Hidden symmetries of scattering amplitudes:

from $\mathcal{N} = 4$ SYM to QCD

Gregory Korchemsky

IPhT, Saclay

Based on work in collaboration with

James Drummond, Johannes Henn, and Emery Sokatchev

arXiv[hep-th]: 0707.0243, 0709.2368, 0712.1223, 0712.4138, 0803.1466, 0807.1095, 0808.0491

Outline

- ✔ General properties of scattering amplitudes
- ✔ Dual conformal invariance – hidden symmetry of planar amplitudes
- ✔ Scattering amplitude/Wilson loop duality in $\mathcal{N} = 4$ SYM
- ✔ Scattering amplitude/Wilson loop duality in QCD

General properties of amplitudes in gauge theories

Tree amplitudes:

- ✔ Are well-defined in $D = 4$ dimensions (free from UV and IR divergences)
- ✔ Respect classical (Lagrangian) symmetries of gauge theory
- ✔ Gluon tree amplitudes are the same in all gauge theories

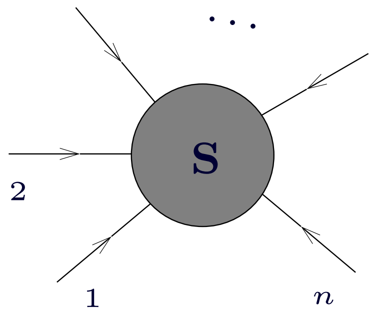
All-loop amplitudes:

- ✔ Loop corrections are not universal (gauge theory dependent)
- ✔ Free from UV divergences (when expressed in terms of renormalized coupling)
- ✔ Suffer from IR divergences $\to$ are not well-defined in $D = 4$ dimensions
- ✔ Some of the classical symmetries (dilatations, conformal boosts,...) are broken

Two main questions in this talk:

- ✔ Do tree amplitudes have hidden symmetry?
- ✔ What happens to this symmetry on loop level?

Gluon scattering amplitudes in $\mathcal{N}=4$ SYM

$\mathcal{A}_n =$ 

- ✗ Quantum numbers of on-shell gluons $|i\rangle = |p_i, h_i, a_i\rangle$: momentum ($(p_i^\mu)^2 = 0$), helicity $(h = \pm 1)$, color (a)
- ✗ Suffer from IR divergences $\mapsto$ require IR regularization
- ✗ Close cousin to QCD gluon amplitudes

✔ Color-ordered planar partial amplitudes

$$\mathcal{A}_n = \mathrm{tr}\left[T^{a_1} T^{a_2} \dots T^{a_n}\right] A_n^{h_1, h_2, \dots, h_n}(p_1, p_2, \dots, p_n) + \text{[Bose symmetry]}$$

- ✗ Color-ordered amplitudes are classified according to their helicity content $h_i = \pm 1$
- ✗ Supersymmetry relations:

$$A^{++\dots+} = A^{-+\dots+} = 0\,, \qquad A^{(\mathrm{MHV})} = A_n^{--+\dots+}\,, \qquad A^{(\mathrm{next-MHV})} = A_n^{---+\dots+}\,, \quad \dots$$

- ✗ The $n = 4$ and $n = 5$ planar gluon amplitudes are all MHV
- ✗ *Weak/strong coupling corrections to all MHV amplitudes in $\mathcal{N} = 4$ SYM are described by a single function of 't Hooft coupling and kinematical invariants!* [Parke,Taylor]

$$A_n^{\mathrm{MHV}} = \delta^{(4)}(p_1 + \dots + p_n) A_n^{(\mathrm{tree})}(p_i, h_i) M_n^{\mathrm{MHV}}(\{s_{ij}\}; \lambda)$$

From MHV amplitudes to MHV superamplitude in $\mathcal{N}=4$ SYM

- ✔ On-shell helicity states in $\mathcal{N}=4$ SYM:

$$G^{\pm} \text{ (gluons } h=\pm 1\text{)}, \qquad \Gamma_A,\ \bar{\Gamma}^A \text{ (gluinos } h=\pm\tfrac{1}{2}\text{)}, \qquad S_{AB} \text{ (scalars } h=0\text{)}$$

- ✔ Can be combined into a single on-shell superstate [Mandelstam],[Brink et el]

$$\Phi(p,\eta) = G^+(p) + \eta^A \Gamma_A(p) + \frac{1}{2}\eta^A\eta^B S_{AB}(p) + \frac{1}{3!}\eta^A\eta^B\eta^C \epsilon_{ABCD}\bar{\Gamma}^D(p) + \frac{1}{4!}\eta^A\eta^B\eta^C\eta^D\epsilon_{ABCD}G^-(p)$$

- ✔ Combine all MHV amplitudes into a single MHV superamplitude [Nair]

$$\mathcal{A}_n^{\mathrm{MHV}} = (\eta_1)^4(\eta_2)^4 \times A\left(G_1^- G_2^- G_3^+ \dots G_n^+\right) + (\eta_1)^4(\eta_2)^2(\eta_3)^2 \times A\left(G_1^- \bar{S}_2\, S_3 \dots G_n^+\right) + \dots$$

- ✔ Spinor helicity formalism: [Xu,Zhang,Chang'87]
  - ✗ commuting spinors: λ^α (helicity -1/2), $\tilde{\lambda}^{\dot\alpha}$ (helicity 1/2)
  - ✗ on-shell momenta:

$$p_i^2 = 0 \quad \Leftrightarrow \quad p_i^{\alpha\dot\alpha} \equiv p_i^\mu (\sigma_\mu)^{\alpha\dot\alpha} = \lambda_i^\alpha\, \tilde{\lambda}_i^{\dot\alpha}$$

Tree MHV superamplitude

- ✔ All MHV amplitudes are combined into a single superamplitude (spinor notations $\langle ij \rangle = \lambda_i^\alpha \lambda_{j\,\alpha}$)

$$\mathcal{A}_n^{\rm MHV}(p_1,\eta_1;\ldots;p_n,\eta_n) = i\,\frac{\delta^{(4)}\left(\sum_{i=1}^n p_i\right)\,\delta^{(8)}\left(\sum_{i=1}^n \lambda_i^\alpha \eta_i^A\right)}{\langle 12\rangle\langle 23\rangle\ldots\langle n1\rangle}$$

- ✔ On-shell $\mathcal{N}=4$ supersymmetry: [Nair]

$$q_\alpha^A = \sum_i \lambda_{i,\alpha}\eta_i^A\,, \qquad \bar q_{A\,\dot\alpha} = \sum_i \tilde\lambda_{i,\dot\alpha}\frac{\partial}{\partial\eta_i^A} \qquad \Longrightarrow \qquad q_\alpha^A\,\mathcal{A}_n^{\rm MHV} = \bar q_{A\,\dot\alpha}\mathcal{A}_n^{\rm MHV} = 0$$

- ✔ (Super)conformal invariance [Witten'03]

$$k_{\alpha\dot\alpha} = \sum_i \frac{\partial^2}{\partial\lambda_i^\alpha\partial\tilde\lambda_i^{\dot\alpha}} \qquad \Longrightarrow \qquad k_{\alpha\dot\alpha}\mathcal{A}_n^{\rm MHV} = 0$$

  Much less trivial to verify for NMHV amplitudes (see forthcoming talks)

- ✔ The MHV superamplitude possesses a much bigger, dual superconformal symmetry

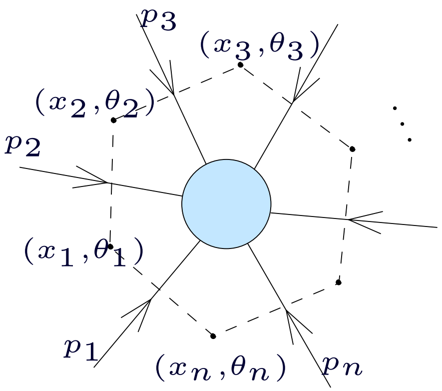
  [Drummond, Henn, GK, Sokatchev]

  acts on the dual coordinates x_i^μ and their superpartners $\theta_{i\,\alpha}^A$

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu\,, \qquad \lambda_i^\alpha\,\eta_i = \theta_i^\alpha - \theta_{i+1}^\alpha$$

Dual $\mathcal{N} = 4$ superconformal symmetry I

✔ Chiral dual superspace $(x_{\alpha\dot\alpha}\,,\, \theta^A_\alpha\,,\, \lambda_\alpha)$:



✗ $p = \sum_{i=1}^{n} p_i = 0 \;\rightarrow\; \quad p_i = x_i - x_{i+1}\,, \quad x_{n+1} = x_1$

✗ $q = \sum_{i=1}^{n} \lambda_i \eta_i = 0 \;\rightarrow\; \quad \lambda_{i\,\alpha}\,\eta_i^A = (\theta_i - \theta_{i+1})^A_\alpha\,, \quad \theta_{n+1} = \theta_1$

✔ The MHV superamplitude in the dual superspace

$$\mathcal{A}_n^{\mathrm{MHV}} = i(2\pi)^4 \frac{\delta^{(4)}\left(x_1 - x_{n+1}\right)\, \delta^{(8)}(\theta_1 - \theta_{n+1})}{\langle 12\rangle\langle 23\rangle \ldots \langle n1\rangle}$$

✔ $\mathcal{N} = 4$ supersymmetry in the dual superspace:

$$Q_{A\,\alpha} = \sum_{i=1}^{n} \frac{\partial}{\partial\theta_i^{A\,\alpha}}\,, \qquad \bar{Q}^A_{\dot\alpha} = \sum_{i=1}^{n} \theta_i^{A\,\alpha} \frac{\partial}{\partial x_i^{\dot\alpha\alpha}}\,, \qquad P_{\alpha\dot\alpha} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i^{\dot\alpha\alpha}}$$

✔ Dual supersymmetry

$$Q_{A\,\alpha}\mathcal{A}_n^{\mathrm{MHV}} = \bar{Q}^A_{\dot\alpha}\mathcal{A}_n^{\mathrm{MHV}} = P_{\alpha\dot\alpha}\mathcal{A}_n^{\mathrm{MHV}} = 0$$

Dual $\mathcal{N}=4$ superconformal symmetry II

- ✔ Super-Poincaré $+$ inversion $=$ conformal supersymmetry:

  - ✗ Inversions in the dual superspace

$$I[\lambda_i^{\alpha}] = (x_i^{-1})^{\dot\alpha\beta}\lambda_{i\beta}\,, \qquad I[\theta_i^{\alpha\,A}] = (x_i^{-1})^{\dot\alpha\beta}\theta_i^{\beta\,A}$$

  - ✗ Neighbouring contractions are dual conformal covariant

$$I[\langle i\, i+1\rangle] = (x_i^2)^{-1}\langle i\, i+1\rangle$$

  - ✗ Impose cyclicity, $x_{n+1}=x_1$, $\theta_{n+1}=\theta_1$, through delta functions. Then, only in $\mathcal{N}=4$,

$$I[\delta^{(4)}(x_1-x_{n+1})] = x_1^8\,\delta^{(4)}(x_1-x_{n+1})$$
$$I[\delta^{(8)}(\theta_1-\theta_{n+1})] = x_1^{-8}\,\delta^{(8)}(\theta_1-\theta_{n+1})$$

- ✔ The tree-level MHV superamplitude is covariant under dual conformal inversions

$$I\left[\mathcal{A}_n^{\mathrm{MHV}}\right] = (x_1^2 x_2^2 \dots x_n^2)\times\mathcal{A}_n^{\mathrm{MHV}}$$

- ✔ ***Dual superconformal covariance is a property of all tree-level superamplitudes (NMHV, N^{2}MHV,...) in $\mathcal{N}=4$ SYM theory*** [Drummond,Henn,GK,Sokatchev]

Does (dual) superconformal symmetry survive loop corrections?

All-loop planar (super)amplitudes can be split into a IR divergent and a finite part

$$\mathcal{A}_n^{(\mathrm{all-loop})} = \mathrm{Div}(1/\epsilon_{\mathrm{IR}})\ [\mathrm{Fin} + O(\epsilon_{\mathrm{IR}})]$$

✔ IR divergences (poles in ϵ_{IR}) exponentiate (in any gauge theory!) [Mueller],[Sen],[Collins],[Sterman],...

$$\mathrm{Div}(1/\epsilon_{\mathrm{IR}}) = \exp\left\{-\frac{1}{2}\sum_{l=1}^{\infty}\lambda^l\left(\frac{\Gamma_{\mathrm{cusp}}^{(l)}}{(l\epsilon_{\mathrm{IR}})^2} + \frac{G^{(l)}}{l\epsilon_{\mathrm{IR}}}\right)\sum_{i=1}^{n}(-s_{i,i+1})^{l\epsilon_{\mathrm{IR}}}\right\}$$

✔ *IR divergences* are in the one-to-one correspondence with *UV divergences* of Wilson loops

[Ivanov,GK,Radyushkin'86]

$\Gamma_{\mathrm{cusp}}(\lambda) = \sum_l \lambda^l \Gamma_{\mathrm{cusp}}^{(l)}$ = *cusp anomalous dimension of Wilson loops*

$G(\lambda) = \sum_l \lambda^l G_{\mathrm{cusp}}^{(l)}$ = *collinear anomalous dimension*

✔ *IR divergences break conformal + dual conformal symmetry*

$$k_{\alpha\dot\alpha}\mathcal{A}_n^{(\mathrm{all-loop})} \neq 0 \qquad \Longrightarrow \qquad \text{(conformal anomaly)}$$

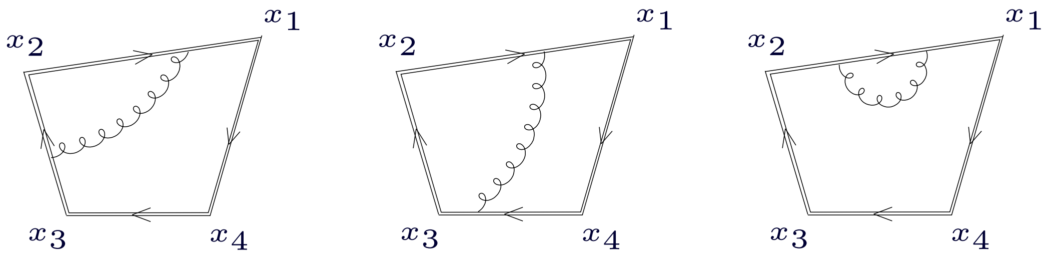
$$K_{\alpha\dot\alpha}\mathcal{A}_n^{(\mathrm{all-loop})} \neq 0 \qquad \Longrightarrow \qquad \text{(dual conformal anomaly)}$$

✔ Dual conformal anomaly can be determined from Wilson loop/scattering amplitude duality

[Drummond,Henn,GK,Sokatchev]

Simplest example:

✔ $n=4$ light-like Wilson loop (with $x_{jk}^2=(x_j-x_k)^2$)

$\ln W(C_4) =$ 

$$=\frac{g^2}{4\pi^2}C_F\left\{-\frac{1}{\epsilon_{\rm UV}{}^2}\left[\left(-x_{13}^2\mu^2\right)^{\epsilon_{\rm UV}}+\left(-x_{24}^2\mu^2\right)^{\epsilon_{\rm UV}}\right]+\frac{1}{2}\ln^2\left(\frac{x_{13}^2}{x_{24}^2}\right)+\text{const}\right\}+O(g^4)$$

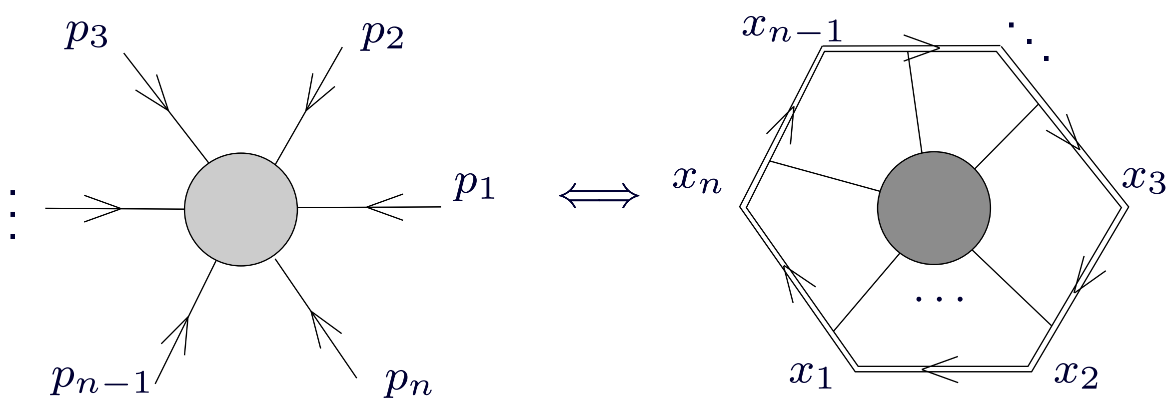
✔ Compare with $n=4$ gluon amplitude

$$\ln\mathcal{A}_4(s,t)=\frac{g^2}{4\pi^2}C_F\left\{-\frac{1}{\epsilon_{\rm IR}{}^2}\left[\left(-s/\mu_{\rm IR}^2\right)^{-\epsilon_{\rm IR}}+\left(-t/\mu_{\rm IR}^2\right)^{-\epsilon_{\rm IR}}\right]+\frac{1}{2}\ln^2\left(\frac{s}{t}\right)+\text{const}\right\}+O(g^4)$$

- ☞ Identity the light-like segments with the on-shell gluon momenta $x_{i,i+1}=p_i$
- ☞ finite $\sim\ln^2(s/t)$ corrections coincide to one loop (constant terms are different)
- ☞ UV div. of the light-like Wilson loop versus IR div. of the gluon amplitude

$$\mu^2:=1/\mu_{\rm IR}^2\,,\qquad \epsilon_{\rm UV}:=-\epsilon_{\rm IR}\qquad\Leftarrow\qquad\begin{cases}\textit{The two objects are defined for different } D=4-2\epsilon\\ \textit{There is a mismatch of } 1/\epsilon \textit{ poles to higher loops}\end{cases}$$

MHV scattering amplitudes/Wilson loop duality II



MHV amplitudes are dual to light-like Wilson loops

$$\ln \mathcal{A}_n^{(\mathrm{MHV})} \sim \ln W(C_n) + O(1/N_c^2)\,, \qquad C_n = \text{light-like } n\text{−(poly)gon}$$

- ✔ At strong coupling, agrees with the BDS ansatz [Alday,Maldacena]
- ✔ At weak coupling, the duality was verified against BDS ansatz for:
  - ✗ $n = 4$ (rectangle) to two loops [Drummond,Henn,GK,Sokatchev]
  - ✗ $n \geq 5$ to one loop [Brandhuber,Heslop,Travaglini]
  - ✗ $n = 5,\ 6$ to two loops [Drummond,Henn,GK,Sokatchev]
  - ✗ $n \geq 7$ to two loops [Anastasiou,Brandhuber,Heslop,Khoze,Spence,Travaglini]]

Wilson loops match the BDS ansatz for $n = 4, 5$ but not for $n \geq 6$

Dual conformal anomaly

Dual conformal symmetry of the amplitudes $\Leftrightarrow$ Conformal symmetry of Wilson loops

Dual conformal anomaly $\Leftrightarrow$ Conformal anomaly of Wilson loops

✔ How could Wilson loops have conformal anomaly in $\mathcal{N}=4$ SYM?

- ✗ Were the Wilson loop well-defined (=finite) in $D=4$ dimensions it would be conformal invariant

$$W(C_n) = W(C_n')$$

- ✗ ... but $W(C_n)$ has cusp UV singularities $\mapsto$ dim.reg. breaks conformal invariance

$$W(C_n) = W(C_n') \times \text{[cusp anomaly]}$$

✔ *All-loop* anomalous conformal Ward identities for the *finite part* of the Wilson loop

$$\ln W(C_n) = F_n^{(WL)} + \text{[UV divergencies]} + O(\epsilon)$$

Under special conformal transformations (boosts), to all orders, [Drummond,Henn,GK,Sokatchev]

$$K^\mu\, F_n \equiv \sum_{i=1}^{n} \left[2x_i^\mu (x_i \cdot \partial_{x_i}) - x_i^2 \partial_{x_i}^\mu\right] F_n = \frac{1}{2}\Gamma_{\rm cusp}(a) \sum_{i=1}^{n} x^\mu_{i,i+1} \ln\left(\frac{x^2_{i,i+2}}{x^2_{i-1,i+1}}\right)$$

The same relations also hold at strong coupling [Alday,Maldacena],[Komargodski]

Finite part of light-like Wilson loops

The consequences of the conformal Ward identity for the finite part of the Wilson loop W_n

✔ $n = 4, 5$ are special: there are no conformal invariants (too few distances due to $x^2_{i,i+1} = 0$)
$\Longrightarrow$ the Ward identity has a *unique all-loop solution* (up to an additive constant)

$$F_4 = \frac{1}{4}\Gamma_{\text{cusp}}(a)\ln^2\left(\frac{x^2_{13}}{x^2_{24}}\right) + \text{ const },$$

$$F_5 = -\frac{1}{8}\Gamma_{\text{cusp}}(a)\sum_{i=1}^{5}\ln\left(\frac{x^2_{i,i+2}}{x^2_{i,i+3}}\right)\ln\left(\frac{x^2_{i+1,i+3}}{x^2_{i+2,i+4}}\right) + \text{ const}$$

Exactly the BDS ansatz for the 4- and 5-point MHV amplitudes!

✔ Starting from $n = 6$ there are conformal invariants in the form of cross-ratios

$$u_1 = \frac{x^2_{13}x^2_{46}}{x^2_{14}x^2_{36}}, \qquad u_2 = \frac{x^2_{24}x^2_{15}}{x^2_{25}x^2_{14}}, \qquad u_3 = \frac{x^2_{35}x^2_{26}}{x^2_{36}x^2_{25}}$$

General solution of the Ward identity contains *an arbitrary function* of the conformal cross-ratios.

✔ Crucial test - go to six points at two loops where the answer is not determined by conformal symmetry. [Drummond,Henn,GK,Sokatchev] [Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]

$$F_6^{(\text{WL})} = F_6^{(\text{MHV})} \neq F_6^{(\text{BDS})}$$

The Wilson loop/scattering amplitude duality holds at $n = 6$ to two loops!

Dual conformal symmetry beyond MHV

$$\mathcal{A}_n(x_i,\lambda_i,\theta_i^A) = \mathcal{A}_n^{\rm MHV} + \mathcal{A}_n^{\rm NMHV} + \mathcal{A}_n^{\rm N^2MHV} + \ldots + \mathcal{A}_n^{\overline{\rm MHV}}$$

- ✔ The tree superamplitude $\mathcal{A}_n^{(\rm tree)}$ is covariant under dual superconformal transformations
- ✔ At loop level, this symmetry becomes anomalous due to IR divergences
- ✔ The dual superconformal symmetry is restored in the ratio of superamplitudes $\mathcal{A}_n$ and $\mathcal{A}_n^{\rm MHV}$

$$\mathcal{A}_n(x_i,\lambda_i,\theta_i^A) = \mathcal{A}_n^{\rm MHV} \times \Big[R_n(x_i,\lambda_i,\theta_i^A) + O(\epsilon)\Big]$$

The ratio function

$$R_n = 1 + R_n^{\rm NMHV} + R_n^{\rm N^2MHV} + \ldots$$

is *IR finite* and, most importantly, it is *superconformal invariant* ! [Drummond,Henn,GK,Sokatchev]

- ✔ Wilson loop/superamplitude duality involves a new ingredient

$$\mathcal{A}_n(x_i,\lambda_i,\theta_i^A)/W_n(x_i) = \mathcal{A}_n^{\rm MHV\,(tree)} \times \left[R_n(x_i,\lambda_i,\theta_i^A) + O(\epsilon)\right]$$

Wilson loop $W_n(x_i)$ takes care of anomalous contribution

"Ratio function" R_n is dual superconformal invariant

What is an operator definition of dual superconformal invariant R_n ?

From $\mathcal{N}=4$ SYM to QCD

Finite part of 4-gluon amplitude in QCD at two loops ($x=-\frac{t}{s}$, $y=-\frac{u}{s}$, $z=-\frac{u}{t}$, $X=\ln x$, $Y=\ln y$, $S=\ln z$) [Glover,Oleari,Tejeda-Yeomans'01]

$$
\begin{aligned}
\mathcal{M}_4^{(\mathrm{QCD})} = \Big\{ \Big(& 48\,\mathrm{Li}_4(x) - 48\,\mathrm{Li}_4(y) - 128\,\mathrm{Li}_4(z) + 40\,\mathrm{Li}_3(x)\,X - 64\,\mathrm{Li}_3(x)\,Y - \tfrac{98}{3}\,\mathrm{Li}_3(x) + 64\,\mathrm{Li}_3(y)\,X - 40\,\mathrm{Li}_3(y)\,Y \\
& + 18\,\mathrm{Li}_3(y) + \tfrac{98}{3}\,\mathrm{Li}_2(x)\,X - \tfrac{16}{3}\,\mathrm{Li}_2(x)\,\pi^2 - 18\,\mathrm{Li}_2(y)\,Y - \tfrac{37}{6}\,X^4 + 28\,X^3\,Y - \tfrac{23}{3}\,X^3 - 16\,X^2\,Y^2 + \tfrac{49}{3}\,X^2\,Y - \tfrac{35}{3}\,X^2\,\pi^2 - \tfrac{38}{3}\,X^2 \\
& - \tfrac{22}{3}\,S\,X^2 - \tfrac{20}{3}\,X\,Y^3 - 9\,X\,Y^2 + 8\,X\,Y\,\pi^2 + 10\,X\,Y - \tfrac{31}{12}\,X\,\pi^2 - 22\,\zeta_3\,X + \tfrac{22}{3}\,S\,X + \tfrac{37}{27}\,X + \tfrac{11}{6}\,Y^4 - \tfrac{41}{9}\,Y^3 - \tfrac{11}{3}\,Y^2\,\pi^2 \\
& - \tfrac{22}{3}\,S\,Y^2 + \tfrac{266}{9}\,Y^2 - \tfrac{35}{12}\,Y\,\pi^2 + \tfrac{418}{9}\,S\,Y + \tfrac{257}{9}\,Y + 18\,\zeta_3\,Y - \tfrac{31}{30}\,\pi^4 - \tfrac{11}{9}\,S\,\pi^2 + \tfrac{31}{9}\,\pi^2 + \tfrac{242}{9}\,S^2 + \tfrac{418}{9}\,\zeta_3 + \tfrac{2156}{27}\,S \\
& - \tfrac{11093}{81} - 8\,S\,\zeta_3 \Big) \tfrac{t^2}{s^2} + \Big(-256\,\mathrm{Li}_4(x) - 96\,\mathrm{Li}_4(y) + 96\,\mathrm{Li}_4(z) + 80\,\mathrm{Li}_3(x)\,X + 48\,\mathrm{Li}_3(x)\,Y - \tfrac{64}{3}\,\mathrm{Li}_3(x) - 48\,\mathrm{Li}_3(y)\,X \\
& + 96\,\mathrm{Li}_3(y)\,Y - \tfrac{304}{3}\,\mathrm{Li}_3(y) + \tfrac{64}{3}\,\mathrm{Li}_2(x)\,X - \tfrac{32}{3}\,\mathrm{Li}_2(x)\,\pi^2 + \tfrac{304}{3}\,\mathrm{Li}_2(y)\,Y + \tfrac{26}{3}\,X^4 - \tfrac{64}{3}\,X^3\,Y - \tfrac{64}{3}\,X^3 + 20\,X^2\,Y^2 \\
& + \tfrac{136}{3}\,X^2\,Y + 24\,X^2\,\pi^2 + 76\,X^2 - \tfrac{88}{3}\,S\,X^2 + \tfrac{8}{3}\,X\,Y^3 + \tfrac{104}{3}\,X\,Y^2 - \tfrac{16}{3}\,X\,Y\,\pi^2 + \tfrac{176}{3}\,S\,X\,Y - \tfrac{136}{3}\,X\,Y - \tfrac{50}{3}\,X\,\pi^2 \\
& - 48\,\zeta_3\,X + \tfrac{2350}{27}\,X + \tfrac{440}{3}\,S\,X + 4\,Y^4 - \tfrac{176}{9}\,Y^3 + \tfrac{4}{3}\,Y^2\,\pi^2 - \tfrac{176}{3}\,S\,Y^2 - \tfrac{494}{9}\,Y\,\pi^2 + \tfrac{5392}{27}\,Y - 64\,\zeta_3\,Y + \tfrac{496}{45}\,\pi^4 \\
& - \tfrac{308}{9}\,S\,\pi^2 + \tfrac{200}{9}\,\pi^2 + \tfrac{968}{9}\,S^2 + \tfrac{8624}{27}\,S - \tfrac{44372}{81} + \tfrac{1864}{9}\,\zeta_3 - 32\,S\,\zeta_3 \Big) \tfrac{t}{u} + \Big(\tfrac{88}{3}\,\mathrm{Li}_3(x) - \tfrac{88}{3}\,\mathrm{Li}_2(x)\,X + 2\,X^4 - 8\,X^3\,Y \\
& - \tfrac{220}{9}\,X^3 + 12\,X^2\,Y^2 + \tfrac{88}{3}\,X^2\,Y + \tfrac{8}{3}\,X^2\,\pi^2 - \tfrac{88}{3}\,S\,X^2 + \tfrac{304}{9}\,X^2 - 8\,X\,Y^3 - \tfrac{16}{3}\,X\,Y\,\pi^2 + \tfrac{176}{3}\,S\,X\,Y - \tfrac{77}{3}\,X\,\pi^2 \\
& + \tfrac{1616}{27}\,X + \tfrac{968}{9}\,S\,X - 8\,\zeta_3\,X + 4\,Y^4 - \tfrac{176}{9}\,Y^3 - \tfrac{20}{3}\,Y^2\,\pi^2 - \tfrac{176}{3}\,S\,Y^2 - \tfrac{638}{9}\,Y\,\pi^2 - 16\,\zeta_3\,Y + \tfrac{5392}{27}\,Y - \tfrac{4}{15}\,\pi^4 - \tfrac{308}{9}\,S\,\pi^2 \\
& - 20\,\pi^2 - 32\,S\,\zeta_3 + \tfrac{1408}{9}\,\zeta_3 + \tfrac{968}{9}\,S^2 - \tfrac{44372}{81} + \tfrac{8624}{27}\,S \Big) \tfrac{t^2}{u^2} + \Big(\tfrac{44}{3}\,\mathrm{Li}_3(x) - \tfrac{44}{3}\,\mathrm{Li}_2(x)\,X - X^4 + \tfrac{110}{9}\,X^3 - \tfrac{22}{3}\,X^2\,Y \\
& + \tfrac{14}{3}\,X^2\,\pi^2 + \tfrac{44}{3}\,S\,X^2 - \tfrac{152}{9}\,X^2 - 10\,X\,Y + \tfrac{11}{2}\,X\,\pi^2 + 4\,\zeta_3\,X - \tfrac{484}{9}\,S\,X - \tfrac{808}{27}\,X + \tfrac{7}{30}\,\pi^4 - \tfrac{31}{9}\,\pi^2 \\
& + \tfrac{11}{9}\,S\,\pi^2 - \tfrac{418}{9}\,\zeta_3 - \tfrac{242}{9}\,S^2 - \tfrac{2156}{27}\,S + 8\,S\,\zeta_3 + \tfrac{11093}{81} \Big) \tfrac{u\,t}{s^2} + \Big(-176\,\mathrm{Li}_4(x) + 88\,\mathrm{Li}_3(x)\,X - 168\,\mathrm{Li}_3(x)\,Y - \ldots
\end{aligned}
$$

No relation to rectangular Wilson loop ... but let us examine the Regge limit $s \gg -t$

Scattering amplitude/Wilson loop duality in QCD

- ✔ Planar four-gluon QCD scattering amplitude in the Regge limit $s \gg -t$ [Schnitzer'76],[Fadin,Kuraev,Lipatov'76]

$$\mathcal{M}_4^{(\mathrm{QCD})}(s,t) \sim (s/(-t))^{\omega_R(-t)} + \ldots$$

The Regge trajectory $\omega_R(-t)$ is known to two loops [Fadin,Fiore,Kotsky'96]

- ✔ The all-loop gluon Regge trajectory in QCD [GK'96]

$$\omega_R(-t) = \frac{1}{2}\int_{(-t)}^{\mu_{\mathrm{IR}}^2} \frac{dk_\perp^2}{k_\perp^2}\Gamma_{\mathrm{cusp}}(a(k_\perp^2)) + \Gamma_R(a(-t)) + [\text{poles in } 1/\epsilon_{\mathrm{IR}}]\,,$$

- ✔ Rectangular Wilson loop in QCD in the Regge limit $|x_{13}^2| \gg |x_{24}^2|$

$$W^{(\mathrm{QCD})}(C_4) \sim \left(x_{13}^2/(-x_{24}^2)\right)^{\omega_{\mathrm{W}}(-x_{24}^2)} + \ldots$$

- ✔ The all-loop Wilson loop 'trajectory' in QCD

$$\omega_{\mathrm{W}}^{(\mathrm{QCD})}(-t) = \frac{1}{2}\int_{(-t)}^{1/\mu_{\mathrm{UV}}^2} \frac{dk_\perp^2}{k_\perp^2}\Gamma_{\mathrm{cusp}}(a(k_\perp^2)) + \Gamma_{\mathrm{W}}(a(-t)) + [\text{poles in } 1/\epsilon_{\mathrm{UV}}]\,,$$

- ✔ *The scattering amplitude/Wilson loop duality relation holds in QCD in the Regge limit only* [GK'96]

$$\ln \mathcal{M}_4^{(\mathrm{QCD})}(s,t) = \ln W^{(\mathrm{QCD})}(C_4) + O(t/s)$$

while in $\mathcal{N}=4$ SYM it is exact for arbitrary t/s!

Conclusions and recent developments

- ✔ MHV amplitudes in $\mathcal{N}=4$ theory
  - ✗ possess the dual conformal symmetry both at weak and at strong coupling
  - ✗ Dual to light-like Wilson loops
- ✔ This symmetry is a part of much bigger dual superconformal symmetry of all planar superamplitudes in $\mathcal{N}=4$ SYM [Drummond,Henn,GK,Sokatchev]
  - ✗ Relates various particle amplitudes with different helicity configurations (MHV, NMHV,...)
  - ✗ Imposes non-trivial constraints on the loop corrections
- ✔ Dual superconformal symmetry is now explained better through the AdS/CFT correspondence by a combined bosonic [Kallosh,Tseytlin] and fermionic T duality symmetry [Berkovits,Maldacena], [Beisert,Ricci,Tseytlin,Wolf]
- ✔ Dual symmetry is also present in QCD but in the Regge limit only ... yet another glimpse of QCD/string duality?!
- ✔ What is the generalisation of the Wilson loop/amplitude duality beyond MHV?