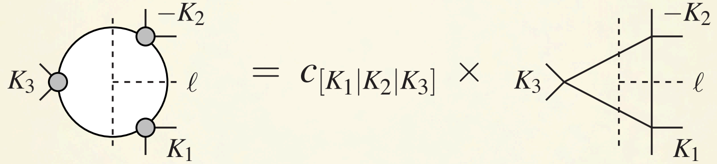
The discontinuity across the leading singularity, *via* quadruple cuts, is **unique**, and corresponds to the **coefficient** of the master **box**



- 4PLE-cut integrand: $I_4(\ell) = A_1^{\text{tree}} \times A_2^{\text{tree}} \times A_3^{\text{tree}} \times A_4^{\text{tree}}$
- Momentum-decomposition ansatz: $\ell^\mu = \alpha\, K_1^\mu + \beta\, K_2^\mu + \gamma\, K_3^\mu + \delta\, \varepsilon^\mu_{\nu\rho\sigma} K_1^\nu K_2^\rho K_3^\sigma$
- Cut-conditions: $D_1 = D_2 = D_3 = D_4 = 0 \qquad \Leftrightarrow$ coefficient constraints
- Solutions: $\ell^\mu_\pm = \alpha_0\, K_1^\mu + \beta_0\, K_2^\mu + \gamma_0\, K_3^\mu + \delta_\pm\, \varepsilon^\mu_{\nu\rho\sigma} K_1^\nu K_2^\rho K_3^\sigma$

$$c_4 = \frac{I_4(\ell_+) + I_4(\ell_-)}{2}$$

Triple-Cut



Forde (2008)

- 3ple-cut integrand: $I_3(\ell) = A_1(\ell) \times A_2(\ell) \times A_3(\ell)$
- Loop-Momentum decomposition:

$$\ell_\mu = \alpha_1\, p_\mu + \alpha_2\, q_\mu + t\, \frac{\langle q|\gamma_\mu|p]}{2} + \frac{1}{t}\, \frac{\langle p|\gamma_\mu|q]}{2}$$

$$p^\mu = \frac{K_1^\mu - (K_1^2/\gamma) K_2^\mu}{1-(K_1^2 K_2^2/\gamma)}\,, \qquad q^\mu = \frac{K_2^\mu - (K_2^2/\gamma) K_1^\mu}{1-(K_1^2 K_2^2/\gamma)}\,, \qquad q^2 = p^2 = 0\,,$$

- Cut-conditions: $D_1 = D_2 = D_3 = 0 \quad \Leftrightarrow \quad$ coefficient constraints

$$\alpha_1 = \frac{K_1^2(\gamma - K_2^2)}{\gamma^2 - K_1^2 K_2^2}\,, \qquad \alpha_2 = \frac{K_2^2(\gamma - K_1^2)}{\gamma^2 - K_1^2 K_2^2}\,, \qquad \gamma = (K_1 \cdot K_2) \pm \sqrt{\Delta}\,, \qquad \Delta = (K_1 \cdot K_2)^2 + K_1^2 K_2^2\,.$$

$$c_{[K_1,K_2,K_3]} \;=\; \frac{1}{2}\, \mathrm{Res}_{t=0} \left\{ I_3(\ell^+) + I_3(\ell^-) \right\}$$

Double-Cut Phase-Space Measure

- 4-dim LIPS Cacahazo, Svrček & Witten

$$\int d^4\Phi = \int d^4\ell_0\, \delta^{(+)}(\ell_0^2)\, \delta^{(+)}((\ell_0-K)^2) = \int \frac{\langle \ell\, d\ell\rangle[\ell\, d\ell]}{\langle \ell|K|\ell]} \int t\, dt\, \delta^{(+)}\left(t - \frac{K^2}{\langle \ell|K|\ell]}\right)$$

$$\Leftarrow \qquad \ell_0^2 = 0\,, \quad \ell_0^\mu = \frac{\langle \ell_0|\gamma^\mu|\ell_0]}{2} \equiv t\,\ell^\mu = t\frac{\langle \ell|\gamma^\mu|\ell]}{2}$$

- D-dim LIPS Anastasiou, Britto, Feng, Kunszt, & P.M.; Britto, Feng; Britto, Feng, & P.M.

$$\int d^{4-2\epsilon}\Phi = \int d\mu^{-2\epsilon} \int d^4\Phi(\mu^2)\,,$$

$$\int d^4\Phi(\mu^2) = \int d^4L\, \delta^{(+)}(L^2 - M_1^2 - \mu^2)\, \delta^{(+)}((L-K)^2 - M_2^2 - \mu^2)$$

$$= \int \frac{\langle \ell\, d\ell\rangle[\ell\, d\ell]}{\langle \ell|K|\ell]} \int t\, dt\, \delta^{(+)}\left(t - \frac{(1-2z_0)K^2}{\langle \ell|K|\ell]}\right)$$

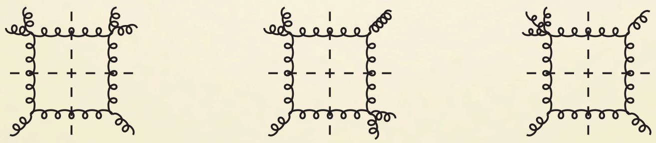
$$\Leftarrow \qquad L = \ell_0 + z_0 K\,, \qquad \text{with } \ell_0^2 = 0\,, \qquad \ell_0^\mu = t\frac{\langle \ell|\gamma^\mu|\ell]}{2} \qquad z_0 = \frac{K^2 + M_1^2 - M_2^2 - \sqrt{\lambda[K^2, M_1^2, M_2^2] - 4\mu^2}}{2K^2}$$

$gg \rightarrow gggg$

- Numerical Result: Ellis, Giele, Zanderighi (2006)
- Analytical Result:

Amplitude	$N=4$	$N=1$	$N=0\vert_{CC}$	$N=0\vert_{rat}$
$(--++++)$	BDDK'94	BDDK'94	BDDK'94	BDK'05, KF'05
$(-+-+++)$	BDDK'94	BDDK'94	BBST'04	BBDFK'06, XYZ'06
$(-++-++)$	BDDK'94	BDDK'94	BBST'04	BBDFK'06, XYZ'06
$(---+++)$	BDDK'94	BBDD'04	BBDI'05, BFM'06	BBDFK'06
$(--+-++)$	BDDK'94	BBCF'05, BBDP'05	BFM'06	XYZ'06
$(-+-+-+)$	BDDK'94	BBCF'05, BBDP'05	BFM'06	XYZ'06

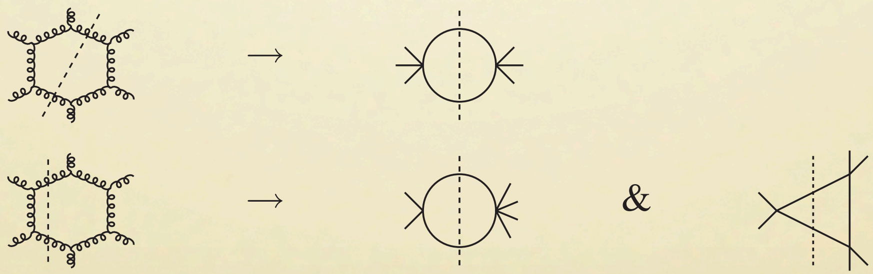
Quadruple Cuts



Bidder, Bjerrum-Bohr, Dunbar & Perkins (2005)

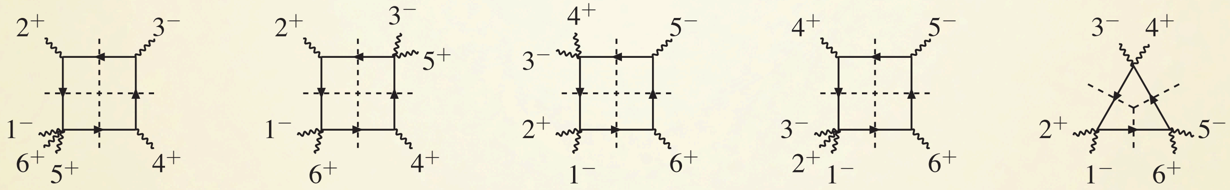
Double Cuts

Britto, Feng & P.M. (2006)



$\gamma\gamma \longrightarrow \gamma\gamma\gamma\gamma$

- Numerical Result: Nagy & Soper (2006); Ossola, Papadopoulous & Pittau (2007)
- Analytical Result: Mahlon (1996); Binoth, Gehrmann, Heinrich & P.M. (2007)



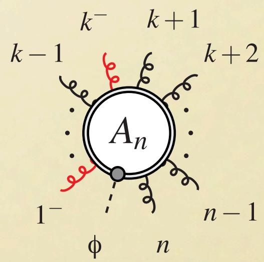
$gg \longrightarrow Hgg\ldots ggg$

▷ Heavy-top limit

- Numerical: H + 4 partons Campbell, Ellis, Zanderighi (2006)
- Analytical: H + n-gluons

$\phi = \frac{1}{2}(H + iA) \Rightarrow A^{\text{tree}}$ (ϕ + n-gluons) $\sim A^{\text{tree}}$(n-gluons) w/out mom. cons. Dixon, Glover & Kohze

- ϕ-nite Berger, Del Duca, Dixon (2006)
- ϕ-MHV amplitudes (nearest neighbour minuses) Badger, Glover, Risager (2007)
- ϕ-MHV amplitudes (generic configuration) Glover, Williams, P.M. (2008)



Double-Cut Phase-Space Measure

- 4-dim LIPS Cachazo, Svrček & Witten

$$\int d^4\Phi = \int d^4\ell_0 \, \delta^{(+)}\left(\ell_0^2\right) \delta^{(+)}\left((\ell_0 - K)^2\right)$$

$$\ell_0^\mu = \frac{\langle \ell_0|\gamma^\mu|\ell_0]}{2} \equiv t\,\ell^\mu = t\frac{\langle \ell|\gamma^\mu|\ell]}{2}$$

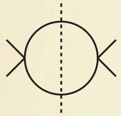
▷ Change of Variables

$$\forall p,q : q^2 = p^2 = 0 \quad \Rightarrow \quad |\ell\rangle \equiv |p\rangle + z|q\rangle \quad \& \quad |\ell] \equiv |p] + \bar{z}|q] \quad \Rightarrow \quad \langle \ell\, d\ell\rangle[\ell\, d\ell] = -\langle q|p|q]\, dz\, d\bar{z}\,,$$

$$\Leftrightarrow \quad \ell_\mu = p_\mu + z\,\bar{z}\,q_\mu + z\,\frac{\langle q|\gamma_\mu|p]}{2} + \bar{z}\,\frac{\langle p|\gamma_\mu|q]}{2}$$

$$\Rightarrow \int d^4\Phi = -2(p\cdot q) \oint_{\bar{z}=z^*} dz \int d\bar{z} \int \frac{t\,dt}{\langle \ell|K|\ell]}\, \delta^{(+)}\left(t - \frac{K^2}{\langle \ell|K|\ell]}\right)$$

- I_2

$\Delta I_2 =$ K  $= \int d^4\ell_0\, \delta\left(\ell_0^2\right) \delta\left((\ell_0 - K)^2\right) = -2(p\cdot q) \oint_{\bar{z}=z^*} dz \int d\bar{z} \int \frac{t\, dt}{\langle \ell | K | \ell]} \delta^{(+)}\left(t - \frac{K^2}{\langle \ell | K | \ell]}\right)$

▷ t-integration

$$\Rightarrow \qquad \Delta I_2 = -(2p\cdot q)\, K^2 \oint_{\bar{z}=z^*} dz \int d\bar{z} \frac{1}{\langle \ell | K | \ell]^2}$$

▷ Change of Variables

$$\forall p, q : q^2 = p^2 = 0 \quad \Rightarrow \quad |\ell\rangle \equiv |p\rangle + z|q\rangle \quad \& \quad |\ell] \equiv |p] + \bar{z}|q]$$

$$\Delta I_2 = (-1)\,(2p\cdot q)\,K^2 \oint dz \int d\bar{z} \frac{1}{\Big(\langle p|K|p] + z\langle q|K|p] + \bar{z}\langle p|K|q] + z\bar{z}\langle q|K|q]\Big)^2}$$

▷ Special choice of p and q

$$K_\mu \equiv p_\mu + q_\mu \;:\; q^2 = p^2 = 0\,, \quad K^2 \equiv 2p\cdot q = 2p\cdot K = 2q\cdot K$$

$$\Delta I_2 = (-1) \oint dz \int d\bar{z} \frac{1}{(1 + z\bar{z})^2}$$

▷ Primitive in $\bar{z}$

$$\Delta I_2 \quad = \quad (-1) \oint dz \int d\bar{z} \, \frac{d}{d\bar{z}} \frac{(-1)}{(1+z\bar{z})\, z} \quad = \quad \oint dz \, \frac{1}{(1+z\bar{z})\, z}$$

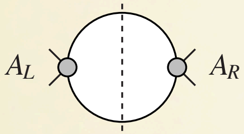
▷ Cauchy Residue Theorem in z would imply the contribution at the pole

$$z = 0 \qquad \Rightarrow \quad \bar{z} = 0$$

With the final result

$$\Delta I_2 \quad = \quad (2\pi i) \times 1$$

Novel Double-Cut Phase Space

$\Delta =$ 

$$= \int d^4\Phi \, A_L^{\rm tree}(\ell_0) \times A_R^{\rm tree}(\ell_0) \, , \qquad \ell_0^\mu = \frac{K^2}{2} \frac{\langle \ell | \gamma^\mu | \ell]}{\langle \ell | K | \ell]}$$

- Change of Variables with *special* p and q :

$$i) \quad q^2 = p^2 = 0$$

$$ii) \quad K_\mu \equiv p_\mu + q_\mu \, , \quad K^2 \equiv 2p\cdot q = 2p \cdot K = 2q \cdot K$$

$$iii) \quad |\ell\rangle \equiv |p\rangle + z|q\rangle \quad \& \quad |\ell] \equiv |p] + \bar{z}|q]$$

$$\Leftrightarrow \ell_0^\mu = \frac{1}{(1+z\bar{z})} \left(p^\mu + z\,\bar{z}\, q^\mu + z \frac{\langle q|\gamma^\mu|p]}{2} + \bar{z} \frac{\langle p|\gamma^\mu|q]}{2} \right)$$

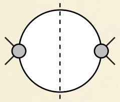
- Simplified parametrization of the Phase-Space

$$\int d^4\Phi = -K^2 \oint_{\bar{z}=z^*} dz \int d\bar{z} \int \frac{t\, dt}{(1+z\bar{z})} \, \delta\!\left(t - \frac{1}{(1+z\bar{z})} \right)$$

Easy Integration in Two Steps

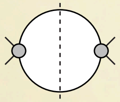
- Double Cut:

After the trivial t-integration



$$= \oint_{\bar{z}=z^*} dz \int d\bar{z}\, f(z,\bar{z}) \,, \qquad f(z,\bar{z}) = \frac{P(z,\bar{z})}{Q(z,\bar{z})}$$

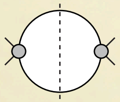
▷ Primitive in $\bar{z}$



$$= \oint dz\, F(z,\bar{z}) \,, \qquad F(z,\bar{z}) = \int d\bar{z}\, f(z,\bar{z}) = F^{\rm rat}(z,\bar{z}) + F^{\rm log}(z,\bar{z})$$

▷ Cauchy Residues in z

$c_{[K]} =$



$$\Big|_{\rm rat} = \oint dz\, F^{\rm rat}(z,z^*) = {\rm Res}_{z=0}\, F^{\rm rat}(z,z^*) + {\rm Res}_{z\neq 0}\, F^{\rm rat}(z,z^*)$$

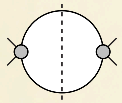
pole @ $z=0$ (pure bubble);
poles @ $z\neq 0$ (triangles reduction)

- The result will NOT depend on the choices of p and q, and it is symmetric under $p \leftrightarrow q$.

Easy Integration in Two Steps

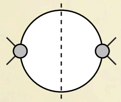
- Double Cut:

After the trivial t-integration



$$= \oint_{\bar{z}=z^*} dz \int d\bar{z}\, f(z,\bar{z})\,, \qquad f(z,\bar{z}) = \frac{P(z,\bar{z})}{Q(z,\bar{z})}$$

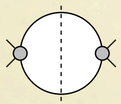
▷ Primitive in $\bar{z}$



$$= \oint dz\, F(z,\bar{z})\,, \qquad F(z,\bar{z}) = \int d\bar{z}\, f(z,\bar{z}) = F^{\rm rat}(z,\bar{z}) + F^{\rm log}(z,\bar{z})$$

▷ Cauchy Residues in z

$$c_{[K]} =$$



$$\Bigg|_{\rm rat} = \oint dz\, F^{\rm rat}(z,z^*) = {\rm Res}_{z=0}\, F^{\rm rat}(z,z^*) + {\rm Res}_{z\neq 0}\, F^{\rm rat}(z,z^*)$$

pole @ $z=0$ (pure bubble);
poles @ $z\neq 0$ (triangles reduction)

- The result will NOT depend on the choices of p and q, and it is symmetric under $p \leftrightarrow q$.

Can we made it simpler?

Hermite Polynomial Reduction

- Hermite Polynomial Reduction   Hermite (1872)

computing the rational part of an integral *without* performing any factorization!

$$\int \frac{4x^2+x-5}{(-5x^4+2x^3-5x^2+5x+1)^2}\,dx = \frac{473010x^3-491204x^2+761105x-522487}{918007(5x^4-2x^3+5x^2-5x-1)} + \int \frac{473010x^2-793204x+1216495}{918007(5x^4-2x^3+5x^2-5x-1)}\,dx$$

$$\int \frac{5x^4+x^3+4x^2+x-5}{(-5x^3-5x^2+2x-5)^4}\,dx = \frac{-1409030590x^2-2218871619x+105794481}{210709362134(5x^3+5x^2-2x+5)} + \frac{-12811755x^2-21656355x+468268}{669201870(5x^3+5x^2-2x+5)^2} + \frac{-14230x^2-5129x-8915}{70845(5x^3+5x^2-2x+5)^3} + \int \frac{-704515295x-1514356324}{105354681067(5x^3+5x^2-2x+5)}\,dx$$

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Contributed by: Sam Blake

Hermite Polynomial Reduction

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Contributed by: Sam Blake

After the "circle circle" of Zoltan and Nima, "Sam S@M" cannot be accidental!

Strategy for Automation

- Hermite Polynomial Reduction

extract the rational part of the $\bar{z}$-integral *without* factorizing the denominator in terms of its roots.

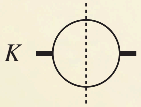
$$F(z,\bar{z}) = \int d\bar{z}\, f(z,\bar{z}) = F^{\rm rat}(z,\bar{z}) + F^{\rm log}(z,\bar{z}) \qquad\qquad f(z,\bar{z}) = \frac{P(z,\bar{z})}{Q(z,\bar{z})}$$

- Exploiting the Pole-Position

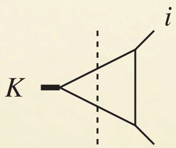
From the knowledge of the analytic form of the coefficients (spinor-integration) we know *apriori* where the *potential poles* are located.

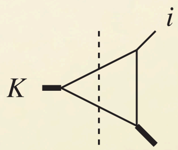
Britto, Feng; Britto, Feng & P.M; Britto, Feng & Yang

▷ poles @ $z = 0$

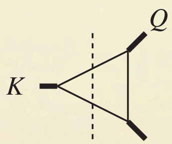


▷ poles @ $z \neq 0$: 1) $\langle \ell\, i \rangle \;\Rightarrow$





2) $\langle \ell | K\, Q | \ell \rangle \;\Rightarrow$



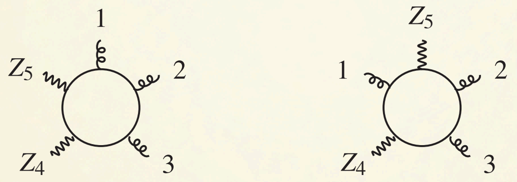
- Emergent BCFW-like construction

▷▷ Sum the residues to *all potential poles* in $F^{\rm rat}$ soon after the HPR ◁◁

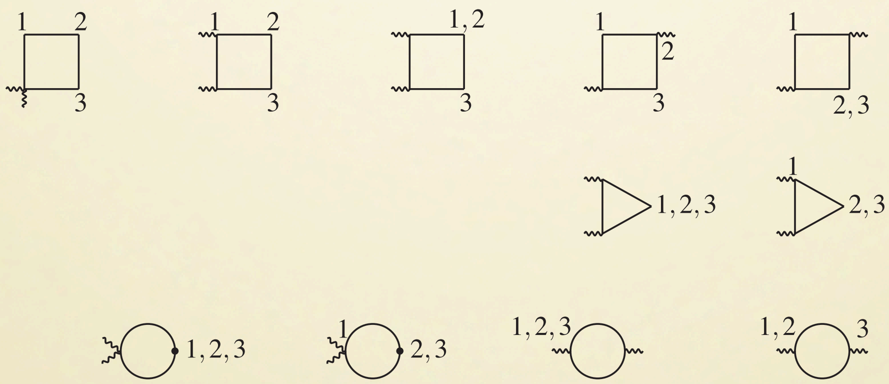
$gg \longrightarrow VV$ jet (N_f)

Binoth & Guffanti ⊕ Britto, Feng & P.M.

- Representative Diagrams



- 11 Master Integrals



▷ sewing (single and double) vector-boson currents and amplitudes!

Berends, Giele, Kujif
Bern, Forde, Kosower, & P.M.
Badger, Glover, Khoze

(Z5,1+ l2+,3+,Z4):Cut[1]

In[1]:= << Spinors`

```
-------  SPINORS @ MATHEMATICA (S@M)  -------

        Version:  S@M 1.0 (29-OCT-2007)

 Authors:
    Daniel Maitre (SLAC),
    Pierpaolo Mastrolia (University of Zurich)

A list of all functions provided by the package
is stored in the variable
    $SpinorsFunctions
```

■

In[137]:= DeclareSpinor[λ, l1, l2, l3, l4, p, η]

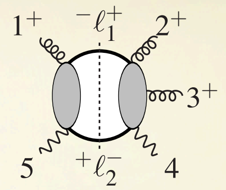
{λ, l1, l2, l3, l4, p, η} added to the list of spinors

In[138]:= P = P1Z5

Out[138]= P1Z5

$$-2(p\cdot q)\oint_{\bar{z}=z^*} dz \int d\bar{z}$$

■



In[139]:= Cut[1] = JMpP[ϵ5, l1, 1, l2] ** JMppP[ϵ4, l2, 3, 2, l1]

Out[139]= $-\dfrac{\langle l1 \mid PZ5 \mid \epsilon5 \mid l1\rangle \langle l2 \mid PZ4 \mid \epsilon4 \mid l2\rangle}{\langle l1 \mid 1\rangle \langle l1 \mid 2\rangle \langle l2 \mid 1\rangle \langle l2 \mid 3\rangle \langle 2 \mid 3\rangle}$

In[140]:= Cut[1] = tIntegrazione[Cut[1], l2, {l1, P}]

Out[140]= $-\dfrac{d\lambda\, s[P1Z5]\, \langle \lambda \mid PZ4 \mid \epsilon4 \mid \lambda\rangle\, [\lambda \mid P1Z5 \mid \epsilon5 \mid PZ5 \mid P1Z5 \mid \lambda]}{\langle \lambda \mid 1\rangle \langle \lambda \mid 3\rangle \langle 2 \mid 3\rangle \langle \lambda \mid P1Z5 \mid \lambda]^2 \langle 1 \mid P1Z5 \mid \lambda] \langle 2 \mid P1Z5 \mid \lambda]}$

In[141]:= DeclareSpinor[4, 5]

{4, 5} added to the list of spinors

In[142]:= Cut[1] = Cut[1] //.
```
  {
   ϵ4 → Sp[4],
   ϵ5 → Sp[5]
  };
```

In[143]:= Cut[1] = SpOpen[Cut[1]]

Out[143]= $\dfrac{d\lambda\, s[P1Z5]\, \langle \lambda \mid 4\rangle \langle \lambda \mid PZ4 \mid 4] \langle 5 \mid P1Z5 \mid \lambda]\, [5 \mid PZ5 \mid P1Z5 \mid \lambda]}{\langle \lambda \mid 1\rangle \langle \lambda \mid 3\rangle \langle 2 \mid 3\rangle \langle \lambda \mid P1Z5 \mid \lambda]^2 \langle 1 \mid P1Z5 \mid \lambda] \langle 2 \mid P1Z5 \mid \lambda]}$

■ Phase-Space definition

In[144]:= Cut[1] = Cut[1] //. dλ → (-1) ** Spab[η, p, η] ** dz ** dbz

Out[144]= $-\dfrac{dbz\, dz\, s[P1Z5]\, \langle \lambda \mid 4\rangle \langle \eta \mid p \mid \eta] \langle \lambda \mid PZ4 \mid 4] \langle 5 \mid P1Z5 \mid \lambda]\, [5 \mid PZ5 \mid P1Z5 \mid \lambda]}{\langle \lambda \mid 1\rangle \langle \lambda \mid 3\rangle \langle 2 \mid 3\rangle \langle \lambda \mid P1Z5 \mid \lambda]^2 \langle 1 \mid P1Z5 \mid \lambda] \langle 2 \mid P1Z5 \mid \lambda]}$

- **Primitive in bar-z**

$$|\ell] \equiv |p] + \bar{z}|q]$$

```
In[145]:= Cut[1] = BSpinorReplace[Cut[1], λ, p + bz * η]

Out[145]= -(dbz dz s[P1Z5] ⟨λ | 4⟩ ⟨η | p | η] ⟨λ | PZ4 | 4]
      (⟨5 | P1Z5 | p] + bz ⟨5 | P1Z5 | η]) ([5 | PZ5 | P1Z5 | p] + bz [5 | PZ5 | P1Z5 | η])) /
    (⟨λ | 1⟩ ⟨λ | 3⟩ ⟨2 | 3⟩ (⟨λ | P1Z5 | p] + bz ⟨λ | P1Z5 | η])^2
      (⟨1 | P1Z5 | p] + bz ⟨1 | P1Z5 | η]) (⟨2 | P1Z5 | p] + bz ⟨2 | P1Z5 | η]))
```

- **Primitive: way 1: Direct Integration**

$$F(z,\bar{z}) = \int d\bar{z}\, f(z,\bar{z}) = F^{\text{rat}}(z,\bar{z}) + F^{\log}(z,\bar{z})$$

```
In[146]:= Primitive = Integrate[Cut[1], bz]

Out[146]= - 1/(⟨λ | 1⟩ ⟨λ | 3⟩ ⟨2 | 3⟩) dbz dz s[P1Z5] ⟨λ | 4⟩ ⟨η | p | η]
    ⟨λ | PZ4 | 4] (-(((⟨λ | P1Z5 | η] ⟨5 | P1Z5 | p] - ⟨λ | P1Z5 | p] ⟨5 | P1Z5 | η])
          (⟨λ | P1Z5 | η] [5 | PZ5 | P1Z5 | p] - ⟨λ | P1Z5 | p] [5 | PZ5 | P1Z5 | η])) /
        ((⟨λ | P1Z5 | η] (⟨λ | P1Z5 | p] + bz ⟨λ | P1Z5 | η]) (⟨λ | P1Z5 | η] ⟨1 | P1Z5 | p] - ⟨λ | P1Z5 | p]
          ⟨1 | P1Z5 | η]) (⟨λ | P1Z5 | η] ⟨2 | P1Z5 | p] - ⟨λ | P1Z5 | p] ⟨2 | P1Z5 | η])) -
      (Log[⟨1 | P1Z5 | p] + bz ⟨1 | P1Z5 | η]] (-⟨1 | P1Z5 | η] ⟨5 | P1Z5 | p] + ⟨1 | P1Z5 | p]
            ⟨5 | P1Z5 | η]) (-⟨1 | P1Z5 | η] [5 | PZ5 | P1Z5 | p] + ⟨1 | P1Z5 | p] [5 | PZ5 | P1Z5 | η])) /
       ((⟨λ | P1Z5 | η] ⟨1 | P1Z5 | p] - ⟨λ | P1Z5 | p] ⟨1 | P1Z5 | η])^2
         (-⟨1 | P1Z5 | η] ⟨2 | P1Z5 | p] + ⟨1 | P1Z5 | p] ⟨2 | P1Z5 | η])) -
      (Log[⟨2 | P1Z5 | p] + bz ⟨2 | P1Z5 | η]] (-⟨2 | P1Z5 | η] ⟨5 | P1Z5 | p] + ⟨2 | P1Z5 | p]
            ⟨5 | P1Z5 | η]) (-⟨2 | P1Z5 | η] [5 | PZ5 | P1Z5 | p] + ⟨2 | P1Z5 | p] [5 | PZ5 | P1Z5 | η])) /
       ((⟨λ | P1Z5 | η] ⟨2 | P1Z5 | p] - ⟨λ | P1Z5 | p] ⟨2 | P1Z5 | η])^2
         (⟨1 | P1Z5 | η] ⟨2 | P1Z5 | p] - ⟨1 | P1Z5 | p] ⟨2 | P1Z5 | η])) +
      (Log[⟨λ | P1Z5 | p] + bz ⟨λ | P1Z5 | η]] (2 ⟨λ | P1Z5 | p] ⟨λ | P1Z5 | η]
           (⟨1 | P1Z5 | η] ⟨2 | P1Z5 | η] ⟨5 | P1Z5 | p] [5 | PZ5 | P1Z5 | p] -
             ⟨1 | P1Z5 | p] ⟨2 | P1Z5 | p] ⟨5 | P1Z5 | η] [5 | PZ5 | P1Z5 | η]) +
          ⟨λ | P1Z5 | η]^2 (-⟨1 | P1Z5 | η] ⟨2 | P1Z5 | p] ⟨5 | P1Z5 | p] [5 | PZ5 | P1Z5 | p] +
             ⟨1 | P1Z5 | p] (-⟨2 | P1Z5 | η] ⟨5 | P1Z5 | p] [5 | PZ5 | P1Z5 | p] + ⟨2 | P1Z5 | p]
                 (⟨5 | P1Z5 | η] [5 | PZ5 | P1Z5 | p] + ⟨5 | P1Z5 | p] [5 | PZ5 | P1Z5 | η]))) +
          ⟨λ | P1Z5 | p]^2 (⟨1 | P1Z5 | p] ⟨2 | P1Z5 | η] ⟨5 | P1Z5 | η] [5 | PZ5 | P1Z5 | η] -
             ⟨1 | P1Z5 | η] (-⟨2 | P1Z5 | p] ⟨5 | P1Z5 | η] [5 | PZ5 | P1Z5 | η] + ⟨2 | P1Z5 | η]
                 (⟨5 | P1Z5 | η] [5 | PZ5 | P1Z5 | p] + ⟨5 | P1Z5 | p] [5 | PZ5 | P1Z5 | η]))))) /
       ((⟨λ | P1Z5 | η] ⟨1 | P1Z5 | p] - ⟨λ | P1Z5 | p] ⟨1 | P1Z5 | η])^2
         (⟨λ | P1Z5 | η] ⟨2 | P1Z5 | p] - ⟨λ | P1Z5 | p] ⟨2 | P1Z5 | η])^2))
```

```
In[147]:= RatPrimitive = Primitive //. Log[x_] → 0
```

$F^{\text{rat}}(z,\bar{z})$

```
Out[147]= (dbz dz s[P1Z5] ⟨λ | 4⟩ ⟨η | p | η] ⟨λ | PZ4 | 4]
    (⟨λ | P1Z5 | η] ⟨5 | P1Z5 | p] - ⟨λ | P1Z5 | p] ⟨5 | P1Z5 | η])
    (⟨λ | P1Z5 | η] [5 | PZ5 | P1Z5 | p] - ⟨λ | P1Z5 | p] [5 | PZ5 | P1Z5 | η])) /
  (⟨λ | 1⟩ ⟨λ | 3⟩ ⟨2 | 3⟩ ⟨λ | P1Z5 | η] (⟨λ | P1Z5 | p] + bz ⟨λ | P1Z5 | η])
    (⟨λ | P1Z5 | η] ⟨1 | P1Z5 | p] - ⟨λ | P1Z5 | p] ⟨1 | P1Z5 | η])
    (⟨λ | P1Z5 | η] ⟨2 | P1Z5 | p] - ⟨λ | P1Z5 | p] ⟨2 | P1Z5 | η]))
```

■ Primitive: Hermite Polynomial Reduction

```
In[148]:= Hresult = HermiteReduce[Numerator[Cut[1]], Denominator[Cut[1]], bz];

In[149]:= Hresult[[1]] - RatPrimitive // Simplify

Out[149]= 0

In[150]:= Cut[1] = Hresult[[1]] //. dbz → 1
```

$F^{\text{rat}}(z,\bar{z})$

```
Out[150]= (dz s[P1Z5] ⟨λ | 4⟩ ⟨η | p | η] ⟨λ | P1Z5 | η]^2 ⟨λ | PZ4 | 4] ⟨5 | P1Z5 | p] [5 | PZ5 | P1Z5 | p] -
     dz s[P1Z5] ⟨λ | 4⟩ ⟨η | p | η] ⟨λ | P1Z5 | p] ⟨λ | P1Z5 | η] ⟨λ | PZ4 | 4]
      ⟨5 | P1Z5 | η] [5 | PZ5 | P1Z5 | p] - dz s[P1Z5] ⟨λ | 4⟩ ⟨η | p | η]
      ⟨λ | P1Z5 | p] ⟨λ | P1Z5 | η] ⟨λ | PZ4 | 4] ⟨5 | P1Z5 | p] [5 | PZ5 | P1Z5 | η] +
     dz s[P1Z5] ⟨λ | 4⟩ ⟨η | p | η] ⟨λ | P1Z5 | p]^2 ⟨λ | PZ4 | 4] ⟨5 | P1Z5 | η] [5 | PZ5 | P1Z5 | η]) /
  (⟨λ | 1⟩ ⟨λ | 3⟩ ⟨2 | 3⟩ ⟨λ | P1Z5 | η] (⟨λ | P1Z5 | p] + bz ⟨λ | P1Z5 | η])
    (⟨λ | P1Z5 | η] ⟨1 | P1Z5 | p] - ⟨λ | P1Z5 | p] ⟨1 | P1Z5 | η])
    (⟨λ | P1Z5 | η] ⟨2 | P1Z5 | p] - ⟨λ | P1Z5 | p] ⟨2 | P1Z5 | η]))
```

Preparing for Cauchy's

$$|\ell\rangle \equiv |p\rangle + z|q\rangle$$

In[151]:= `Cut[1] = ASpinorReplace[Cut[1], λ, p + z * η]`

Out[151]=
$$\Big(\text{dz s}[\text{P1Z5}]\,(\langle p|4\rangle + z\langle\eta|4\rangle)\,\langle\eta|p|\eta]\,(\langle p|\text{P1Z5}|\eta] + z\langle\eta|\text{P1Z5}|\eta])^2\,(\langle p|\text{PZ4}|4] + z\langle\eta|\text{PZ4}|4])\,\langle 5|\text{P1Z5}|p]\,[5|\text{PZ5}|\text{P1Z5}|p] - \text{dz s}[\text{P1Z5}]\,(\langle p|4\rangle + z\langle\eta|4\rangle)\,\langle\eta|p|\eta]\,(\langle p|\text{P1Z5}|p] + z\langle\eta|\text{P1Z5}|p])\,(\langle p|\text{P1Z5}|\eta] + z\langle\eta|\text{P1Z5}|\eta])\,(\langle p|\text{PZ4}|4] + z\langle\eta|\text{PZ4}|4])\,\langle 5|\text{P1Z5}|\eta]\,[5|\text{PZ5}|\text{P1Z5}|p] - \text{dz s}[\text{P1Z5}]\,(\langle p|4\rangle + z\langle\eta|4\rangle)\,\langle\eta|p|\eta]\,(\langle p|\text{P1Z5}|p] + z\langle\eta|\text{P1Z5}|p])\,(\langle p|\text{P1Z5}|\eta] + z\langle\eta|\text{P1Z5}|\eta])\,(\langle p|\text{PZ4}|4] + z\langle\eta|\text{PZ4}|4])\,\langle 5|\text{P1Z5}|p]\,[5|\text{PZ5}|\text{P1Z5}|\eta] + \text{dz s}[\text{P1Z5}]\,(\langle p|4\rangle + z\langle\eta|4\rangle)\,\langle\eta|p|\eta]\,(\langle p|\text{P1Z5}|p] + z\langle\eta|\text{P1Z5}|p])^2\,(\langle p|\text{PZ4}|4] + z\langle\eta|\text{PZ4}|4])\,\langle 5|\text{P1Z5}|\eta]\,[5|\text{PZ5}|\text{P1Z5}|\eta]\Big)\Big/$$
$$\Big((\langle p|1\rangle + z\langle\eta|1\rangle)\,(\langle p|3\rangle + z\langle\eta|3\rangle)\,\langle 2|3\rangle\,(\langle p|\text{P1Z5}|\eta] + z\langle\eta|\text{P1Z5}|\eta])\,(\langle p|\text{P1Z5}|p] + z\langle\eta|\text{P1Z5}|p] + \text{bz}\,(\langle p|\text{P1Z5}|\eta] + z\langle\eta|\text{P1Z5}|\eta]))\,((\langle p|\text{P1Z5}|\eta] + z\langle\eta|\text{P1Z5}|\eta])\,\langle 1|\text{P1Z5}|p] - (\langle p|\text{P1Z5}|p] + z\langle\eta|\text{P1Z5}|p])\,\langle 1|\text{P1Z5}|\eta])\,((\langle p|\text{P1Z5}|\eta] + z\langle\eta|\text{P1Z5}|\eta])\,\langle 2|\text{P1Z5}|p] - (\langle p|\text{P1Z5}|p] + z\langle\eta|\text{P1Z5}|p])\,\langle 2|\text{P1Z5}|\eta])\Big)$$

- ## Special choice of η and p

  - ### following identities

$$K_\mu \equiv p_\mu + q_\mu\,, \quad K^2 \equiv 2p\cdot q = 2p\cdot K = 2q\cdot K$$

In[152]:=
```
Cut[1] = Cut[1] //. {
      Spab[p, P, η] → 0,
      Spab[η, P, p] → 0,
      Spab[η, P, η] → s[P],
      Spab[p, P, p] → s[P],
      Spaa[p, η] → s[P] / Spbb[η, p]
    } // Factor;
```

In[153]:=
```
Cut[1] = Cut[1] //. {
    (sp : (Spaa | Spbb | Spab)) [L1__, P, p] → sp[L1, p + η, p],
    (sp : (Spaa | Spbb | Spab)) [L1__, P, η] → sp[L1, p + η, η],
    (sp : (Spaa | Spbb | Spab)) [p, P, L1__] → sp[p, p + η, L1],
    (sp : (Spaa | Spbb | Spab)) [η, P, L1__] → sp[η, p + η, L1]
  };
```

In[154]:=
```
Cut[1] = SpOpen[Cut[1], p];
Cut[1] = SpOpen[Cut[1], η];
```

In[156]:=
```
Cut[1] = Cut[1] //. {
      Spaa[p, η] → s[P] / Spbb[η, p]
    } // Factor;
```

In[157]:=
```
Cut[1] = Cut[1] // Factor
```

Out[157]=

$$-\left(\mathrm{dz}\,(\langle p\,|\,4\rangle + z\,\langle \eta\,|\,4\rangle)\,(\langle p\,|\,5\rangle + z\,\langle \eta\,|\,5\rangle)\,(\langle p\,|\,\mathrm{PZ4}\,|\,4] + z\,\langle \eta\,|\,\mathrm{PZ4}\,|\,4])\,(\langle p\,|\,\mathrm{PZ5}\,|\,5] + z\,\langle \eta\,|\,\mathrm{PZ5}\,|\,5])\right) \Big/ \left(z\,(1+\mathrm{bz}\,z)\,(\langle p\,|\,1\rangle + z\,\langle \eta\,|\,1\rangle)^2\,(\langle p\,|\,2\rangle + z\,\langle \eta\,|\,2\rangle)\,(\langle p\,|\,3\rangle + z\,\langle \eta\,|\,3\rangle)\,\langle 2\,|\,3\rangle\right)$$

Residues in z

```
In[158]:= c2[1] = TakeResidue[Cut[1], z, bz] //. dz → 1;
```

check = 0

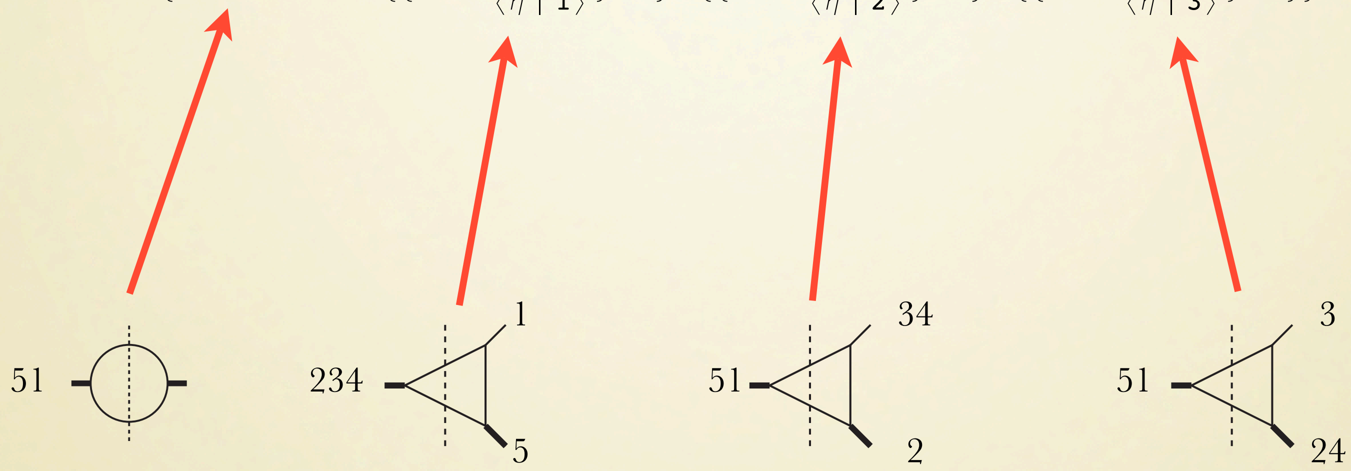
$$\text{TruePoleList}=\left\{\{\{z\to 0\},1\},\left\{\left\{z\to-\frac{\langle p\mid 1\rangle}{\langle\eta\mid 1\rangle}\right\},2\right\},\left\{\left\{z\to-\frac{\langle p\mid 2\rangle}{\langle\eta\mid 2\rangle}\right\},1\right\},\left\{\left\{z\to-\frac{\langle p\mid 3\rangle}{\langle\eta\mid 3\rangle}\right\},1\right\}\right\}$$

Residues in z

```
In[158]:= c2[1] = TakeResidue[Cut[1], z, bz] //. dz → 1;
```

check = 0

$$\text{TruePoleList}=\left\{\{\{z\to 0\},1\},\left\{\left\{z\to -\frac{\langle p\,|\,1\rangle}{\langle \eta\,|\,1\rangle}\right\},2\right\},\left\{\left\{z\to -\frac{\langle p\,|\,2\rangle}{\langle \eta\,|\,2\rangle}\right\},1\right\},\left\{\left\{z\to -\frac{\langle p\,|\,3\rangle}{\langle \eta\,|\,3\rangle}\right\},1\right\}\right\}$$



In[159]:=
```
c2[1] = Map[Factor, c2[1]]
```

Out[159]=

$$-\frac{\langle p|4\rangle\langle p|5\rangle\langle p|\mathrm{PZ4}|4]\,\langle p|\mathrm{PZ5}|5]}{\langle p|1\rangle^{2}\langle p|2\rangle\langle p|3\rangle\langle 2|3\rangle}+$$

$$\Big((-\langle p|4\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|5\rangle)(-\langle\eta|1\rangle\langle p|\mathrm{PZ4}|4]+\langle p|1\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|1\rangle\langle p|\mathrm{PZ5}|5]+\langle p|1\rangle\langle\eta|\mathrm{PZ5}|5])[1|p]\,[1|\eta]\Big)\Big/\Big(\langle p|1\rangle\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])^{2}\Big)-$$

$$\Big((-\langle p|4\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|5\rangle)(-\langle\eta|1\rangle\langle p|\mathrm{PZ4}|4]+\langle p|1\rangle\langle\eta|\mathrm{PZ4}|4])\langle\eta|\mathrm{PZ5}|5]\,[1|\eta]\Big)\Big/\Big(\langle p|1\rangle\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])\Big)-$$

$$\Big((-\langle p|4\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|5\rangle)\langle\eta|\mathrm{PZ4}|4](-\langle\eta|1\rangle\langle p|\mathrm{PZ5}|5]+\langle p|1\rangle\langle\eta|\mathrm{PZ5}|5])[1|\eta]\Big)\Big/\Big(\langle p|1\rangle\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])\Big)-$$

$$\Big((-\langle p|4\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|4\rangle)\langle\eta|5\rangle(-\langle\eta|1\rangle\langle p|\mathrm{PZ4}|4]+\langle p|1\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|1\rangle\langle p|\mathrm{PZ5}|5]+\langle p|1\rangle\langle\eta|\mathrm{PZ5}|5])[1|\eta]\Big)\Big/\Big(\langle p|1\rangle\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])\Big)-$$

$$\Big(\langle\eta|4\rangle(-\langle p|5\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|5\rangle)(-\langle\eta|1\rangle\langle p|\mathrm{PZ4}|4]+\langle p|1\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|1\rangle\langle p|\mathrm{PZ5}|5]+\langle p|1\rangle\langle\eta|\mathrm{PZ5}|5])[1|\eta]\Big)\Big/\Big(\langle p|1\rangle\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])\Big)+$$

$$\Big(\langle\eta|3\rangle(-\langle p|4\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|5\rangle)(-\langle\eta|1\rangle\langle p|\mathrm{PZ4}|4]+\langle p|1\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|1\rangle\langle p|\mathrm{PZ5}|5]+\langle p|1\rangle\langle\eta|\mathrm{PZ5}|5])[1|\eta]\Big)\Big/\Big(\langle p|1\rangle\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)^{2}\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])\Big)+$$

$$\Big(\langle\eta|2\rangle(-\langle p|4\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|5\rangle)(-\langle\eta|1\rangle\langle p|\mathrm{PZ4}|4]+\langle p|1\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|1\rangle\langle p|\mathrm{PZ5}|5]+\langle p|1\rangle\langle\eta|\mathrm{PZ5}|5])[1|\eta]\Big)\Big/\Big(\langle p|1\rangle\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)^{2}(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])\Big)+$$

$$\Big((-\langle p|4\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|5\rangle)(-\langle\eta|1\rangle\langle p|\mathrm{PZ4}|4]+\langle p|1\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|1\rangle\langle p|\mathrm{PZ5}|5]+\langle p|1\rangle\langle\eta|\mathrm{PZ5}|5])[1|\eta]\Big)\Big/\Big(\langle p|1\rangle^{2}\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])\Big)-$$

$$\Big((-\langle p|4\rangle\langle\eta|2\rangle+\langle p|2\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|2\rangle+\langle p|2\rangle\langle\eta|5\rangle)(-\langle\eta|2\rangle\langle p|\mathrm{PZ4}|4]+\langle p|2\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|2\rangle\langle p|\mathrm{PZ5}|5]+\langle p|2\rangle\langle\eta|\mathrm{PZ5}|5])[2|\eta]\Big)\Big/\Big(\langle p|2\rangle(\langle p|2\rangle\langle\eta|1\rangle-\langle p|1\rangle\langle\eta|2\rangle)^{2}(-\langle p|3\rangle\langle\eta|2\rangle+\langle p|2\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|2\rangle[2|p]+\langle\eta|2\rangle[2|\eta])\Big)-$$

$$\Big((-\langle p|4\rangle\langle\eta|3\rangle+\langle p|3\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|3\rangle+\langle p|3\rangle\langle\eta|5\rangle)(-\langle\eta|3\rangle\langle p|\mathrm{PZ4}|4]+\langle p|3\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|3\rangle\langle p|\mathrm{PZ5}|5]+\langle p|3\rangle\langle\eta|\mathrm{PZ5}|5])[3|\eta]\Big)\Big/\Big(\langle p|3\rangle(\langle p|3\rangle\langle\eta|1\rangle-\langle p|1\rangle\langle\eta|3\rangle)^{2}(\langle p|3\rangle\langle\eta|2\rangle-\langle p|2\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|3\rangle[3|p]+\langle\eta|3\rangle[3|\eta])\Big)$$

to be cleaned up with “Schouten id’s”

In[159]:= `c2[1] = Map[Factor, c2[1]]`

Out[159]=

$$-\frac{\langle p|4\rangle\langle p|5\rangle\langle p|\mathrm{PZ4}|4]\langle p|\mathrm{PZ5}|5]}{\langle p|1\rangle^2\langle p|2\rangle\langle p|3\rangle\langle 2|3\rangle}+$$

$$\Big((-\langle p|4\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|5\rangle)(-\langle\eta|1\rangle\langle p|\mathrm{PZ4}|4]+\langle p|1\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|1\rangle\langle p|\mathrm{PZ5}|5]+\langle p|1\rangle\langle\eta|\mathrm{PZ5}|5])[1|p][1|\eta]\Big)\Big/\Big(\langle p|1\rangle\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])^2\Big)-$$

$$\Big((-\langle p|4\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|5\rangle)(-\langle\eta|1\rangle\langle p|\mathrm{PZ4}|4]+\langle p|1\rangle\langle\eta|\mathrm{PZ4}|4])\langle\eta|\mathrm{PZ5}|5][1|\eta]\Big)\Big/\Big(\langle p|1\rangle\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])\Big)-$$

$$\Big((-\langle p|4\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|5\rangle)\langle\eta|\mathrm{PZ4}|4](-\langle\eta|1\rangle\langle p|\mathrm{PZ5}|5]+\langle p|1\rangle\langle\eta|\mathrm{PZ5}|5])[1|\eta]\Big)\Big/\Big(\langle p|1\rangle\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])\Big)-$$

$$\Big((-\langle p|4\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|4\rangle)\langle\eta|5\rangle(-\langle\eta|1\rangle\langle p|\mathrm{PZ4}|4]+\langle p|1\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|1\rangle\langle p|\mathrm{PZ5}|5]+\langle p|1\rangle\langle\eta|\mathrm{PZ5}|5])[1|\eta]\Big)\Big/\Big(\langle p|1\rangle\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])\Big)-$$

$$\Big(\langle\eta|4\rangle(-\langle p|5\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|5\rangle)(-\langle\eta|1\rangle\langle p|\mathrm{PZ4}|4]+\langle p|1\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|1\rangle\langle p|\mathrm{PZ5}|5]+\langle p|1\rangle\langle\eta|\mathrm{PZ5}|5])[1|\eta]\Big)\Big/\Big(\langle p|1\rangle\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])\Big)+$$

$$\Big(\langle\eta|3\rangle(-\langle p|4\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|5\rangle)(-\langle\eta|1\rangle\langle p|\mathrm{PZ4}|4]+\langle p|1\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|1\rangle\langle p|\mathrm{PZ5}|5]+\langle p|1\rangle\langle\eta|\mathrm{PZ5}|5])[1|\eta]\Big)\Big/\Big(\langle p|1\rangle\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)^2\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])\Big)+$$

$$\Big(\langle\eta|2\rangle(-\langle p|4\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|5\rangle)(-\langle\eta|1\rangle\langle p|\mathrm{PZ4}|4]+\langle p|1\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|1\rangle\langle p|\mathrm{PZ5}|5]+\langle p|1\rangle\langle\eta|\mathrm{PZ5}|5])[1|\eta]\Big)\Big/\Big(\langle p|1\rangle\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)^2(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])\Big)+$$

$$\Big((-\langle p|4\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|5\rangle)(-\langle\eta|1\rangle\langle p|\mathrm{PZ4}|4]+\langle p|1\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|1\rangle\langle p|\mathrm{PZ5}|5]+\langle p|1\rangle\langle\eta|\mathrm{PZ5}|5])[1|\eta]\Big)\Big/\Big(\langle p|1\rangle^2\langle\eta|1\rangle(-\langle p|2\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|2\rangle)(-\langle p|3\rangle\langle\eta|1\rangle+\langle p|1\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|1\rangle[1|p]+\langle\eta|1\rangle[1|\eta])\Big)-$$

$$\Big((-\langle p|4\rangle\langle\eta|2\rangle+\langle p|2\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|2\rangle+\langle p|2\rangle\langle\eta|5\rangle)(-\langle\eta|2\rangle\langle p|\mathrm{PZ4}|4]+\langle p|2\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|2\rangle\langle p|\mathrm{PZ5}|5]+\langle p|2\rangle\langle\eta|\mathrm{PZ5}|5])[2|\eta]\Big)\Big/\Big(\langle p|2\rangle(\langle p|2\rangle\langle\eta|1\rangle-\langle p|1\rangle\langle\eta|2\rangle)^2(-\langle p|3\rangle\langle\eta|2\rangle+\langle p|2\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|2\rangle[2|p]+\langle\eta|2\rangle[2|\eta])\Big)-$$

$$\Big((-\langle p|4\rangle\langle\eta|3\rangle+\langle p|3\rangle\langle\eta|4\rangle)(-\langle p|5\rangle\langle\eta|3\rangle+\langle p|3\rangle\langle\eta|5\rangle)(-\langle\eta|3\rangle\langle p|\mathrm{PZ4}|4]+\langle p|3\rangle\langle\eta|\mathrm{PZ4}|4])(-\langle\eta|3\rangle\langle p|\mathrm{PZ5}|5]+\langle p|3\rangle\langle\eta|\mathrm{PZ5}|5])[3|\eta]\Big)\Big/\Big(\langle p|3\rangle(\langle p|3\rangle\langle\eta|1\rangle-\langle p|1\rangle\langle\eta|3\rangle)^2(\langle p|3\rangle\langle\eta|2\rangle-\langle p|2\rangle\langle\eta|3\rangle)\langle 2|3\rangle(\langle p|3\rangle[3|p]+\langle\eta|3\rangle[3|\eta])\Big)$$

Valid for any Z's-polarization

Conclusion

- Novel Double-Cut Integration
  - Derived from the spinor-integration
  - Special decomposition of the loop-momentum on the cut
  - Simplified Contour-integral of rational functions in two variables
  - No subtractions required
  - No PV-tensor reduction required
- Straightforward computation of 2-point coefficients
  - Hermite Polynomial Reduction (w.r.t. one variable)
    - to find directly the rational term of a primitive
  - Cauchy's residue (w.r.t. the second variable)
    - The contour contains a pole in zero (bubble) and other finite-poles (triangle)
- Automation: HPR & apriori knowledge of the pole-positions
- Similar structure to BCFW: but the residue at zero has to be added to the others

Progress

Numerical Unitarity

Analytic Unitarity