Amplitudes, Wilson loops and dual superconformal symmetry

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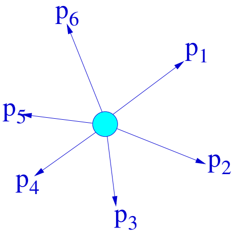
based on work with Babis Anastasiou, Andi Brandhuber, Valya Khoze, Bill Spence, Gabriele Travaglini
arXiv:0707.1153,0807.4097,0902.2245 and work to appear soon

Introduction

- Duality between two objects in $\mathcal{N}$=4 Super Yang-Mills:



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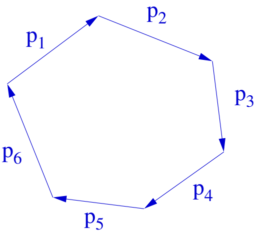

Gluon amplitudes = Wilson loops

- Vast simplification of the computation of amplitudes

Example

We compute all MHV 2-loop gluon scattering amplitudes (assuming the conjectured duality) for any *n*.

1. The duality
   - The evidence so far...
   - Wilson loop calculations - 1 loop
   - Wilson loop calculations - 2 loop

2. Results of two-loop computations
   - 6 points
   - 7 points
   - 8 points

3. Dual superconformal symmetry of the entire S-matrix
   - at tree level
   - at one loop

Outline

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MHV amplitudes

- Colour-stripped planar $L - loop$ MHV amplitudes $A_n^{(L)}$

L-loop amplitude

$$A_n^{(L)} \;= A_n^{\text{tree}}\, \mathcal{M}_n^{(L)}(p_i)$$

- $\mathcal{M}_n^{(L)}$ is a scalar function of the external momentum p_i only.
- In the first part of this talk we will focus on $\mathcal{M}_n^{(L)}$ for the MHV amplitude
- Amplitudes are infrared divergent: we regularise by dimensional regularisation and work in $d = 4 - 2\epsilon$ dimensions

L-loop amplitude

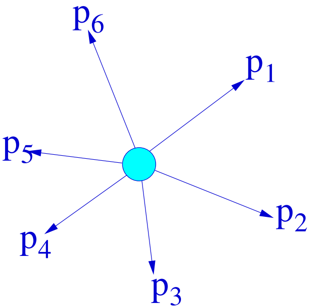
The BDS conjecture [Anastasiou Bern Dixon Kosower 2003, Bern Dixon Smirnov 2005]

The BDS formula: an all-loop expression for any n

$$\log\left(\mathcal{M}_n(\epsilon)\right) = \sum_{L=1}^{\infty} a^L \left(f_{\mathcal{A}}^{(L)}(\epsilon)\mathcal{M}_n^{(1)}(L\epsilon) + C^{(L)} \right) + O(\epsilon)$$

- 'a' is the 't Hooft coupling
- Here $f_{\mathcal{A}}^{(L)}(\epsilon) = f_0^{(L)} + f_1^{(L)}\epsilon + f_2^{(L)}\epsilon^2$ where $f_i^{(L)}$ is a number.
- needs modification from $n = 6$ points...

Amplitude/Wilson loop duality



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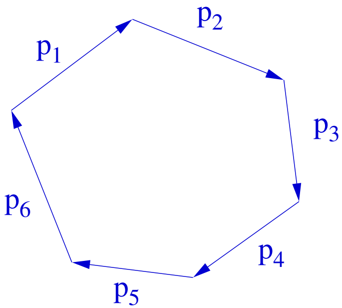


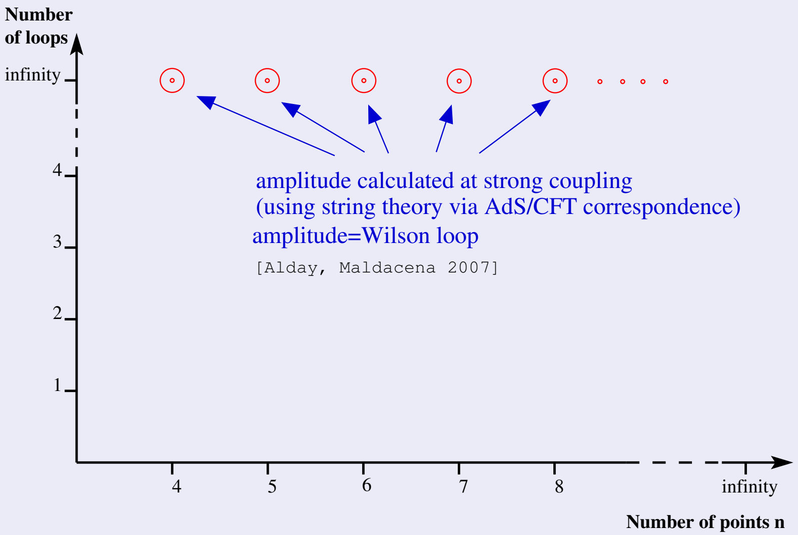
amplitude $\mathcal{M}_n$ '=' $\langle W[\mathcal{C}_n]\rangle$

$(d = 4 - 2\epsilon)$ $(d = 4 + 2\epsilon)$

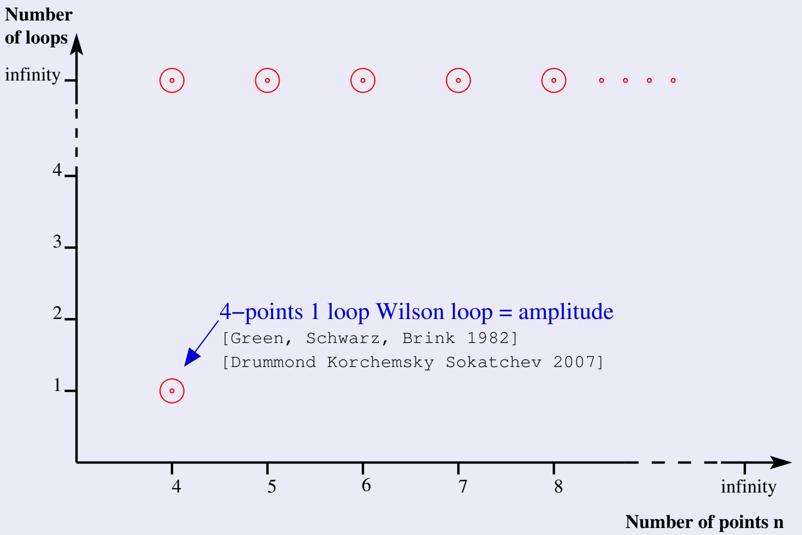
- Wilson loop over the polygonal contour $\mathcal{C}_n$

$$W[\mathcal{C}] := \frac{1}{2} \mathrm{Tr}\, \mathcal{P} \exp \left[ig \oint_{\mathcal{C}} d\tau \Big(A_\mu(x(\tau))\dot{x}^\mu(\tau) \Big) \right]$$

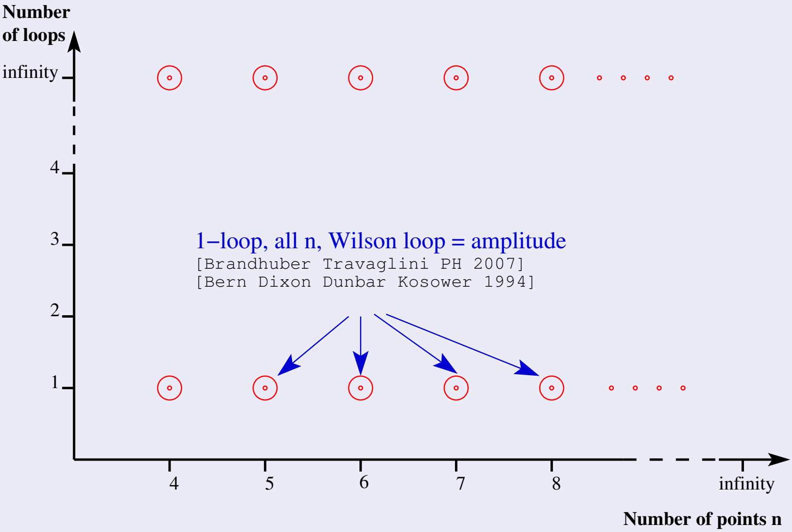
Evidence so far...



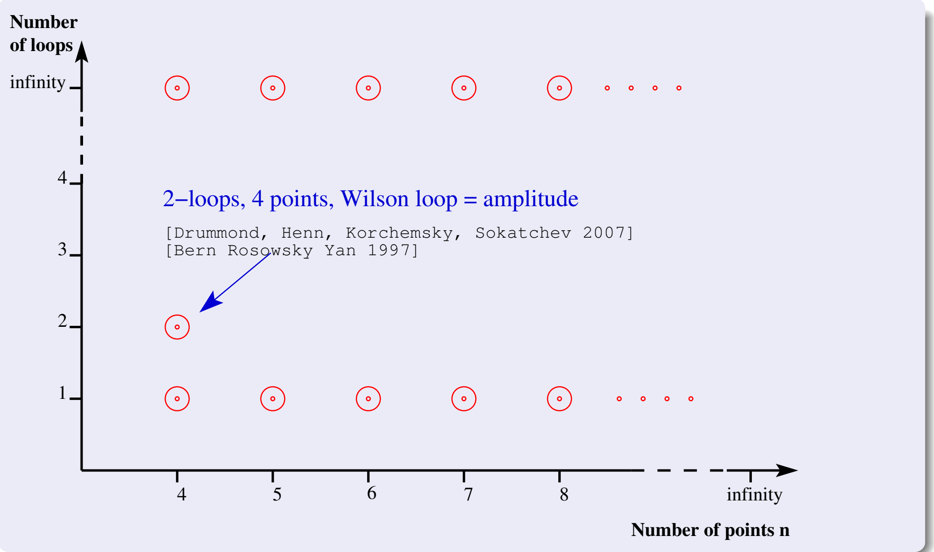
Evidence so far...



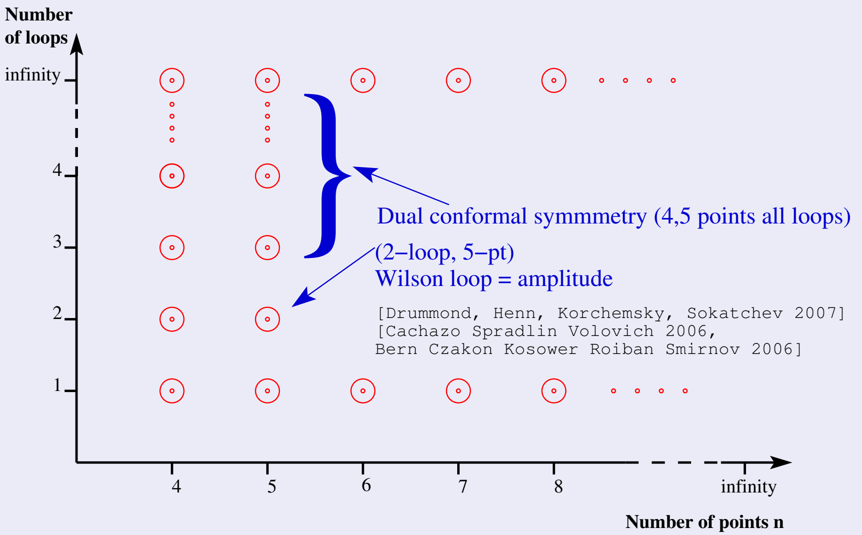
Evidence so far...



Evidence so far...



Evidence so far...



Dual conformal invariance

[Drummond Henn Korchemsky Sokatchev 2007]

- 4-, 5-point Wilson loop is completely determined by conformal symmetry as the ABDK/BDS conjecture

$$\log\Big(W_n(\epsilon)\Big) = \sum_{L=1}^{\infty} a^L\, f_W^{(L)}(\epsilon) W_n^{(1)}(L\epsilon) + C_w(a) + O(\epsilon)$$

- $\Rightarrow$ the 4-,5-point amplitude determined similarly by new conjectured symmetry 'dual conformal symmetry'

$$\log\Big(\mathcal{M}_n(\epsilon)\Big) \;=\; \sum_{L=1}^{\infty} a^L\, f_{\mathcal{A}}^{(L)}(\epsilon)\mathcal{M}_n^{(1)}(L\epsilon) + C_{\mathcal{A}}(a) + O(\epsilon)$$

- beyond 5-points there might exist a non-zero conformally invariant remainder function $\mathcal{R}_n^W, \mathcal{R}_n^{\mathcal{A}}$

Remainder function

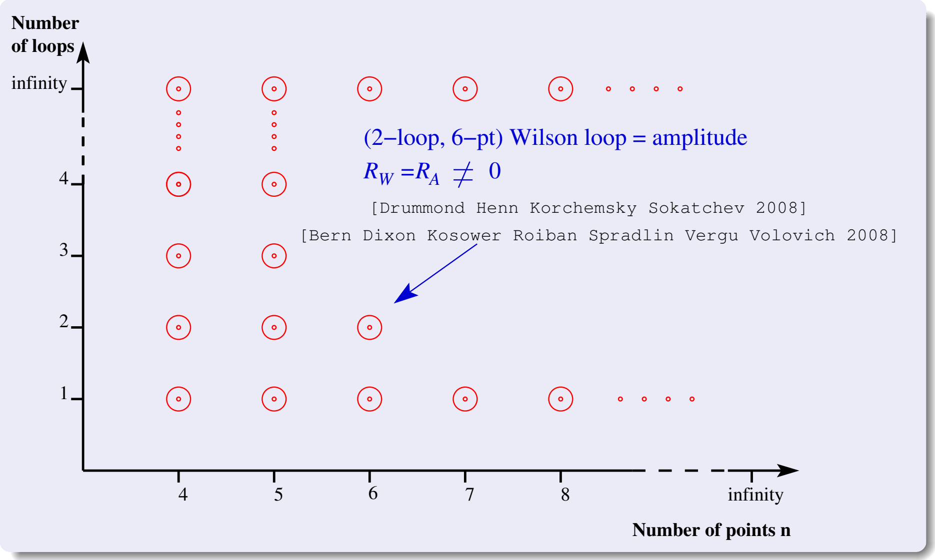
$n \geq 6$

$$\log\left(\mathcal{M}_n(\epsilon)\right) = \sum_{L=1}^{\infty} a^L f_{\mathcal{A}}^{(L)}(\epsilon)\mathcal{M}_n^{(1)}(L\epsilon) + C_{\mathcal{A}}(a) + \mathcal{R}_n^{\mathcal{A}}(p_i; a) + O(\epsilon)$$

$$\log\left(W_n(\epsilon)\right) = \sum_{L=1}^{\infty} a^L f_{W}^{(L)}(\epsilon)W_n^{(1)}(L\epsilon) + C_{W}(a) + \mathcal{R}_n^{W}(p_i; a) + O(\epsilon)$$

- non-zero remainder function found for the two-loop six-point amplitude and the Wilson loop [Drummond Henn Korchemsky Sokatchev 2008, Bern Dixon Kosower Roiban Spradlin Vergu Volovich 2008]

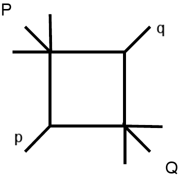
Evidence so far...



Wilson loop calculations, 1-loop

- the expression for the $n-point$ amplitude and for the WL are very closely related:

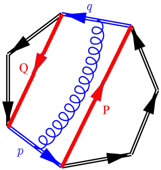
Amplitude $= \quad \sum_{p,q}$



[Bern Dixon Dunbar Kosower 1994]

$$=$$

WL $= \quad \sum_{p,q}$

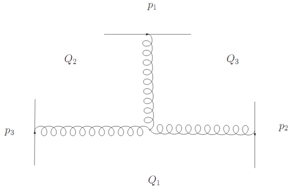


[Brandhuber Travaglini PH 2007]

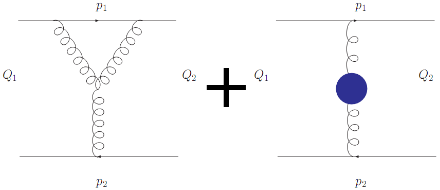
$$P = \sum_{k=p+1}^{q-1} k \qquad Q = \sum_{k=q+1}^{p-1} k$$

2-loop n-point Wilson loop (log of)

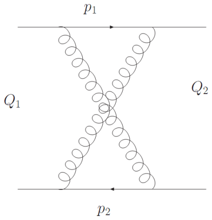
Only four new "master" integrals to be computed for all *n*



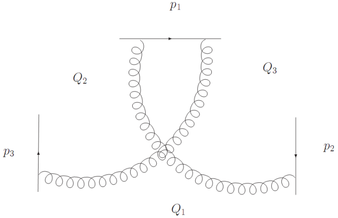
$$f_H(p_1, p_2, p_3; Q_1, Q_2, Q_3)$$



$$f_Y(p_1, p_2; Q_1, Q_2)$$



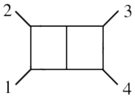
$$f_X(p_1, p_2; Q_1, Q_2)$$



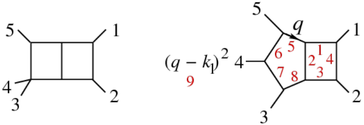
$$f_C(p_1, p_2, p_3; Q_1, Q_2, Q_3)$$

(Compare with amplitude (parity even part))

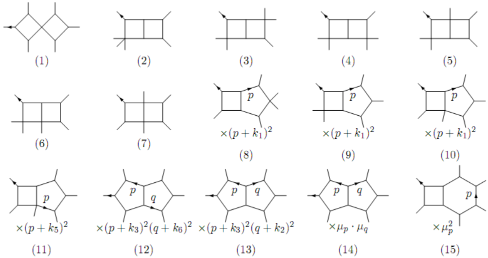
$n = 4$



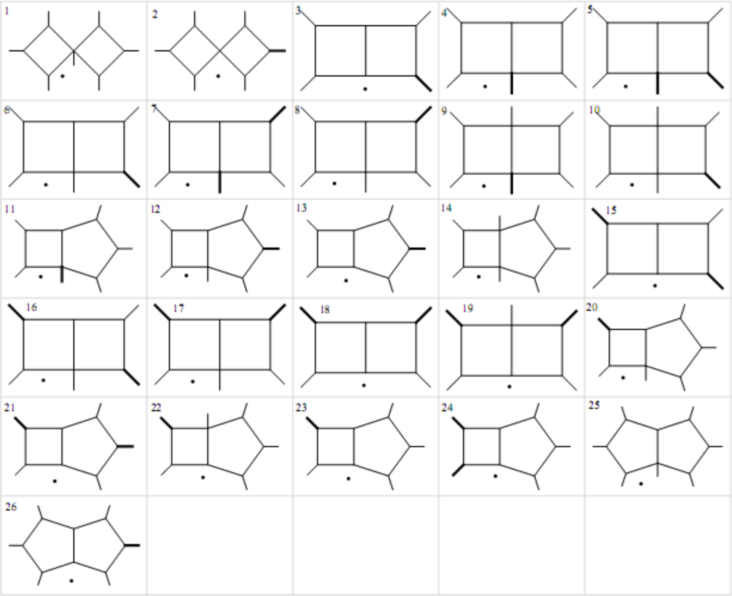
$n = 5$



$n = 6$



$n = 7$ [Vergu]



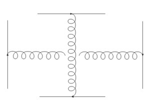
Complete 2-loop Wilson loop

- The logarithm of the complete n-sided Wilson loop is given in terms of the four new master diagrams together with the one loop diagram $f_P(p_i, p_j; Q_{ji}, Q_{ij})$ as

$$\sum_{1 \le i<j<k \le n} \Big[f_H(p_i, p_j, p_k; Q_{jk}, Q_{ki}, Q_{ij}) + f_C(p_i, p_j, p_k; Q_{jk}, Q_{ki}, Q_{ij}) + f_C(p_j, p_k, p_i; Q_{ki}, Q_{ij}, Q_{jk}) + f_C(p_k, p_i, p_j; Q_{ij}, Q_{jk}, Q_{ki}) \Big]$$

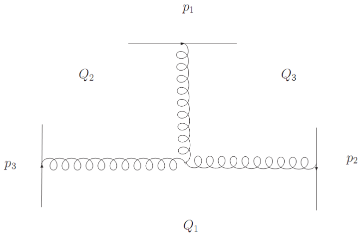
$$+ \sum_{1 \le i<j \le n} \Big[f_X(p_i, p_j; Q_{ji}, Q_{ij}) + f_Y(p_i, p_j; Q_{ji}, Q_{ij}) + f_Y(p_j, p_i; Q_{ij}, Q_{ji}) \Big]$$

$$+ \sum_{1 \le i<k<j<l \le n} (-1/2) f_P(p_i, p_j; Q_{ji}, Q_{ij}) f_P(p_k, p_l; Q_{lk}, Q_{kl})$$



Comment

- UV singularities ($1/\epsilon$) of these diagrams depend on whether $Q_i = 0$ or not, ie whether legs can be adjacent



- Eg f_H has a
  - $1/\epsilon^2$ singularity if $Q_1 = Q_2 = 0$, $Q_3 \neq 0$,
  - $1/\epsilon$ singularity if $Q_1 = 0$, $Q_2, Q_3 \neq 0$
  - finite if $Q_1, Q_2, Q_3 \neq 0$.

Precise correspondence at 2 loops

Amplitude

$$\mathcal{M}_n^{(2)}(\epsilon) - \frac{1}{2}\left(\mathcal{M}_n^{(1)}(\epsilon)\right)^2 = f_A^{(2)}(\epsilon)\mathcal{M}_n^{(1)}(2\epsilon) + C_A^{(2)} + \mathcal{R}_n^{\mathcal{A}}(p_i)$$

Wilson loop: our definition of the WL remainder

$$W_n^{(2)}(\epsilon) - \frac{1}{2}\left(W_n^{(1)}(\epsilon)\right)^2 = f_W^{(2)}(\epsilon)W_n^{(1)}(2\epsilon) + C_w^{(2)} + \mathcal{R}_n^{W}(p_i)$$

- Note from now on $\mathcal{R}_n^{\mathcal{A}}(p_i)$, $\mathcal{R}_n^{W}(p_i)$ will denote the two-loop remainder functions

The correspondence

$$\mathcal{R}_n^A = \mathcal{R}_n^W$$

- DHKS definition contained a previously unknown constant shift
- $f_{2,W}^{(2)}$ and $C_W^{(2)}$ (and hence $\mathcal{R}_n^W$) are uniquely defined by writing the 4- and 5- sided WL (for which $\mathcal{R}_n^W = 0$) as

after correction of a constant in the two-loop four-point result of DHKS

$$w_4^{(2)}(\epsilon) = f_{\mathrm{W}}^{(2)}(\epsilon)\, w_4^{(1)}(2\epsilon) + C_{\mathrm{W}}^{(2)} ,$$
$$w_5^{(2)}(\epsilon) = f_{\mathrm{W}}^{(2)}(\epsilon)\, w_5^{(1)}(2\epsilon) + C_{\mathrm{W}}^{(2)} ,$$
$$f_{\mathrm{W}}^{(2)}(\epsilon) = -\zeta_2 + 7\zeta_3\,\epsilon - 5\zeta_4\,\epsilon^2 \qquad C_{\mathrm{W}}^{(2)} = -\frac{1}{2}\zeta_2^2$$

- cf $f_{\mathcal{A}}^{(2)}(\epsilon) = -\zeta_2 - \zeta_3\,\epsilon - \zeta_4\,\epsilon^2 \qquad C_{\mathcal{A}}^{(2)} = -\frac{1}{2}\zeta_2^2$

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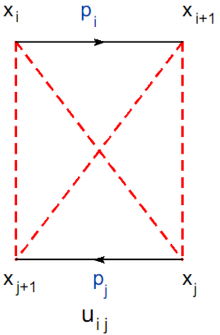
Computations at $n = 6, 7, 8...$

- Using the numerical techniques of [Anastasiou Beerli Daleo (2007,2008), Lazopoulos Melnikov Petriello (2007), Anastasiou Melnikov Petriello (2005)] we compute the 2-loop master integrals
- Computations of WL performed for $n = 4, 5, 6, 7, 8 \rightarrow$ considerable amount of data collected.
- Verified that the remainder function is conformally invariant as shown by DHKS and that it has cyclic and "parity" symmetry
- We have verified that the the WL does not care about the Gram determinant constraint on $p_i.p_j$
- $\Rightarrow n(n-3)/2$ independent kinematic invariants, of which the remainder function depends on $n(n-5)/2$ independent on-shell conformally invariant cross-ratios

Basis of cross-ratios



$$u_{ij} = \frac{x_{ij+1}^2 x_{ji+1}^2}{x_{ij}^2 x_{i+1j+1}^2}$$

- here x_i are the positions of the vertices of the WL ie $x_i - x_{i+1} = p_i$

Hexagon computations

- 3 cross-ratios

$$u_{36} = \frac{x_{31}^2 x_{46}^2}{x_{36}^2 x_{41}^2} := u_1 \ , \ u_{14} = \frac{x_{15}^2 x_{24}^2}{x_{14}^2 x_{25}^2} := u_2 \ , \ u_{25} = \frac{x_{26}^2 x_{35}^2}{x_{25}^2 x_{36}^2} := u_3$$

- remainder function $\rightarrow \mathcal{R}(u_1, u_2, u_3)$

Hexagon Calculations

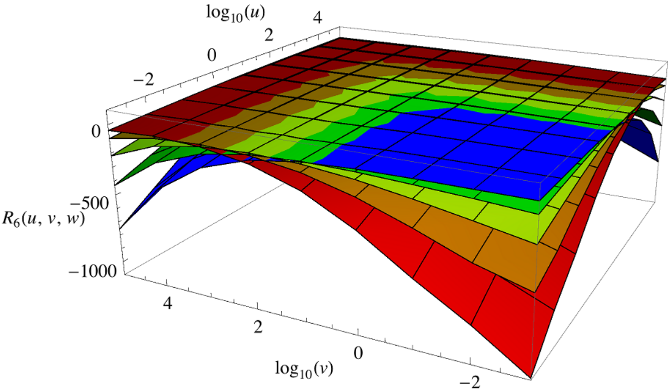
- Checks of conformal invariance of the Remainder (previously done by DHKS/BDKSVV):

(u_1, u_2, u_3)	$\mathcal{R}_6^{\mathrm{WL}}(A)$	$\mathcal{R}_6^{\mathrm{WL}}(B)$	$\mathcal{R}_6^{\mathrm{WL}}(C)$
$(1/9, 1/9, 1/9)$	5.18056	5.18096	5.18102
$(1/4, 1/4, 1/4)$	1.08916	1.08916	1.08919
$(1, 1, 1)$	-2.70814	-2.7066	-2.70657
$(100, 100, 100)$	-2.09134	-2.09204	-2.09228

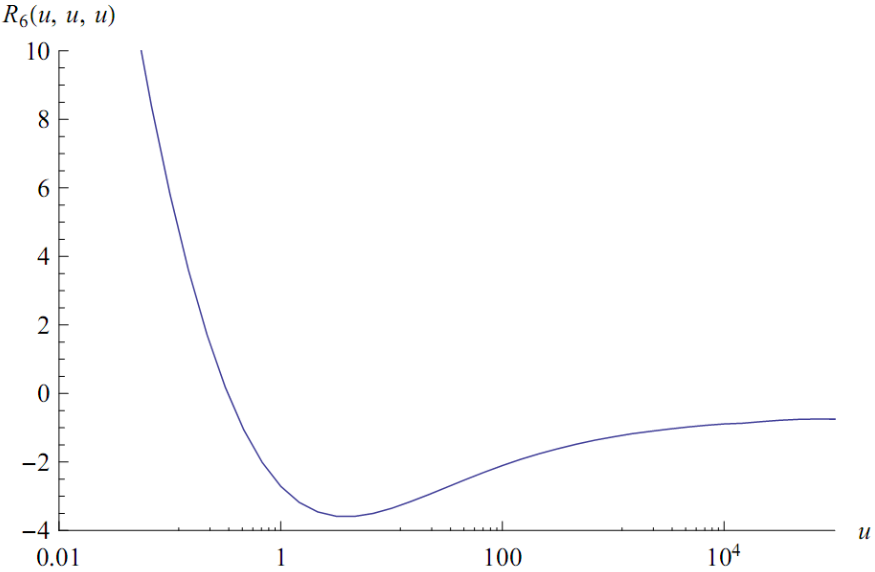
(A), (B), (C) are three different
but conformally equivalent kinematics.

6-pnt Wilson loop

- $\mathcal{R}_6^W$ with $u_1 = u,\ u_2 = v,\ u_3 = w$
- $w = 1$ blue, $w = 10$ green, $w = 100$ yellow, $w = 1000$ orange, $w = 10000$ red



Plot of $\mathcal{R}_6(u, u, u)$



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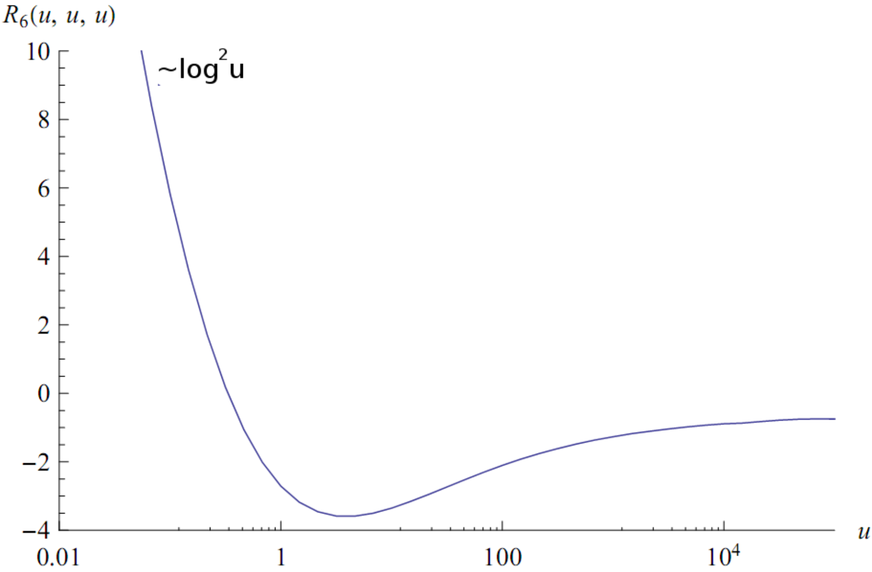
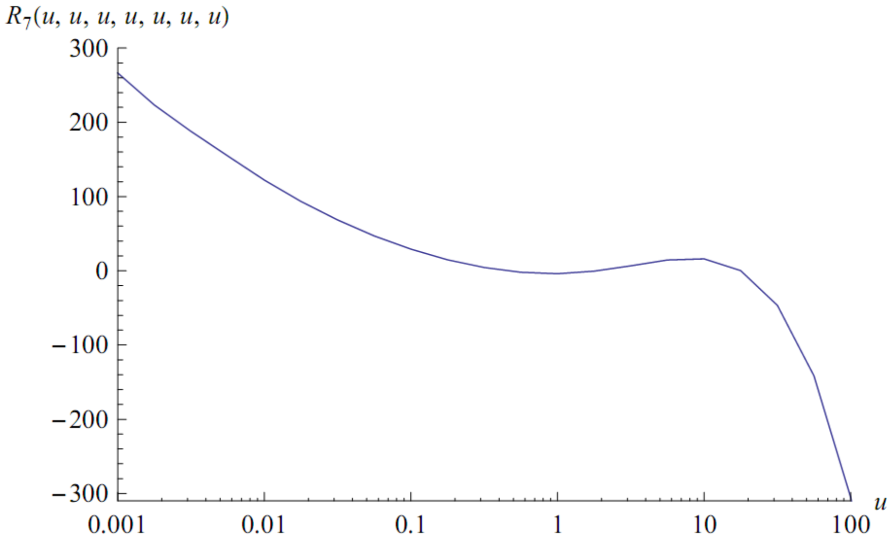


Seven-point results

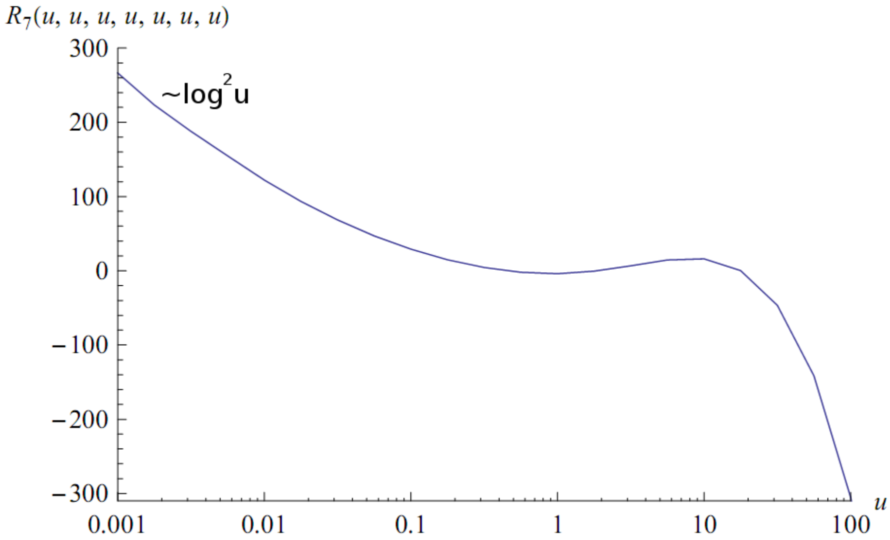
- Fourteen kinematic invariants in total.
- Seven conformal cross ratios. Conformal invariance:

$(u_{14}, u_{25}, u_{36}, u_{47}, u_{15}, u_{26}, u_{37})$	$\mathcal{R}_7^{\mathrm{WL}}(A)$	$\mathcal{R}_7^{\mathrm{WL}}(B)$
$(1, 1, 1, 1, 1, 1, 1)$	-3.85627	-3.85732
$(1/4, 1/4, 1/4, 1/4, 1/4, 1/4, 1/4)$	8.13063	8.13272
$(1/4, 1, 1, 1/4, 1, 1, 1)$	-4.40748	-4.40651
$(1, 1/4, 1, 1, 1/4, 1, 1)$	-4.40657	-4.40056
$(1, 1, 1/4, 1, 1, 1/4, 1)$	-4.40654	-4.40559
$(1, 1, 1, 1/4, 1, 1, 1/4)$	-4.40746	-4.40617
$(1, 1/2, 1, 1, 1, 1/4, 1)$	-4.27219	-4.27108
$(1, 1/4, 1, 1, 1, 1/2, 1)$	-4.27224	-4.27049
$(1/4, 1, 1/4, 1, 1, 1, 1)$	-4.63668	-4.63696

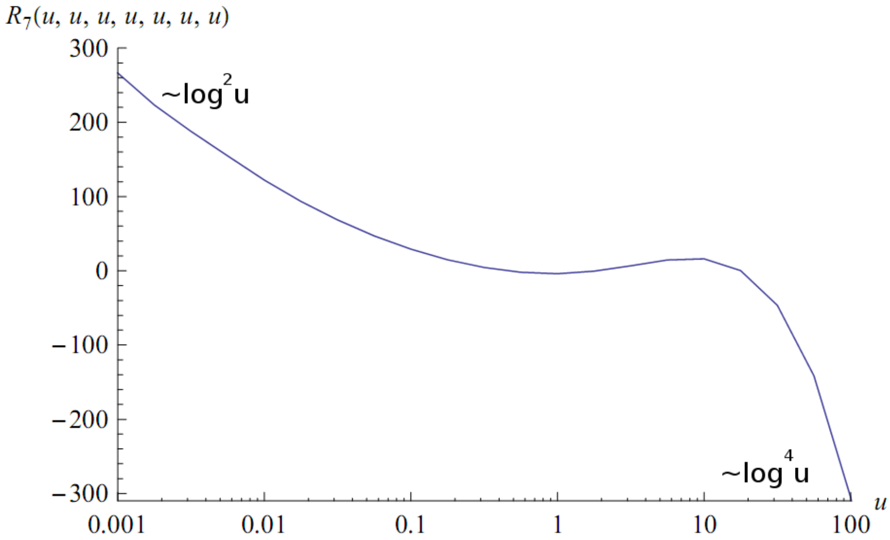
Plot of $\mathcal{R}_7(u, u, u, u, u, u, u)$



Plot of $\mathcal{R}_7(u, u, u, u, u, u, u)$



Plot of $\mathcal{R}_7(u,u,u,u,u,u,u)$



Collinear limits

- $p_{n-1} \to zP$, $p_n \to (1-z)P$

- unmodified ABDK/BDS conjecture for the amplitude has the correct simple collinear limits
- Therefore $\mathcal{R}_n(u)$ must have trivial simple collinear limits

$$\mathcal{R}_n \to \mathcal{R}_{n-1}$$

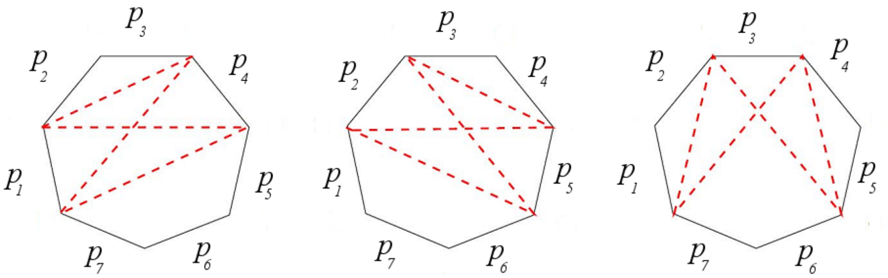
- We verify this for $n = 6, 7, 8$ (with no constant shifts)
- Makes predictions for DHKS limits

$$\mathcal{R}_n^{WL} = \mathcal{R}_n^{DHKS} - n\pi^4/48 \qquad \mathcal{R}_6^{DHKS} \to \pi^4/8 = 12.1761..$$

In more detail for 7-points...

- Simple collinear limit

$$p_6 = x_6 - x_7 = zP \ , \qquad p_7 = x_7 - x_1 = (1-z)P$$

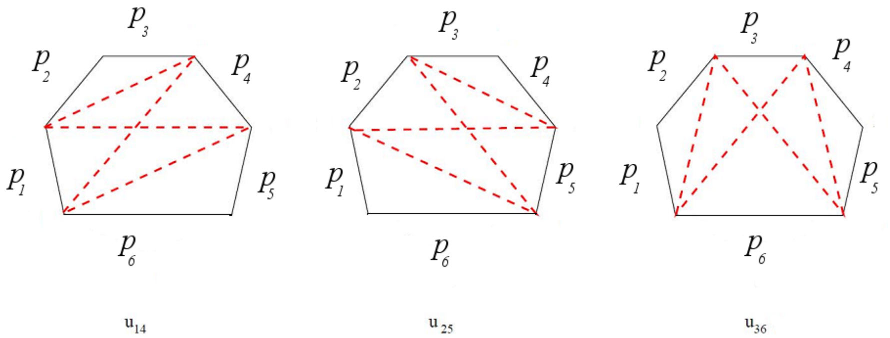


u_{14} $\qquad$ u_{25} $\qquad$ $u_{36}\ u_{37}$

$$u_{14}^{(7)} \longrightarrow u_{14}^{(6)} = u_2 \ , \qquad u_{25}^{(7)} \longrightarrow u_{25}^{(6)} = u_3 \ , \qquad u_{36}^{(7)}\ u_{37}^{(7)} \longrightarrow u_{36}^{(6)} = u_1$$

- Simple collinear limit

$$p_6 = x_6 - x_7 = zP \ , \qquad p_7 = x_7 - x_1 = (1-z)P$$



$$u_{14}^{(7)} \longrightarrow u_{14}^{(6)} = u_2 \ , \qquad u_{25}^{(7)} \longrightarrow u_{25}^{(6)} = u_3 \ , \qquad u_{36}^{(7)}\, u_{37}^{(7)} \longrightarrow u_{36}^{(6)} = u_1$$

- Simple collinear limit

$$u_{14} = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}\,, \quad u_{25} = \frac{x_{26}^2 x_{35}^2}{x_{25}^2 x_{36}^2}\,,$$

$$u_{36} = \frac{x_{46}^2}{x_{36}^2}\frac{z x_{13}^2 + (1-z) x_{36}^2}{z x_{14}^2 + (1-z) x_{46}^2}\,, \quad u_{47} = \frac{z x_{14}^2}{z x_{14}^2 + (1-z) x_{46}^2}\,, \quad u_{15} = 0$$

$$u_{26} = \frac{(1-z) x_{36}^2}{z x_{13}^2 + (1-z) x_{36}^2}\,, \quad u_{37} = \frac{x_{13}^2}{x_{14}^2}\frac{z x_{14}^2 + (1-z) x_{46}^2}{z x_{13}^2 + (1-z) x_{36}^2}\,.$$

$$\boxed{\begin{aligned} &\to \mathcal{R}_7^{\mathrm{WL}}\left(u_{14}, u_{25}, u_{36}, \frac{1-u_{36}}{1-u_{37}u_{36}}, 0, \frac{1-u_{37}}{1-u_{37}u_{36}}, u_{37}\right) \\ &= \mathcal{R}_6^{\mathrm{WL}}(u_{37}u_{36}, u_{14}, u_{25})\,. \end{aligned}}$$

- Triple collinear limit

$$\mathcal{R}_n \rightarrow \mathcal{R}_{n-2} + \mathcal{R}_6(\bar{u}_1, \bar{u}_2, \bar{u}_3)$$

[Bern Dixon Kosower Roiban Spradlin Vergu Volovich]

$$p_5 = x_5 - x_6 = z_1 P\,, \quad p_6 = x_6 - x_7 = z_2 P\,, \quad p_7 = x_7 - x_1 = z_3 P$$

$$\bar{u}_1 = \frac{1}{1-z_3}\frac{x_{57}^2}{x_{15}^2}\,, \quad \bar{u}_2 = \frac{1}{1-z_1}\frac{x_{16}^2}{x_{15}^2}\,, \quad \bar{u}_3 = \frac{z_1 z_3}{(1-z_1)(1-z_3)}$$

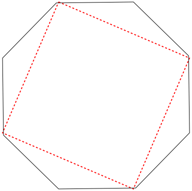
- We get

$$\boxed{\mathcal{R}_7^{\mathrm{WL}}\left(0, \frac{1-\sqrt{\bar{u}_3\kappa}}{1-\bar{u}_3}, \sqrt{\bar{u}_3\kappa}, \bar{u}_1, \bar{u}_2, \sqrt{\bar{u}_3/\kappa}, \frac{1-\sqrt{\bar{u}_3/\kappa}}{1-\bar{u}_3}\right) = \mathcal{R}_6^{\mathrm{WL}}(\bar{u}_1, \bar{u}_2, \bar{u}_3)}$$

- Exactly the same constraint as the single collinear limit

Eight points

- 20 kinematic invariants:
- 12 cross-ratios: u_{ii+3} $\quad i = 1..8 \qquad u_{ii+4} \quad i = 1..4$
- Unlike 6,7 points we can not write
  kinematic invariants = two-particle invariants + cross-ratios
- cross-ratios depending only on two-particle invariants



$$\frac{x_{13}^2 x_{57}^2}{x_{35}^2 x_{71}^2}$$

- Instead simply choose 8 simple- and multi-particle invariants independent of the u's to fix

$$(m_1, m_2, \ldots, m_8) = x_{i+5\; i+8}^2,\; x_{i\; i+4}^2 \;, \qquad i = 1, \ldots, 4 \;,$$

- Check conformal invariance: $u_{ij} = 1$

$(m_1, \ldots, m_8)$	$\mathcal{R}_8^{\mathrm{WL}}$
$(-1, -1, -1, -1, -1, -1, -1, -1)$	-4.603
$(-2, -2, -2, -2, -2, -2, -2, -2)$	-4.602
$(-1, -2, -4, -8, -1, -2, -4, -8)$	-4.605
$(-5, -3, -5, -3, -1, -3, -5, -7)$	-4.605

- Check conformal invariance:
  $u = (2, 3, 4, 1/2, 1/3, 1/4, 1/5, 1, 1/5, 1/6, 1/7, 1/8)$

$(m_1, \ldots, m_8)$	$\mathcal{R}_8^{\mathrm{WL}}$
$(-2, -3, -4, -1, -5, -6, -7, -8)$	5.993
$(-1/3, -1/4, -1/9, -1/2, -1/8, -1/7, -1/6, -1)$	5.984

Also checked collinear limit at 8 points

$$p_7 = x_7 - x_8 = zP \ , \qquad p_8 = x_8 - x_1 = (1-z)P$$

We find

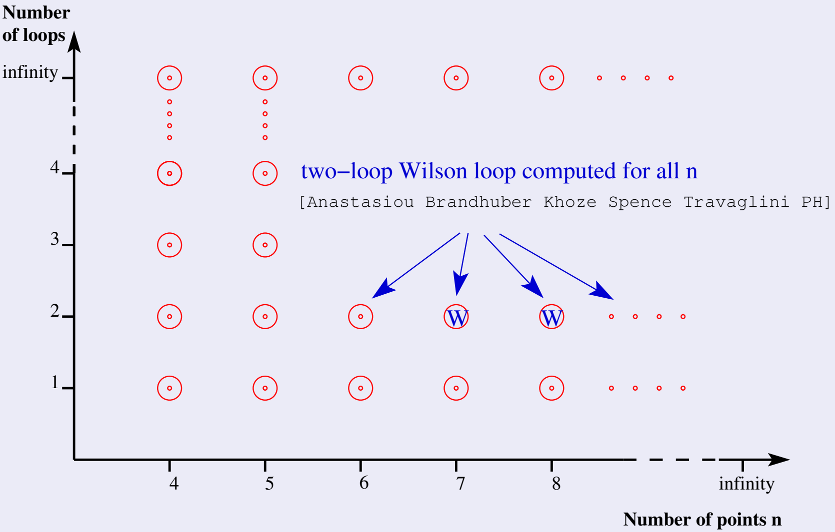
$$\mathcal{R}_8^{\mathrm{WL}}(u_{14}, u_{25}, u_{36}, u_{47}, u_{58}, u_{16}, u_{27}, u_{38}, u_{15}, u_{26}, u_{37}, u_{48}) \to$$
$$\mathcal{R}_8^{\mathrm{WL}}\left(u_{14}, u_{25}, u_{36}, u_{47}, u_{58}^*, 0, u_{27}^*, u_{38}, u_{15}, u_{26}, u_{37}, u_{48}^*\right)$$
$$= \mathcal{R}_7^{\mathrm{WL}}(u_{14}, u_{25}, u_{36}, u_{47}u_{48}^*, u_{15}, u_{26}, u_{37}u_{38}) \ ,$$

$$u_{27} = \frac{-1 + u_{38}}{-1 + u_{37}u_{38}},$$
$$u_{58} = \frac{-1 + u_{37}u_{38} + u_{37}u_{47} - u_{37}u_{38}u_{47}}{-1 + u_{37}u_{38}},$$
$$u_{48} = \frac{-1 + u_{37}}{-1 + u_{37}u_{38} + u_{37}u_{47} - u_{37}u_{38}u_{47}}.$$

$$\mathcal{R}_8^{\mathrm{WL}} \to \mathcal{R}_7^{\mathrm{WL}}$$

Summary of WL results

- Summary: the number of distinct integrals one needs to evaluate the 2-loop n-gon WL is independent of n
- We can compute all n-sided polygonal light-like Wilson loops at two loops
- Assuming the amplitude/Wilson loop duality continues to hold, we can compute two-loop planar MHV amplitudes for any number of points
- We have computed these for 6,7,8 points and performed some non-trivial tests (eg colinear limits and dual conformal invariance.)


Number
of loops
infinity
4
3
2
1
two–loop Wilson loop computed for all n
[Anastasiou Brandhuber Khoze Spence Travaglini PH]
W
W
4
5
6
7
8
infinity
Number of points n

Outline

1. The duality
   - The evidence so far...
   - Wilson loop calculations - 1 loop
   - Wilson loop calculations - 2 loop
2. Results of two-loop computations
   - 6 points
   - 7 points
   - 8 points
3. Dual superconformal symmetry of the entire S-matrix
   - at tree level
   - at one loop

Dual superconformal symmetry of the entire S-matrix:

[Drummond Henn Korchemsky Sokatchev]

- So far we have considered MHV amplitudes only
- What about other helicities, other particles?
- What about the MHV tree-level amplitude?[Drummond Henn Korchemsky Sokatchev 2008]
- Superconformal transformations?

Conjecture [Drummond Henn Korchemsky Sokatchev]

$\mathcal{A}_n = \mathcal{A}_{n,\mathrm{MHV}}\,\mathcal{R}$ $\qquad$ $\mathcal{R}$ is dual superconformally invariant

Dual symmetry at tree-level

[Brandhuber Travaglini PH]

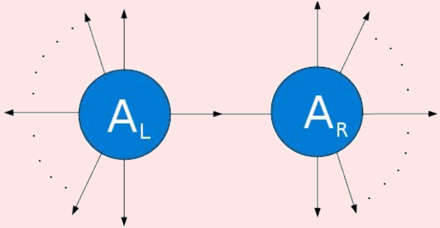
- Use the BCFW recursion relation [Britto Cachazo Feng Witten 2005]
- Superspace version:

  [Bianchi Elvang Freedman, Arkani-Hamed Cachazo Kaplan, Brandhuber Travaglini PH]

dual superconformal covariance at tree level

Proof by induction using superBCFW[Brandhuber Travaglini PH]

$$\mathcal{A} = \sum$$



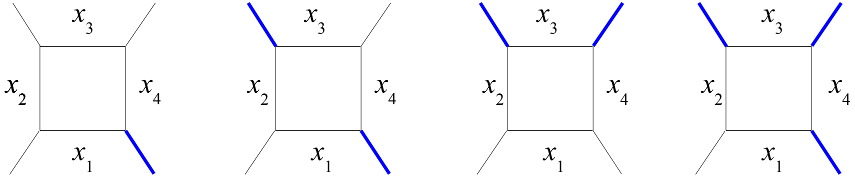
- each individual BCFW diagram is separately covariant
- Solution to the recursion relation [Drummond Henn]

Dual superconformal symmetry at one loop

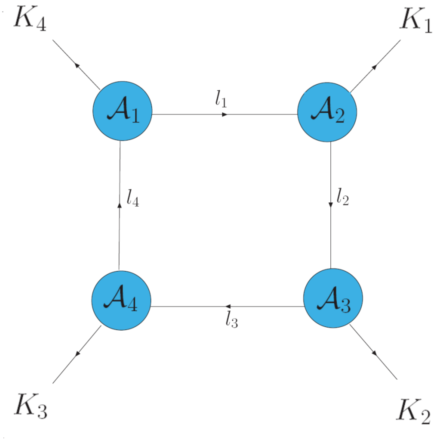
- What can we say at one loop?
- All one loop amplitudes written in terms of boxes [Bern Dixon Dunbar Kosower 1994]

$$\mathcal{A}^{1-\text{loop}} = \sum_{\{i,j,k,l\}} c(i,j,k,l)\, F(i,j,k,l)$$

- $F(1,2,3,4) = \int \frac{d^D x_5}{(2\pi)^D} \frac{\sqrt{R}}{x_{51}^2 x_{52}^2 x_{53}^2 x_{54}^2}$ where $\sqrt{R} \rightarrow x_{13}^2 x_{24}^2 - x_{23}^2 x_{41}^2$



- The box coefficients $c(i, j, k, l)$ can be calculated from tree-level amplitudes using quadruple cuts [Britto Cachazo Feng 2005]



Supercoefficients are dual superconformal covariant
[Brandhuber Travaglini PH, Drummond Henn Korchemsky Sokatchev]

New constraints on box supercoefficients

To appear soon [Brandhuber Travaglini PH]

- So far we have considered tree amplitudes and one loop box *supercoefficients* which are IR finite
- Now we use anomalous dual conformal transformation to find new constraints on box coefficients

Anomalous dual conformal transformation

[Drummond Henn Korchemsky Sokatchev 2008]

$$K^{\mu} \mathcal{A}_n^{1-\text{loop}} = -2\epsilon \, \mathcal{A}_n^{\text{tree}} \sum_{i=1}^{n} x_{i+1}^{\mu} x_{i\,i+2}^2 \, J(x_{i\,i+2}^2)$$

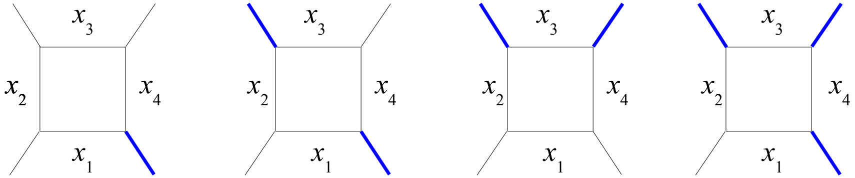
$$J(x^2) := \frac{1}{\epsilon^2}(-x^2)^{-\epsilon-1} \qquad J(x^2, y^2) := \frac{1}{\epsilon^2} \frac{(-x^2)^{-\epsilon} - (-y^2)^{-\epsilon}}{(-x^2) - (-y^2)}$$

are 1-mass and 2-mass triangles.

Approach: perform dual conformal transformation on generic box:

$$K^\mu F \sim 4\epsilon \int \frac{d^D x_5}{(2\pi)^D} \frac{\sqrt{R}}{x_{51}^2 x_{52}^2 x_{53}^2 x_{54}^2} x_5^\mu$$

- Evaluate RHS via PV reduction



Dual conformal transformation of boxes

$$
\begin{aligned}
\frac{1}{4}K^\mu F^{0\mathrm{m}} &= -\epsilon\left[(x_1+x_3)^\mu x_{24}^2 J(x_{24}^2) + (x_2+x_4)^\mu x_{13}^2 J(x_{13}^2)\right] , \\
\frac{1}{4}K^\mu F_1^{1\mathrm{m}} &= \epsilon\Big\{-x_1^\mu x_{24}^2 J(x_{24}^2, x_{41}^2) + x_2^\mu\Big[x_{41}^2 J(x_{24}^2, x_{41}^2) - x_{13}^2 J(x_{13}^2)\Big] \\
&+ x_3^\mu\Big[x_{41}^2 J(x_{13}^2, x_{41}^2) - x_{24}^2 J(x_{24}^2)\Big] - x_4^\mu x_{13}^2 J(x_{13}^2, x_{41}^2)\Big\} \\
\frac{1}{4}K^\mu F_{13}^{2\mathrm{me}} &= \epsilon\Big\{x_1^\mu\Big[-x_{24}^2 J(x_{24}^2, x_{41}^2) + x_{23}^2 J(x_{13}^2, x_{23}^2)\Big] + x_2^\mu\Big[-x_{13}^2 J(x_{13}^2, x_{23}^2) + x_{41}^2 J(x_{24}^2, x_{41}^2)\Big] \\
&+ x_3^\mu\Big[-x_{24}^2 J(x_{24}^2, x_{23}^2) + x_{41}^2 J(x_{13}^2, x_{41}^2)\Big] + x_4^\mu\Big[-x_{13}^2 J(x_{13}^2, x_{41}^2) + x_{23}^2 J(x_{24}^2, x_{23}^2)\Big]\Big\} , \\
\frac{1}{4}K^\mu F_{14}^{2\mathrm{mh}} &= \epsilon\Big\{-x_1^\mu x_{24}^2 J(x_{24}^2, x_{41}^2) \\
&+ x_2^\mu\Big[x_{41}^2 J(x_{24}^2, x_{41}^2) - x_{13}^2 J(x_{13}^2) + x_{34}^2 J(x_{24}^2, x_{34}^2)\Big] \\
&- x_3^\mu x_{24}^2 J(x_{24}^2, x_{34}^2)\Big]\Big\} , \\
\frac{1}{4}K^\mu F_{134}^{3\mathrm{m}} &= \epsilon\Big\{x_1^\mu\Big[x_{23}^2 J(x_{23}^2, x_{13}^2) - x_{24}^2 J(x_{24}^2, x_{41}^2)\Big] + x_2^\mu\Big[x_{41}^2 J(x_{41}^2, x_{24}^2) - x_{13}^2 J(x_{13}^2, x_{23}^2)\Big]\Big\} .
\end{aligned}
$$

transformation of 1 loop amplitude

$$K^\mu \mathcal{A}_n^{\text{1-loop}} = \sum_{\{i,j,k,l\}} c(i,j,k,l)\, K^\mu F(i,j,k,l)$$

$$= \epsilon \sum_{i,k} \mathcal{E}(i,k) \times \left[x_{i-1}^\mu\, x_{ik}^2\, J(x_{ik}^2, x_{i-1\,k}^2) - x_i^\mu\, x_{i-1\,k}^2\, J(x_{ik}^2, x_{i-1\,k}^2) \right]$$

$$= -2\epsilon\, \mathcal{A}_n^{\text{tree}} \sum_{i=1}^{n} x_{i+1}^\mu x_{i\,i+2}^2\, J(x_{i\,i+2}^2)$$

- the combinations of two-mass and one-mass triangles appearing in the middle line are linearly independent for different i, k $\Rightarrow$

box coefficient constraints

$$\mathcal{E}(i,i-2) \equiv -\mathcal{E}(i-1,i) = -2\,\mathcal{A}_n^{\text{tree}}\,, \qquad i = 1\ldots n\,,$$

$$\mathcal{E}(i,k) = 0\,, \qquad i = 1\ldots n\,,\quad k = i+2,\ldots,i-3\,.$$

$$\mathcal{E}(i,k) := \sum_{j=k+1}^{i+n-2} c(i,k,j,i-1) - \sum_{j=i+1}^{k-1} c(i,j,k,i-1)\,,$$

- these give $n(n-4)$ independent constraints on the box coefficients
- they determine the 1-mass, 2-mass easy and half of the 2-mass hard coefficients (in terms of the other 2mh and the 3m coefficients).

Compare with IR consistency

[Giele Glover 1992, Kunszt Signer Trócsányi 1994]

$$\mathcal{A}_n^{1-\text{loop}}\Big|_{\text{IR}} = -\mathcal{A}_n^{\text{tree}} \sum_{i=1}^{n} \frac{(-x_{ii+2}^2)^{-\epsilon}}{\epsilon^2} ,$$

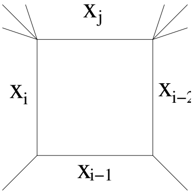
- equate the coefficients of the two-particle and multi-particle infrared divergent terms

$$\begin{aligned}(-x_{ii+2}^2)^{-\epsilon}/\epsilon^2 : &\quad \mathcal{E}(i, i+2) + \mathcal{E}(i+2, i) - \mathcal{E}(i+3, i) = -2\mathcal{A}_{\text{tree}} , \\ (-x_{ik}^2)^{-\epsilon}/\epsilon^2 : &\quad \mathcal{E}(i, k) + \mathcal{E}(k, i) - \mathcal{E}(i+1, k) - \mathcal{E}(k+1, i) = 0\end{aligned}$$

- $n(n-3)/2$ equations: determine the 1m and 2me boxes in terms of the rest[Bern Dixon Kosower 2004]
- conformal equations: also gives half of the 2mh coefficients
- conformal equations are simpler

- combination of infrared equations [Roiban Spradlin Volovich 2005]

$$\sum_{j=i+1}^{i+n-3}$$



$$= 2\,\mathcal{A}_n^{\text{tree}}$$

- appears in the BCF context
- somewhat complicated to prove from IR equations[Arkani-Hamed Cachazo Kaplan]
- naturally appears as simply

$$\mathcal{E}(i, i-2) = -2\mathcal{A}_n^{\text{tree}}$$

NMHV dual conformal invariance

- BDK computed the NMHV one-loop n-point gluon amplitudes[Bern Dixon Kosower 2004]
- DHKS found all NMHV n-point amplitudes as manifesty supersymmetric superamplitudes [Drummond Henn Korchemsky Sokatchev 2008]
- DHKS proved dual conformal invariance of these explicitly for $n = 6, 7$ (and also checked it for $n = 8, 9$)
- We extend these results and prove dual conformal invariance of the NMHV superamplitude for all n

Coefficients in terms of 'R'

[Bern Dixon Kosower 2004, Drummond Henn Korchemsky Sokatchev 2008]

- Coefficients can all be written in terms of dual conformal covariant objects R

$$c^{3\mathrm{m}}(r,s,t) = R_{rst}$$

$$c^{2\mathrm{mh}}(r,t) = R_{r,r+2,t} + R_{r+1,t,r}$$

$$c^{2\mathrm{me}}(r,s) = \sum_{u,v=s+1}^{r-1} R_{r,u,v} + \sum_{u,v=r+1}^{s-1} R_{s,u,v}$$

$$\begin{aligned} c^{1\mathrm{m}}(r-2) &= c^{2\mathrm{me}}(r,r-2) + R_{r-1,r+1,r-2} \\ &= \sum_{u,v=r+1}^{r+n-3} R_{r-2,u,v} + R_{r-1,r+1,r-2} \end{aligned}$$

  where R_{ruv} is only defined for $u \geq r+2$, $v \geq u+2$
- these satisfy $R_{r+2,s,r+1} = R_{r,r+2,s}$

- the RSV combination of IR equations

$$\mathcal{E}(i, i-2) \equiv -\sum_{u,v=i}^{i-3} R_{i-2uv} - \sum_{u,v=i-1}^{i-4} R_{i-3uv} = 2\mathcal{A}_{\text{tree}}$$

from this, for n odd we immediately get

$$\sum_{u,v=3}^{n} R_{1uv} = \sum_{u,v=2}^{n-1} R_{nuv} \; ,$$

(n even, use collinear limit arguments)

- important identity leading to the NMHV superamplitude (previously conjectured) [Drummond Henn Korchemsky Sokatchev 2008]

- This equation does not involve momentum conservation, therefore it is true independent of the number of points
- using cyclicity leads to

new equations

$$\sum_{u,v=r+2}^{s} R_{ruv} = \sum_{u,v=r+1}^{s-1} R_{suv} \qquad r+5 \leq s \leq r+n-1 \, .$$

- stronger than equations coming from the equality of two inequivalent representations of the 2me coefficient
- weaker equations used to prove dual superconformality for $n \leq 9$[Drummond Henn Korchemsky Sokatchev 2008]

stronger equation $\Rightarrow$ NMHV superamplitudes are dual superconformal for all n

Summary of superconformal section

- Tree-level dual conformal invariance proved
- supercoefficients at one loop: dual conformal invariance proved
- NMHV one loop dual conformal invariance at one loop proved
- Additional restrictions on one loop coefficients from assuming dual conformal invariance

Future directions

- amplitude calculation at $n \geq 7$-points needed![Vergu]
- analytic determination of 6-pnt amplitude/Wilson loop
- More direct/complete proof of dual superconformal invariance (eg MHV diagrams)
- Proof of WL/amplitude duality
- Generalisations of WL to NMHV amplitudes etc. [Berkovitz Maldacena]
- Generalisations to other theories (QCD)
- Understanding the role of infinite new symmetries (from integrability) [Beisert Ricci Tseytlin Wolf, Berkovitz Maldacena, Drummond Henn Plefka]