

# All-order results for infrared and collinear singularities

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Workshop on gauge and string amplitudes  
IPPP, Durham, March 30 - April 3, 2009

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# Introduction

# Ancient History

JULY 15, 1937

PHYSICAL REVIEW

VOLUME 52

## Note on the Radiation Field of the Electron

F. BLOCH AND A. NORDSIECK\*

*Stanford University, California*

(Received May 14, 1937)

Previous methods of treating radiative corrections in non-stationary processes such as the scattering of an electron in an atomic field or the emission of a  $\beta$ -ray, by an expansion in powers of  $e^2/\hbar c$ , are defective in that they predict infinite low frequency corrections to the transition probabilities. This difficulty can be avoided by a method developed here which is based on the alternative assumption that  $e^2\omega/mc^2$ ,  $\hbar\omega/mc^2$  and  $\hbar\omega/c\Delta p$  ( $\omega$ =angular frequency of radiation,  $\Delta p$ =change in momentum of electron) are small compared to unity. In contrast to the expansion in powers of  $e^2/\hbar c$ , this permits the transition to the classical limit  $\hbar=0$ .

External perturbations on the electron are treated in the Born approximation. It is shown that for frequencies such that the above three parameters are negligible the quantum mechanical calculation yields just the directly reinterpreted results of the classical formulae, namely that the total probability of a given change in the motion of the electron is unaffected by the interaction with radiation, and that the mean number of emitted quanta is infinite in such a way that the mean radiated energy is equal to the energy radiated classically in the corresponding trajectory.

A remarkable achievement, before quantum field theory was born.

# Modern History

## Factorization

$$\mathcal{M}_{\{r_i\}}^{[f]} \left( \beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = \sum_{L=1}^{N^{[f]}} \mathcal{M}_L^{[f]} \left( \beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) (c_L)_{\{r_i\}}$$

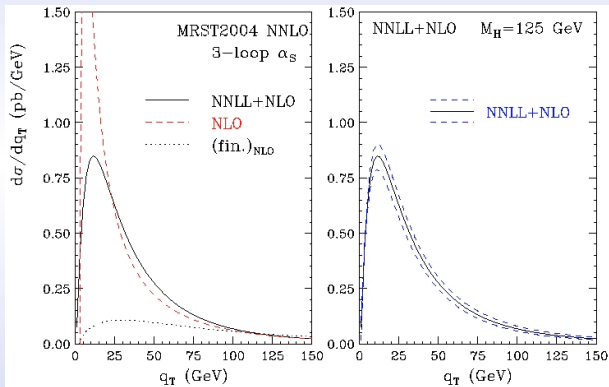
$$\begin{aligned} \mathcal{M}_L^{[f]} \left( \beta_j, \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) &= \prod_{i=1}^{n+2} J^{[i]} \left( \frac{Q'^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \\ &\times S_{LI}^{[f]} \left( \beta_j, \frac{Q'^2}{\mu^2}, \frac{Q'^2}{Q^2}, \alpha_s(\mu^2), \epsilon \right) H_I^{[f]} \left( \beta_j, \frac{Q^2}{\mu^2}, \frac{Q'^2}{Q^2}, \alpha_s(\mu^2) \right), \end{aligned}$$

## Progress

- ▶ Exponentiation applies to **non-abelian** gauge theories.
- ▶ Exponentiation extends to **collinear** divergences.
- ▶ Exponentiation is performed at the **amplitude** level.
- ▶ An optimal **regularization scheme** is used.
- ▶ A **long history**: (Mueller; Collins; Sen; Botts and Sterman; LM and Sterman; Kidonakis, Oderda and Sterman; Catani; Sterman and Tejeda-Yeomans; ...)

# Motivation: LHC phenomenology

Higgs boson spectrum at LHC (M. Grazzini, hep-ph/0512025)



Predictions for the  $q_T$  spectrum of Higgs bosons produced via gluon fusion at the LHC, with and without resummation.



# Motivation: gauge field theories

- ▶ Understanding long-distance singularities to all orders provides a window into non-perturbative effects.
- ▶ The structure of long-distance singularities is universal for all massless gauge theories.
- ▶ A very special theory has emerged as a theoretical laboratory:  
 $\mathcal{N} = 4$  Super Yang-Mills.
  - ▶ It is conformal invariant:  $\beta_{\mathcal{N}=4}(\alpha_s) = 0$ .
  - ▶ Exponentiation of IR/C poles in scattering amplitudes simplifies.
  - ▶ AdS/CFT suggests a 'simple' description at strong coupling, in the planar limit.
  - ▶ Exponentiation has been observed for MHV amplitudes up to five legs.
  - ▶ Higher-point amplitudes are strongly constrained by (super)conformal symmetry.
  - ▶ A string calculation at strong coupling matches the perturbative result.
  - ▶ Amplitudes admit a dual description in terms of polygonal Wilson loops.
  - ▶ Integrability leads to possibly exact expressions for anomalous dimensions.



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(You Guys *et al.*)

## Tools: dimensional regularization

Nonabelian exponentiation of **IR/C** poles requires  **$d$ -dimensional** evolution equations. The **running coupling** in  $d = 4 - 2\epsilon$  obeys

$$\mu \frac{\partial \bar{\alpha}}{\partial \mu} \equiv \beta(\epsilon, \bar{\alpha}) = -2\epsilon \bar{\alpha} + \hat{\beta}(\bar{\alpha}) \quad , \quad \hat{\beta}(\bar{\alpha}) = -\frac{\bar{\alpha}^2}{2\pi} \sum_{n=0}^{\infty} b_n \left(\frac{\bar{\alpha}}{\pi}\right)^n .$$

The **one-loop** solution is

$$\bar{\alpha}(\mu^2, \epsilon) = \alpha_s(\mu_0^2) \left[ \left(\frac{\mu^2}{\mu_0^2}\right)^\epsilon - \frac{1}{\epsilon} \left(1 - \left(\frac{\mu^2}{\mu_0^2}\right)^\epsilon\right) \frac{b_0}{4\pi} \alpha_s(\mu_0^2) \right]^{-1} .$$

The  $\beta$  function develops an **IR free** fixed point, so that  $\bar{\alpha}(0, \epsilon) = 0$  for  $\epsilon < 0$ . The **Landau pole** is at

$$\mu^2 = \Lambda^2 \equiv Q^2 \left(1 + \frac{4\pi\epsilon}{b_0\alpha_s(Q^2)}\right)^{-1/\epsilon} .$$

- ▶ Integrations over the **scale of the coupling** can be **analytically** performed.
- ▶ All infrared and collinear poles arise **by integration** of  $\alpha_s(\mu^2, \epsilon)$ .

## Tools: factorization

All factorizations separating dynamics at different energy scales lead to resummation of logarithms of the ratio of scales.

- ▶ Renormalization group logarithms.

Renormalization factorizes cutoff dependence

$$G_0^{(n)}(p_i, \Lambda, g_0) = \prod_{i=1}^n Z_i^{1/2}(\Lambda/\mu, g(\mu)) G_R^{(n)}(p_i, \mu, g(\mu)) ,$$

$$\frac{dG_0^{(n)}}{d\mu} = 0 \quad \rightarrow \quad \frac{d \log G_R^{(n)}}{d \log \mu} = - \sum_{i=1}^n \gamma_i(g(\mu)) .$$

- ▶ RG evolution resums  $\alpha_s^n(\mu^2) \log^n(Q^2/\mu^2)$  into  $\alpha_s(Q^2)$ .

**Note:** Factorization is the difficult step! It requires a diagrammatic analysis

- ▶ all-order power counting (UV, IR, collinear ...);
- ▶ implementation of gauge invariance via Ward identities.

## Tools: factorization

- ▶ Collinear factorization logarithms.

Mellin moments of partonic DIS structure functions factorize

$$\tilde{F}_2 \left( N, \frac{Q^2}{m^2}, \alpha_s \right) = \tilde{C} \left( N, \frac{Q^2}{\mu_F^2}, \alpha_s \right) \tilde{f} \left( N, \frac{\mu_F^2}{m^2}, \alpha_s \right)$$

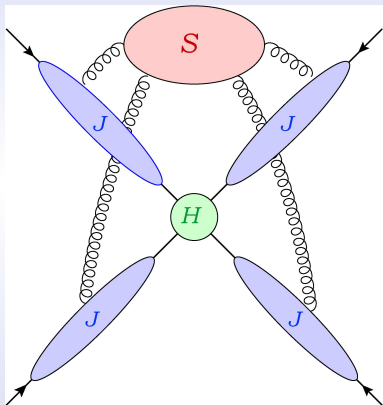
$$\frac{d\tilde{F}_2}{d\log \mu_F} = 0 \quad \rightarrow \quad \frac{d \log \tilde{f}}{d \log \mu_F} = \gamma_N(\alpha_s) .$$

- ▶ Altarelli-Parisi evolution resums collinear logarithms into evolved parton distributions (or fragmentation functions).

**Note:** Sudakov (double) logarithms are more difficult.

- ▶ A double factorization is required: hard vs. collinear vs. soft. Gauge invariance plays a key role in the decoupling.
- ▶ After identification of the relevant modes, effective field theory can be used (SCET).

# Sudakov factorization



Leading regions for Sudakov factorization.

- ▶ Divergences arise in **fixed-angle** amplitudes from **leading regions** in loop momentum space.
- ▶ Soft gluons factorize both from **hard (easy)** and from **collinear (intricate)** virtual exchanges.
- ▶ Jet functions  $J$  represent **color singlet** evolution of external hard partons.
- ▶ The soft function  $S$  is a **matrix** mixing the available **color representations**.
- ▶ In the **planar limit** soft exchanges are **confined** to wedges:  $S \propto \mathbf{I}$ .
- ▶ In the **planar limit**  $S$  can be reabsorbed defining jets  $J$  as square roots of elementary form factors.
- ▶ Beyond the planar limit  $S$  is determined by an anomalous dimension matrix  $\Gamma_S$ .
- ▶ Phenomenological applications to jet and heavy quark production at hadron colliders.

# Form Factors and Planar Amplitudes

(with L. Dixon and G. Sterman)

# Gauge theory form factors

Consider as an example the **quark form factor**

$$\Gamma_{\mu}(p_1, p_2; \mu^2, \epsilon) \equiv \langle 0 | J_{\mu}(0) | p_1, p_2 \rangle = \bar{v}(p_2) \gamma_{\mu} u(p_1) \Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) .$$

- ▶ The form factor obeys the **evolution equation**

$$Q^2 \frac{\partial}{\partial Q^2} \log \left[ \Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] = \frac{1}{2} \left[ K \left( \epsilon, \alpha_s(\mu^2) \right) + G \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] ,$$

- ▶ **Renormalization group invariance** requires

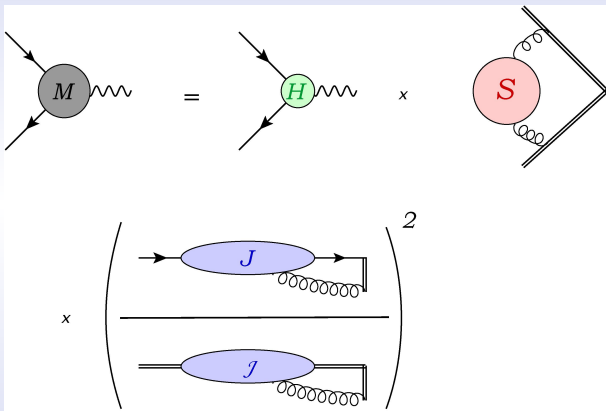
$$\mu \frac{dG}{d\mu} = -\mu \frac{dK}{d\mu} = \gamma_K \left( \alpha_s(\mu^2) \right) .$$

$\gamma_K(\alpha_s)$  is the **cusplike anomalous dimension** (G. Korchemsky and A. Radyushkin; ...).

- ▶ **Dimensional regularization** provides a **trivial initial condition** for evolution if  $\epsilon < 0$  (for **IR** regularization).

$$\bar{\alpha}(\mu^2 = 0, \epsilon < 0) = 0 \rightarrow \Gamma \left( 0, \alpha_s(\mu^2), \epsilon \right) = \Gamma \left( 1, \bar{\alpha}(0, \epsilon), \epsilon \right) = 1 .$$

## Detailed factorization



Operator factorization for the Sudakov form factor, with subtractions.



# Operator definitions

The **functional form** of this graphical factorization is

$$\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = H\left(\frac{Q^2}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) \times \mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon\right) \\ \times \prod_{i=1}^2 \left[ \frac{J\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right)}{\mathcal{J}\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)} \right].$$

We introduced **factorization vectors**  $n_i^\mu$ , with  $n_i^2 \neq 0$ , to define the **jets**,

$$J\left(\frac{(p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) u(p) = \langle 0 | \Phi_n(\infty, 0) \psi(0) | p \rangle.$$

where  $\Phi_n$  is the **Wilson line** operator along the direction  $n^\mu$ .

$$\Phi_n(\lambda_2, \lambda_1) = P \exp \left[ ig \int_{\lambda_1}^{\lambda_2} d\lambda n \cdot A(\lambda n) \right].$$

The jet  $J$  has **collinear** divergences only along  $p$ .

# Operator definitions

The soft function  $\mathcal{S}$  is the eikonal limit of the massless form factor

$$\mathcal{S}(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon) = \langle 0 | \Phi_{\beta_2}(\infty, 0) \Phi_{\beta_1}(0, -\infty) | 0 \rangle .$$

Soft-collinear regions are subtracted dividing by eikonal jets  $\mathcal{J}$ .

$$\mathcal{J}\left(\frac{(\beta_1 \cdot n_1)^2}{n_1^2}, \alpha_s(\mu^2), \epsilon\right) = \langle 0 | \Phi_{n_1}(\infty, 0) \Phi_{\beta_1}(0, -\infty) | 0 \rangle ,$$

- ▶  $\mathcal{S}$  and  $\mathcal{J}$  are pure counterterms in dimensional regularization.
- ▶  $\beta_i$ -dependence of  $\mathcal{S}$  and  $\mathcal{J}$  violates rescaling invariance of Wilson lines.  
⇒ It arises from double poles, associated with  $\gamma_K$ .
- ▶ A single pole function where the cusp anomaly cancels is

$$\bar{\mathcal{S}}(\rho_{12}, \alpha_s(\mu^2), \epsilon) \equiv \frac{\mathcal{S}(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon)}{\prod_{i=1}^2 \mathcal{J}\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)}$$

It can only depend on the scaling variable

$$\rho_{12} \equiv \frac{(\beta_1 \cdot \beta_2)^2 n_1^2 n_2^2}{(\beta_1 \cdot n_1)^2 (\beta_2 \cdot n_2)^2} .$$

## Jet evolution

The full form factor does not depend on the factorization vectors  $n_i^\mu$ .

Defining  $x_i \equiv (\beta_i \cdot n_i)^2 / n_i^2$ ,

$$x_i \frac{\partial}{\partial x_i} \log \Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) = 0.$$

This dictates the evolution of the jet  $J$ , through a ' $K + G$ ' equation

$$\begin{aligned} x_i \frac{\partial}{\partial x_i} \log J_i &= - x_i \frac{\partial}{\partial x_i} \log H + x_i \frac{\partial}{\partial x_i} \log \mathcal{J}_i \\ &\equiv \frac{1}{2} \left[ \mathcal{G}_i(x_i, \alpha_s(\mu^2), \epsilon) + \mathcal{K}(\alpha_s(\mu^2), \epsilon) \right], \end{aligned}$$

Imposing RG invariance of the form factor

$$\gamma_{\overline{S}}(\rho_{12}, \alpha_s) + \gamma_H(\rho_{12}, \alpha_s) + 2\gamma_J(\alpha_s) = 0.$$

leads to the final evolution equation

$$Q \frac{\partial}{\partial Q} \log \Gamma = \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \log H - \gamma_{\overline{S}} - 2\gamma_J + \sum_{i=1}^2 (\mathcal{G}_i + \mathcal{K}).$$

## Collinear evolution

It is **useful** to establish a connection to conventional **collinear factorization**. Define a **parton-in-parton** distribution as

$$\phi_{q/q}(x, \epsilon) = \frac{1}{4N_c} \int \frac{d\lambda}{2\pi} e^{-i\lambda x p \cdot \beta} \langle p | \bar{\psi}_q(\lambda\beta) \gamma \cdot \beta \Phi_\beta(\lambda, 0) \psi_q(0) | p \rangle,$$

The **virtual contribution** can be isolated. At the **amplitude level** it is

$$\bar{\Gamma}_{q/q} \left( \frac{p \cdot \beta}{\mu}, \alpha_s(\mu^2), \epsilon \right) \equiv \langle 0 | \Phi_\beta(\infty, 0) \psi_q(0) | p \rangle,$$

Comparing **factorizations** for this amplitude and the jet  **$J$** , and enforcing **Altarelli-Parisi** evolution one finds

$$\frac{[J((\beta_p \cdot n)^2/n^2, \epsilon)]_{\text{pole}}}{\mathcal{J}((\beta_p \cdot n)^2/n^2, \epsilon)} = \frac{\bar{\Gamma}_{q/q}(\beta_p \cdot \beta, \epsilon)}{\bar{\Gamma}_{q/q}^{\text{eik}}(\beta_p \cdot \beta, \epsilon)} = \exp \left[ \frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} B_\delta^{[q]}(\bar{\alpha}(\xi^2, \epsilon)) \right],$$

where  $B_\delta^{[q]}$  is the **virtual** part of the **splitting function**,

$$P_{ii}(x) = \frac{\gamma_K^{[i]}(\alpha_s)}{2} \left[ \frac{1}{1-x} \right]_+ + B_\delta^{[i]}(\alpha_s) \delta(1-x) + \mathcal{O}((1-x)^0).$$

## Results for Sudakov form factors

- ▶ In dimensional regularization ( $\epsilon < 0$ ) one has the boundary value  $\Gamma(0, \epsilon) = 1$ . Then

$$\log \left[ \Gamma(Q^2, \epsilon) \right] = \frac{1}{2} \int_0^{-Q^2} \frac{d\xi^2}{\xi^2} \left[ K(\epsilon) + G(\bar{\alpha}(\xi^2), \epsilon) + \frac{1}{2} \int_{\xi^2}^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_K(\bar{\alpha}(\lambda^2)) \right]$$

- ▶ The functions  $K$  and  $\gamma_K$  are not independent

$$\mu \frac{d}{d\mu} K(\epsilon, \alpha_s) = -\gamma_K(\alpha_s) \quad \implies \quad K(\epsilon, \alpha_s(\mu^2)) = -\frac{1}{2} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_K(\bar{\alpha}(\lambda^2, \epsilon)) .$$

- ▶ The form factor can be written in terms of just  $G$  and  $\gamma_K$ ,

$$\Gamma(Q^2, \epsilon) = \exp \left\{ \frac{1}{2} \int_0^{-Q^2} \frac{d\xi^2}{\xi^2} \left[ G(-1, \bar{\alpha}(\xi^2, \epsilon), \epsilon) - \frac{1}{2} \gamma_K(\bar{\alpha}(\xi^2, \epsilon)) \log \left( \frac{-Q^2}{\xi^2} \right) \right] \right\} .$$

Recall: in general poles up to  $\alpha_s^n / \epsilon^{n+1}$  appear in the exponent.

## Implications

- ▶ The exponent is **not affected** by the **Landau pole** for  $\epsilon < 0$ .  $\Gamma$  is an **analytic function** of the coupling and  $\epsilon$ . At **one loop** in QCD

$$\begin{aligned} \log \Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) &= \log \Gamma \left( 1, \alpha_s(Q^2), \epsilon \right) \\ &= -\frac{1}{b_0} \left\{ \frac{\gamma_K^{(1)}}{\epsilon} \text{Li}_2 \left[ \frac{a(Q^2)}{a(Q^2) + \epsilon} \right] + 2G^{(1)}(\epsilon) \log \left[ 1 + \frac{a(Q^2)}{\epsilon} \right] \right\}. \end{aligned}$$

- ▶ The **ratio** of the **timelike** to the **spacelike** form factor admits a simple representation

$$\log \left[ \frac{\Gamma(Q^2, \epsilon)}{\Gamma(-Q^2, \epsilon)} \right] = i\frac{\pi}{2}K(\epsilon) + \frac{i}{2} \int_0^\pi \left[ G(\bar{\alpha}(e^{i\theta}Q^2), \epsilon) - \frac{i}{2} \int_0^\theta d\phi \gamma_K(\bar{\alpha}(e^{i\phi}Q^2)) \right]$$

- ▶ **Infinities** are confined to a **phase** given by  $\gamma_K$ .  
⇒ The **modulus** of the ratio is **finite**, and **physically relevant** for resummed EW annihilation processes.

## Form factors in $\mathcal{N} = 4$ SYM

- ▶ In  $d = 4 - 2\epsilon$  conformal invariance is broken and  $\beta(\alpha_s) = -2\epsilon\alpha_s$ .
- ▶ All integrations are trivial. The exponent has only double and single poles to all orders (Z. Bern, L. Dixon, A. Smirnov).

$$\begin{aligned}\log \left[ \Gamma \left( \frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon \right) \right] &= -\frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{\alpha_s(\mu^2)}{\pi} \right)^n \left( \frac{\mu^2}{-Q^2} \right)^{n\epsilon} \left[ \frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon} \right] \\ &= -\frac{1}{2} \sum_{n=1}^{\infty} \left( \frac{\alpha_s(Q^2)}{\pi} \right)^n e^{-i\pi n\epsilon} \left[ \frac{\gamma_K^{(n)}}{2n^2\epsilon^2} + \frac{G^{(n)}(\epsilon)}{n\epsilon} \right],\end{aligned}$$

- ▶ In the planar limit this captures all singularities of fixed-angle amplitudes in  $\mathcal{N} = 4$  SYM.
- ▶ The analytic continuation yields a finite result in four dimensions, arguably exact.

$$\left| \frac{\Gamma(Q^2)}{\Gamma(-Q^2)} \right|^2 = \exp \left[ \frac{\pi^2}{4} \gamma_K \left( \alpha_s(Q^2) \right) \right].$$

## Characterizing $G(\alpha_s, \epsilon)$

The single pole function  $G(\alpha_s, \epsilon)$  is a sum of anomalous dimensions

$$G(\alpha_s, \epsilon) = \beta(\epsilon, \alpha_s) \frac{\partial}{\partial \alpha_s} \log H - \gamma_{\bar{S}} - 2\gamma_J + \sum_{i=1}^2 G_i,$$

In  $d = 4 - 2\epsilon$  finite remainders can be neatly exponentiated

$$C(\alpha_s(Q^2), \epsilon) = \exp \left[ \int_0^{Q^2} \frac{d\xi^2}{\xi^2} \left\{ \frac{d \log C(\bar{\alpha}(\xi^2, \epsilon), \epsilon)}{d \ln \xi^2} \right\} \right] \equiv \exp \left[ \frac{1}{2} \int_0^{Q^2} \frac{d\xi^2}{\xi^2} G_C(\bar{\alpha}(\xi^2, \epsilon), \epsilon) \right]$$

The soft function exponentiates like the full form factor

$$S(\alpha_s(\mu^2), \epsilon) = \exp \left\{ \frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \left[ G_{\text{eik}}(\bar{\alpha}(\xi^2, \epsilon)) - \frac{1}{2} \gamma_K(\bar{\alpha}(\xi^2, \epsilon)) \log \left( \frac{\mu^2}{\xi^2} \right) \right] \right\}.$$

$G(\alpha_s, \epsilon)$  is then simply related to collinear splitting functions and to the eikonal approximation

$$G(\alpha_s, \epsilon) = 2B_{\bar{S}}(\alpha_s) + G_{\text{eik}}(\alpha_s) + G_{\bar{H}}(\alpha_s, \epsilon),$$

**Note:**  $G_{\bar{H}}$  does not generate poles; it vanishes in  $\mathcal{N} = 4$  SYM. □



## Single logarithms in resummation

A deeper characterization of infrared and collinear **single poles** in **amplitudes** should be reflected in **single logarithms** in **cross sections**.

For a resummation as in the **Drell-Yan** cross section

$$\widehat{\omega}_{\overline{\text{MS}}}(N) = \left| \frac{\Gamma(Q^2, \epsilon)}{\phi_V(Q^2, \epsilon)} \right|^2 \exp \left[ F_{\text{DY}}(\alpha_s) + \int_0^1 dz \frac{z^{N-1} - 1}{1-z} \left\{ 2 \int_{Q^2}^{(1-z)^2 Q^2} \frac{d\mu^2}{\mu^2} A(\alpha_s(\mu^2)) + D(\alpha_s((1-z)^2 Q^2)) \right\} \right].$$

the function  $D(\alpha_s)$  should be **related** to  $B_\delta(\alpha_s)$  and to  $G(\alpha_s)$ . Indeed,

$$A(\alpha_s) = \gamma_K(\alpha_s)/2,$$

$$D(\alpha_s) = 4B_\delta(\alpha_s) - 2\tilde{G}(\alpha_s) + \hat{\beta}(\alpha_s) \frac{d}{d\alpha_s} F_{\text{DY}}(\alpha_s),$$

$$B(\alpha_s) = B_\delta(\alpha_s) - \tilde{G}(\alpha_s) + \hat{\beta}(\alpha_s) \frac{d}{d\alpha_s} F_{\text{DIS}}(\alpha_s),$$

where  $B(\alpha_s)$  is the **single log** function for **DIS** resummation, and  $\tilde{G}(\alpha_s)$  is derived recursively from  $G(\alpha_s)$  (Laenen, LM; Ravindran *et al*; Moch, Vermaseren, Vogt).

# Beyond the Planar Limit

(with E. Gardi)

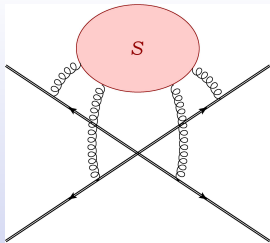
## Factorization at fixed angle

Fixed-angle scattering amplitudes in any massless gauge theory can also be factorized into **hard**, **jet** and **soft** functions.

$$\mathcal{M}_L(p_i/\mu, \alpha_s(\mu^2), \epsilon) = \mathcal{S}_{LK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon) H_K\left(\frac{p_i \cdot p_j}{\mu^2}, \frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right) \\ \times \prod_{i=1}^n \left[ J_i\left(\frac{(p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right) / \mathcal{J}_i\left(\frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right) \right],$$

The **soft** function is now a **matrix**, mixing the available **color tensors**.

$$(c_L)_{\{\alpha_k\}} \mathcal{S}_{LK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon) \\ = \sum_{\{\eta_k\}} \langle 0 | \prod_{i=1}^n [\Phi_{\beta_i}(\infty, 0)_{\alpha_k, \eta_k}] | 0 \rangle (c_K)_{\{\eta_k\}},$$



Soft exchanges mix color structures.

# Soft anomalous dimensions

The soft function  $\mathcal{S}$  obeys a matrix RG evolution equation

$$\mu \frac{d}{d\mu} \mathcal{S}_{IK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon) = -\Gamma_{IJ}^{\mathcal{S}}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon) \mathcal{S}_{JK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon),$$

- **Note:**  $\Gamma^{\mathcal{S}}$  is singular due to overlapping UV and collinear poles.

As before,  $\mathcal{S}$  is a pure counterterm. In dimensional regularization, then

$$\mathcal{S}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon) = P \exp \left[ -\frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \Gamma^{\mathcal{S}}(\beta_i \cdot \beta_j, \alpha_s(\xi^2), \epsilon), \epsilon \right].$$

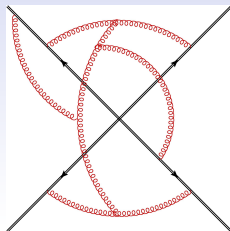
Double poles cancel in the reduced soft function

$$\bar{\mathcal{S}}_{LK}(\rho_{ij}, \alpha_s(\mu^2), \epsilon) = \frac{S_{LK}(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon)}{\prod_{i=1}^n \mathcal{J}_i \left( \frac{(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon \right)}$$

- $\bar{\mathcal{S}}$  must depend on rescaling invariant variables,  $\rho_{ij} \equiv \frac{n_i^2 n_j^2 (\beta_i \cdot \beta_j)^2}{(\beta_i \cdot n_i)^2 (\beta_j \cdot n_j)^2}$ .
- The anomalous dimension  $\Gamma^{\bar{\mathcal{S}}}(\rho_{ij}, \alpha_s)$  for the evolution of  $\bar{\mathcal{S}}$  is finite.

## Surprising simplicity

- ▶  $\Gamma^{\mathcal{S}}$  can be computed from UV poles of  $\mathcal{S}$
- ▶ Non-abelian eikonal exponentiation selects the relevant diagrams: webs
- ▶  $\Gamma^{\mathcal{S}}$  appears highly complex at high orders.



A web contributing to  $\Gamma^{\mathcal{S}}$ .

The two-loop calculation (M. Aybat, L. Dixon, G. Sterman) leads to a surprising result: for any number of light-like eikonal lines

$$\Gamma_S^{(2)} = \frac{\kappa}{2} \Gamma_S^{(1)} \quad \kappa = \left( \frac{67}{18} - \zeta(2) \right) C_A - \frac{10}{9} T_F C_F.$$

- ▶ No new kinematic dependence; no new matrix structure.
- ▶  $\kappa$  is the two-loop coefficient of  $\gamma_K$ , rescaled by the appropriate Casimir,

$$\gamma_K^{(i)}(\alpha_s) = C^{(i)} \left[ 2 \frac{\alpha_s}{\pi} + \kappa \left( \frac{\alpha_s}{\pi} \right)^2 \right] + \mathcal{O}(\alpha_s^3).$$

# Kinematics and cusps

Eikonal jets illustrate how the cusp anomaly dictates kinematic dependence.

- ▶  $\mathcal{J}_i$  obeys its own ' $K + G$ ' equation, and it is a pure counterterm. Thus

$$\mathcal{J}_i(x_i, \alpha_s(\mu^2), \epsilon) = \exp \left[ \frac{1}{4} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \left( G_{\mathcal{J}_i}(x_i, \alpha_s(\xi^2), \epsilon) - \frac{1}{2} \gamma_K^{(i)}(\alpha_s(\xi^2), \epsilon) \ln \frac{\mu^2}{\xi^2} \right) \right],$$

- ▶ The factorization of the form factor links  $\mathcal{J}_i$  to  $\gamma_K^{(i)}$

$$x_i \frac{\partial}{\partial x_i} \ln \mathcal{J}_i(x_i, \alpha_s(\mu^2), \epsilon) = \frac{1}{2} \mathcal{K}(\epsilon, \alpha_s) = -\frac{1}{8} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \gamma_K^{(i)}(\alpha_s(\xi^2), \epsilon).$$

- ▶  $G_{\mathcal{J}_i}$  must then be proportional to  $\gamma_K^{(i)}$  and linear in  $\ln x_i$ ,

$$G_{\mathcal{J}_i}(x_i, \alpha_s) = -\frac{1}{2} \gamma_K^{(i)}(\alpha_s) \ln(x_i) + \delta_{\mathcal{J}_i}(\alpha_s).$$

- ▶ Kinematic dependence of eikonal jets is completely solved

$$\mathcal{J}_i(x_i, \alpha_s(\mu^2), \epsilon) = \exp \left\{ \frac{1}{4} \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \left[ \delta_{\mathcal{J}_i}(\alpha_s(\lambda^2), \epsilon) - \frac{1}{2} \gamma_K^{(i)}(\alpha_s(\lambda^2), \epsilon) \ln \left( \frac{x_i \mu^2}{\lambda^2} \right) \right] \right\}.$$

## Factorization constraints

For general **multi-leg** amplitudes, the conclusion is **less straightforward**

- ▶ For **eikonal jets**, **no**  $\beta_i$  dependence is possible **except** through the cusp.
- ▶ For  $n \geq 4$  legs, rescaling-invariant combinations of  $\beta_i$ 's **exist**:

$$\rho_{ijkl} \equiv (\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l) / (\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)$$

Consider however the anomalous dimension for the **reduced** soft function

$$\Gamma_{IJ}^{\bar{S}}(\rho_{ij}, \alpha_s(\mu^2)) = \Gamma_{IJ}^S(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon) - \delta_{IJ} \sum_{k=1}^n \gamma_{\mathcal{J}_k} \left( \frac{(\beta_k \cdot n_k)^2}{n_k^2}, \alpha_s(\mu^2), \epsilon \right).$$

This poses **strong constraints** on the **soft matrix**. Indeed

- ▶ **Off-diagonal** terms in  $\Gamma^S$  must be **finite**, and must depend **only** on the cross-ratios  $\rho_{ijkl}$ .
- ▶ **Singular** terms in  $\Gamma^S$  must be **diagonal** and proportional to  $\gamma_K$ .
- ▶ **Finite** diagonal terms must **conspire** to construct  $\rho_{ij}$ 's combining  $\beta_i \cdot \beta_j$  with  $x_i = (\beta_i \cdot n_i)^2 / n_i^2$ .

## Factorization constraints

The **constraints** can be formalized simply by using the **chain rule**.

$\Gamma^{\bar{S}}$  depends on  $x_i$  in a **simple** way.

$$x_i \frac{\partial}{\partial x_i} \Gamma_{IJ}^{\bar{S}}(\rho_{ij}, \alpha_s) = -\delta_{IJ} x_i \frac{\partial}{\partial x_i} \gamma_{\mathcal{J}}(x_i, \alpha_s, \epsilon) = -\frac{1}{4} \gamma_K^{(i)}(\alpha_s) \delta_{IJ}.$$

This leads to a **linear equation** for the dependence of  $\Gamma^{\bar{S}}$  on  $\rho_{ij}$

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma_{MN}^{\bar{S}}(\rho_{ij}, \alpha_s) = \frac{1}{4} \gamma_K^{(i)}(\alpha_s) \delta_{MN} \quad \forall i,$$

- ▶ The equation relates  $\Gamma^{\bar{S}}$  to  $\gamma_K$  to all orders in perturbation theory  
⇒ and should remain true at strong coupling as well.
- ▶ It correlates **color** and **kinematics** for any number of hard partons.
- ▶ It admits a **unique solution** for amplitudes with up to three hard partons.  
⇒ For  $n \geq 4$  hard partons, functions of  $\rho_{ijkl}$  solve the **homogeneous equation**.



# Minimal solution

The cusp anomalous dimension exhibits Casimir scaling up to three loops.

- ▶  $\gamma_K^{(i)}(\alpha_s) = C_i \hat{\gamma}_K(\alpha_s)$  with  $C_i$  the quadratic Casimir and  $\hat{\gamma}_K(\alpha_s)$  universal.

Denoting with  $\tilde{\gamma}_K^{(i)}$  possible terms violating Casimir scaling, we write

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\bar{S}}(\rho_{ij}, \alpha_s) = \frac{1}{4} \left[ C_i \hat{\gamma}_K(\alpha_s) + \tilde{\gamma}_K^{(i)}(\alpha_s) \right] \quad \forall i,$$

By linearity, using the color generator notation, the scaling term yields

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma_{\text{Q.C.}}^{\bar{S}}(\rho_{ij}, \alpha_s) = \frac{1}{4} T_i \cdot T_i \hat{\gamma}_K(\alpha_s), \quad \forall i$$

An all-order solution is the dipole formula (E. Gardi, LM; T. Becher, M. Neubert)

$$\Gamma^{\bar{S}}(\rho_{ij}, \alpha_s) = -\frac{1}{8} \hat{\gamma}_K(\alpha_s) \sum_{j \neq i} \ln(\rho_{ij}) T_i \cdot T_j + \frac{1}{2} \hat{\delta}_{\bar{S}}(\alpha_s) \sum_i T_i \cdot T_i,$$

as easily checked using color conservation,  $\sum_i T_i = 0$ .

Note: all known results for massless gauge theories are of this form.

## Beyond the minimal solution?

- ▶ For  $n \geq 4$  the constraint equation does not **uniquely** determine  $\Gamma^{\bar{S}}$ : the homogeneous equation has **nontrivial** solutions

$$\sum_{j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma^{\bar{S}}(\rho_{ij}, \alpha_s) = 0 \quad \Leftrightarrow \quad \Gamma^{\bar{S}} = \Gamma^{\bar{S}}(\rho_{ijkl}, \alpha_s) .$$

- ▶ The **cusp anomalous dimension** may violate **Casimir scaling** starting at **four loops**. This would **add** a contribution  $\Gamma_{\text{H.C.}}^{\bar{S}}$  satisfying

$$\sum_{j, j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Gamma_{\text{H.C.}}^{\bar{S}}(\rho_{ij}, \alpha_s) = \frac{1}{4} \tilde{\gamma}_K^{(i)}(\alpha_s), \quad \forall i .$$

- ▶ Evidence is **accumulating** in favor of the **minimal solution**

- ▶ A **nontrivial** function of  $\rho_{ijkl}$  **fails** to appear in  $\Gamma^{\bar{S}}$  at **two loops**.

$$\tilde{\mathbf{H}}_{[\bar{1}]} = \sum_{j,k,l} \sum_{a,b,c} i f_{abc} T_j^a T_k^b T_l^c \ln(\rho_{ijkl}) \ln(\rho_{iklj}) \ln(\rho_{iljk}) ,$$

- ▶ The minimal solution **holds** for a **subset** of diagrams at **three loops** (**L. Dixon**).
- ▶ **Symmetry** arguments, using **collinear limits**, suggest that functions of  $\rho_{ijkl}$  do not appear at **three loops** (**T. Becher and M. Neubert**).

# Perspective

- ▶ After  $\mathcal{O}(10^2)$  years, **soft** and **collinear** singularities in massless gauge theories are **still a fertile field of study**.
  - ⇒ We are **probing** the **all-order** structure of the **nonabelian exponent**.
  - ⇒ All-order results **constrain** and **test** fixed order calculations.
  - ⇒ Understanding **singularities** has **phenomenological** applications through **resummation**.
- ▶ **Factorization** theorems ⇒ **Evolution** equations ⇒ **Exponentiation**.
- ▶ **Dimensional continuation** is the **simplest** and **most elegant** regulator.
  - ⇒ Transparent **mapping** **UV** ↔ **IR** for ‘**pure counterterm**’ functions.
- ▶ **Remarkable** simplifications in  $\mathcal{N} = 4$  **SYM** point to **exact results**.
- ▶ Only **three functions**,  $\gamma_K$ ,  $G_{\text{eik}}$  and  $B_\delta$  determine **all** singularities in the **planar limit**, and possibly **beyond**.
- ▶ **Factorization** and classical **rescaling invariance** severely **constrain** soft anomalous dimensions to **all orders** and for **any number** of legs.
- ▶ A simple **dipole formula** may encode **all** soft singularities for **any** massless gauge theory.

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