International workshop on gauge and string amplitudes

IPPP, Durham

March 31, 2009

Single cuts from double cuts

Ruth Britto

CEA/Saclay

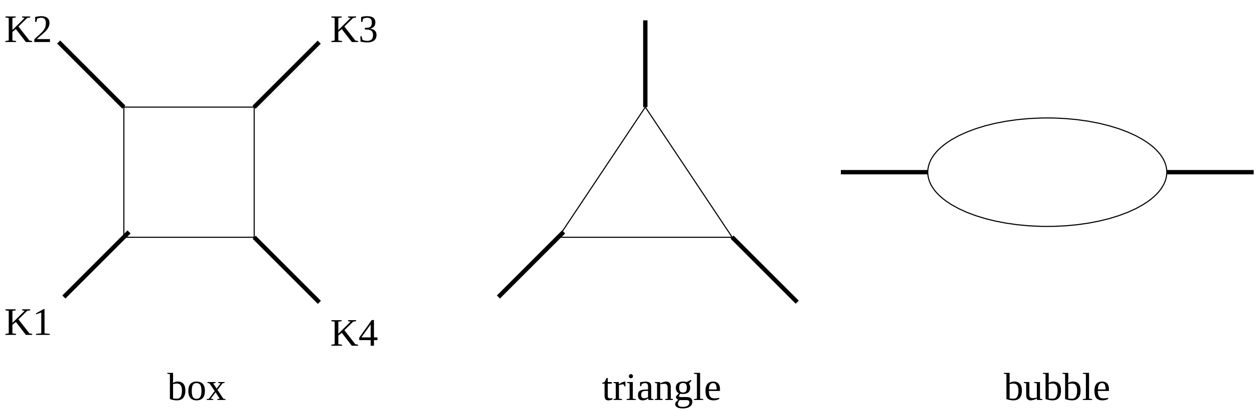
with B. Feng, to appear

One-Loop Amplitudes

In 4 dimensional massless theories, Passarino-Veltman reduction brings the one-loop amplitude to the form

$$A = \sum_i d_i \,(\mathrm{box}) + \sum_i c_i \,(\mathrm{triangle}) + \sum_i b_i \,(\mathrm{bubble}) + \mathrm{rational}$$

where expressions for scalar bubble, scalar triangle and scalar box integrals are known explicitly. (in dim. reg.: Bern, Dixon, Kosower)



One-Loop Amplitudes

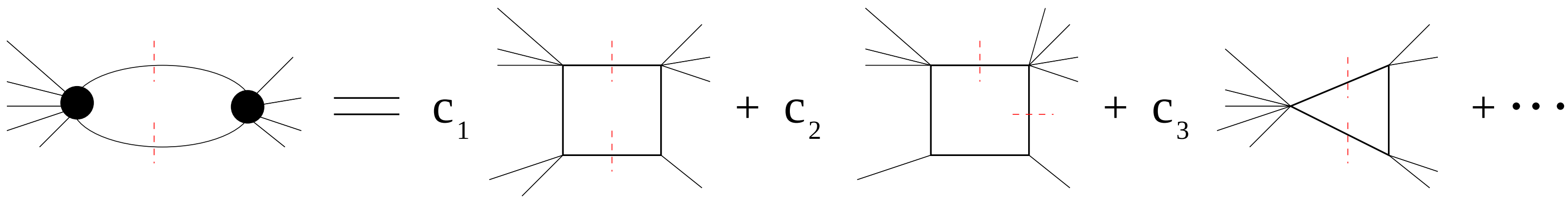
In $D = 4 - 2\epsilon$ dimensions, and allowing for internal masses, the result of reduction is

$$\begin{aligned} A \quad = \quad & \sum_i e_i \text{ (pentagon)} + \sum_i d_i \text{ (box)} + \sum_i c_i \text{ (triangle)} \\ & + \sum_i b_i \text{ (bubble)} + \sum_i a_i \text{ (tadpole)} \end{aligned}$$

Amplitudes from unitarity cuts

$$C = \Delta A^{1-\text{loop}} = \sum c_i \, \Delta I_i$$

Tree level input.



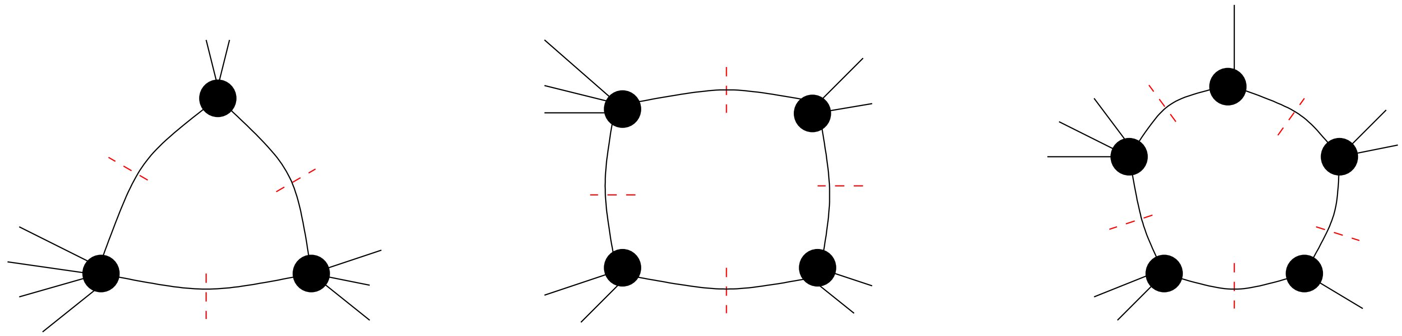
Matching cuts can suffice to determine reduction coefficients! Logarithms with unique arguments. "CUT-CONSTRUCTIBILITY"

(Bern, Dixon, Dunbar, Kosower 1994)

But: we still get several coefficients together in the same equation.

Recent techniques separate coefficients systematically.

One tool: Generalized Unitarity

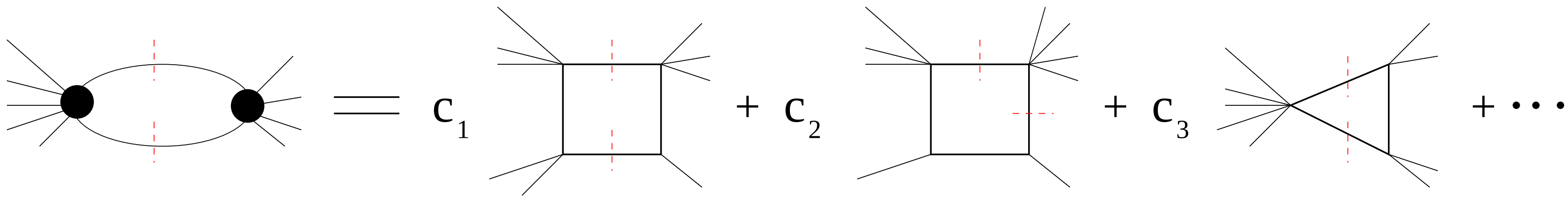


Leading singularities isolate some integrals, then look at subleading singularities, etc.

Work in 4 or D dimensions.

Bern, Dixon, Dunbar, Kosower; RB, Cachazo, Feng; Ossola, Papadopoulos, Pittau; Mastrolia; Forde; Kilgore; Ellis, Giele, Kunszt, Melnikov; Badger

“Ordinary” Unitarity Cuts



Alternatively, we can use ordinary D-dimensional cuts with analytic properties to isolate coefficients. E.g. “spinor integration.”

Formulas given for pentagon, box, triangle, bubble coefficients...

Anastasiou, RB, Cachazo, Feng, Kunszt, Mastrolia, Yang

But: tadpoles drop out of ordinary cuts! They only survive single cuts.

I will show a way to still use double cuts for tadpole coefficients.

OPP algorithm

(Ossola, Papadopoulos, Pittau 2006)

Decompose at the integrand level.

$$A^{1-\text{loop}} = \int d^{4-2\epsilon}q\, I(q)$$

$$I(q) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

$$\bar{D}_i = (q + p_0 + K_i)^2 - M_i^2 - \mu^2$$

where $\mu^2 = -\widetilde{q}^2$.

Expand $I(q)$ in terms of the master integrands,

$$I^{(i)} = \frac{1}{\bar{D}_i}, \qquad I^{(i,j)} = \frac{1}{\bar{D}_i \bar{D}_j}, \qquad I^{(i,j,r)} = \frac{1}{\bar{D}_i \bar{D}_j \bar{D}_r}, \qquad \ldots$$

along with “spurious” terms.

$$
\begin{aligned}
I = \frac{N}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \quad = \quad & \sum_i [a(i) + \widetilde{a}(q;i)] I^{(i)} + \sum_{i<j} [b(i,j) + \widetilde{b}(q;i,j)] I^{(i,j)} \\
& + \sum_{i<j<r} [c(i,j,r) + \widetilde{c}(q;i,j,r)] I^{(i,j,r)} \\
& + \sum_{i<j<r<s} [d(i,j,r,s) + \widetilde{d}(q;i,j,r,s)] I^{(i,j,r,s)} \\
& + \sum_{i<j<r<s<t} e(i,j,r,s,t) I^{(i,j,r,s,t)}
\end{aligned}
$$

The spurious terms are polynomials, given explicitly, and are defined so that

$$
\int d^4q \ \widetilde{a}(q;i) I^{(i)} = 0, \qquad \int d^4q \ \widetilde{b}(q;i,j) I^{(i,j)} = 0, \qquad \ldots
$$

The OPP algorithm (in 4d) is to solve $\bar{D}_i(q) = 0$ numerically, multiply through, and substitute the solutions...

We will use this expansion differently.

Example: 4d box coefficients

$$I = \frac{N}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} = \sum_i [a(i) + \widetilde{a}(q;i)] I^{(i)} + \sum_{i<j} [b(i,j) + \widetilde{b}(q;i,j)] I^{(i,j)}$$
$$+ \sum_{i<j<r} [c(i,j,r) + \widetilde{c}(q;i,j,r)] I^{(i,j,r)}$$
$$+ \sum_{i<j<r<s} [d(i,j,r,s) + \widetilde{d}(q;i,j,r,s)] I^{(i,j,r,s)}$$

Multiply through by $\bar{D}_0 \bar{D}_1 \bar{D}_2 \bar{D}_3$, then plug in $q^{\pm}_{0123}$, the two solutions to

$$0 = \bar{D}_0(q) = \bar{D}_1(q) = \bar{D}_2(q) = \bar{D}_3(q).$$

$$I(q^{\pm}_{0123}) = d(0,1,2,3) + \widetilde{d}(0,1,2,3)$$
$$\widetilde{d}(i,j,r,s) = \epsilon(q, K_1, K_2, K_3)$$

cf. Cachazo's "geometric basis."

Tadpole coefficients

Recall:

$$\bar{D}_i = (q + p_0 + K_i)^2 - M_i^2 - \mu^2$$

Introduce an auxiliary denominator factor D_K:

$$\bar{D}_K = (q + p_0 + K)^2 - M_K^2 - \mu^2$$

For now, K and M_K^2 are just variables.

Plan: Unitarity cut of the propagators D_0 and D_K.

Then, decouple effects of D_K. Result resembles single cut.

OPP expansion with auxiliary factor D_K

$$
\begin{aligned}
I_K^{(1)} &= \frac{I}{\bar{D}_K} \\
&= \sum_i [a(i) + \widetilde{a}(q;i)] I^{(K,i)} + \sum_{i<j} [b(i,j) + \widetilde{b}(q;i,j)] I^{(K,i,j)} \\
&+ \sum_{i<j<r} [c(i,j,r) + \widetilde{c}(q;i,j,r)] I^{(K,i,j,r)} \\
&+ \sum_{i<j<r<s} [d(i,j,r,s) + \widetilde{d}(q;i,j,r,s)] I^{(K,i,j,r,s)} \\
&+ \sum_{i<j<r<s<t} e(i,j,r,s,t) I^{(K,i,j,r,s,t)}
\end{aligned}
$$

OPP expansion with auxiliary factor D_K

$$\begin{aligned}
I_K^{(2)} \quad = \quad & \sum_i [a_K(i) + \widetilde{a}_K(q;i)] I^{(i)} + \sum_{i<j} [b_K(i,j) + \widetilde{b}_K(q;i,j)] I^{(i,j)} \\
& + \sum_{i<j<r} [c_K(i,j,r) + \widetilde{c}_K(q;i,j,r)] I^{(i,j,r)} + \cdots \\
& + [a_K(K) + \widetilde{a}_K(q;K)] I^{(K)} + \sum_j [b_K(K,j) + \widetilde{b}_K(q;K,j)] I^{(K,j)} \\
& + \sum_{j<r} [c_K(K,j,r) + \widetilde{c}_K(q;K,j,r)] I^{(K,j,r)} + \cdots
\end{aligned}$$

Integrate in a specific cut channel (D_0 and D_K)

$$I_K^{(1)} = \frac{I}{\bar{D}_K} = \sum_i [a(i) + \widetilde{a}(q;i)] I^{(K,i)} + \sum_{i<j} [b(i,j) + \widetilde{b}(q;i,j)] I^{(K,i,j)} + \cdots$$

The "spurious" terms in $I_K^{(1)}$ will *not* drop out.

$$\begin{aligned} I_K^{(2)} = & \sum_i [a_K(i) + \widetilde{a}_K(q;i)] I^{(i)} + \sum_{i<j} [b_K(i,j) + \widetilde{b}_K(q;i,j)] I^{(i,j)} \\ & + \sum_{i<j<r} [c_K(i,j,r) + \widetilde{c}_K(q;i,j,r)] I^{(i,j,r)} + \cdots \\ & + [a_K(K) + \widetilde{a}_K(q;K)] I^{(K)} + \sum_j [b_K(K,j) + \widetilde{b}_K(q;K,j)] I^{(K,j)} \\ & + \sum_{j<r} [c_K(K,j,r) + \widetilde{c}_K(q;K,j,r)] I^{(K,j,r)} + \cdots \end{aligned}$$

The terms with tildes in $I_K^{(2)}$ are truly spurious.

After the cut integral (D_0 and D_K)

$$\int_{C_{0K}} I_K \quad = \quad b_K(K,0) \int_{C_{0K}} I^{(K,0)} + \sum_i c_K(K,0,i) \int_{C_{0K}} I^{(K,0,i)} + \cdots$$

The cuts of the master integrals are independent.

So

$$b_K(K,0) = a(0) + [\text{contributions from } \widetilde{a}(q;0),\ \widetilde{b}(q;0,i),\ \widetilde{c}(q;0,i,j),\ \widetilde{d}(q;0,i,j,r)]$$
$$c_K(K,0,i) = b(0,i) + [\text{contributions from } \widetilde{b}(q;0,i),\ \widetilde{c}(q;0,i,j),\ \widetilde{d}(q;0,i,j,r)]$$

We'd like to decouple the "spurious contributions" as much as possible.

Use double cuts to get $c_K(K,0,i),\ b(0,i),\ b_K(K,0)$. Then solve the equations for $a(0)$.

Conditions for decoupling (most) spurious terms

$$K \cdot K_i = 0, \qquad \forall i \tag{1}$$

$$M_K^2 = M_0^2 + K^2. \tag{2}$$

Condition (2) can be taken as a definition of M_K^2.

Condition (1) is treated formally.

Then there is only one spurious term that survives, and we can solve the equations for $a(0)$.

Procedure

1. Start from the single-cut expression $A^{\rm tree}_{1-cut}$ obtained by cutting $\bar{D}_0$. Set aside any explicit μ^2.

2. Construct the true integrand $I = A^{\rm tree}_{1-cut}/\bar{D}_0$ and the auxiliary integrand $I_K = A^{\rm tree}_{1-cut}/(\bar{D}_K \bar{D}_0)$.

3. Use $\int_{C_{0K}} I_K$ to evaluate the auxiliary bubble coefficient $b_K(K,0)$ and all the auxiliary triangle coefficients $c_K(K,0,i)$.

4. Use $\int_{C_{0K_i}} I$ to evaluate all the true bubble coefficients $b(0,i)$.

5. The tadpole coefficient is given by imposing the conditions (1) and (2) in the following expression.

$$a(0) = b_K(K,0) + \sum_i \frac{K_i^2 - M_i^2 + M_0^2}{4K_i^2} \left[c_K(K,0,i) - b(0,i) \right]\Big|_{\mu^2} .$$

Analysis of spurious terms

$$b_K(K,0) = a(0) + [\text{contributions from } \widetilde{a}(q;0),\ \widetilde{b}(q;0,i),\ \widetilde{c}(q;0,i,j),\ \widetilde{d}(q;0,i,j,r)]$$

OPP have given all the spurious terms explicitly.

We look at each of them in turn, extracting the auxiliary bubble coefficients.

One-point spurious terms, $\widetilde{a}(q;0)$

$$I = \frac{2\widetilde{\ell} \cdot R_1}{\bar{D}_0}$$

$$I_K = \frac{2\widetilde{\ell} \cdot R_1}{D_K D_0}$$

Auxiliary bubble coefficient:

$$C[D_0, D_K] = \frac{(K \cdot R_1)(K^2 + M_0^2 - M_K^2)}{K^2}$$

$R_1 = k, n, \ell_7, \ell_8$. Convenient to choose $k = K$.

Decoupling condition:

$$K^2 + M_0^2 - M_K^2 = 0.$$

Two-point spurious terms, $\widetilde{b}(q;0,i)$

$$I_{21} \quad = \quad \frac{(2\widetilde{\ell}\cdot R_1)}{D_0 D_i}$$

$$I_{21}^{D_K} \quad = \quad \frac{(2\widetilde{\ell}\cdot R_1)}{D_K D_0 D_i}.$$

Bubble coefficient:

$$C[D_0, D_K] \quad = \quad \frac{-(K\cdot K_i)(K\cdot R_1) + K^2(K_i\cdot R_1)}{(K\cdot K_i)^2 - K^2 K_i^2}.$$

$R_1 = \ell_7, \ell_8, n$. By construction, $K_i \cdot R_1 = 0$.

Decoupling condition:

$$K\cdot K_i \quad = \quad 0.$$

Two-point spurious terms, $\widetilde{b}(q;0,i)$, cont.

$$I_{22} = \frac{(2\widetilde{\ell}\cdot R_1)(2\widetilde{\ell}\cdot R_2)}{D_0 D_i}, \qquad I_{22}^{D_K} = \frac{(2\widetilde{\ell}\cdot R_1)(2\widetilde{\ell}\cdot R_2)}{D_0 D_K D_i}$$

Bubble coefficient:

$$C[D_0, D_K] = \frac{A(R_1\cdot R_2) + B_{11,12}(K_i\cdot R_1)(K_i\cdot R_2) + B_{21,22}(K\cdot R_1)(K\cdot R_2)}{2((K_i\cdot K)^2 - K_i^2 K^2)^2} + \frac{B_{11,22}(K_i\cdot R_1)(K\cdot R_2) + B_{12,21}(K_i\cdot R_2)(K\cdot R_1)}{2((K_i\cdot K)^2 - K_i^2 K^2)^2}$$

where

$$\alpha = \frac{K^2 + M_0^2 - M_K^2}{K^2}, \quad \alpha_1 = \frac{K_i^2 + M_0^2 - M_i^2}{K_i^2}$$

$$A = K^2((K_i\cdot K)^2 - K_i^2 K^2)(\alpha(K\cdot K_i) - \alpha_1 K_i^2)$$

$$B_{11,12} = 3K^2(\alpha(K\cdot K_i) - \alpha_1 K_i^2)$$

$$B_{21,22} = -2(K\cdot K_i)^2(\alpha(K\cdot K_i) + \alpha_1 K_i^2) - \alpha_1 K^2 (K_i^2)^2 + 5\alpha K^2 K_i^2 (K\cdot K_i)$$

$$B_{11,22} = B_{12,21} = -K^2(\alpha(K\cdot K_i)^2 + 2\alpha K^2 K_i^2 - 3\alpha_1 K_i^2(K\cdot K_i))$$

5 two-point spurious terms with quadratic numerators:

$R_1 = R_2 = \ell_7;$

$R_1 = R_2 = \ell_8;$

$R_1 = \ell_7,\ R_2 = n;$

$R_1 = \ell_8,\ R_2 = n;$

$(\widetilde{\ell} \cdot n)^2 - ((\widetilde{\ell} \cdot K_i)^2 - K_i^2 \widetilde{\ell}^2)/3.$

First 4 give zero under the decoupling conditions.

The 5th gives:

$$C_{\widetilde{b}_{00}} = \frac{K_i^2 + M_0^2 - M_i^2}{12}.$$

So we'll have to calculate its coefficient, $\widetilde{b}_{00}$.

It will come from $c_K(K, 0, i)$.

Three- and four-point spurious terms

Check case by case that all remaining spurious terms decouple under the same conditions.

Solve for $\widetilde{b}_{00}$

Apply the same decoupling conditions in $c_K(K, 0, i)$.

Recall:

$$\begin{aligned} b_K(K,0) &= a(0) + \sum_i \left(\frac{K_i^2 - M_i^2 + M_0^2}{12} \right) \widetilde{b}_{00}(0,i) \\ c_K(K,0,i) &= b(0,i) + [\text{contributions from } \widetilde{b}(q;0,i),\ \widetilde{c}(q;0,i,j),\ \widetilde{d}(q;0,i,j,r)] \end{aligned}$$

Now find the auxiliary triangle coefficients in spurious terms, looking for $\widetilde{b}_{00}$ specifically.

Two-point spurious terms

Only the same spurious term, $(\widetilde{\ell} \cdot n)^2 - ((\widetilde{\ell} \cdot K_i)^2 - K_i^2 \widetilde{\ell}^2)/3$, survives the decoupling conditions:

$$C[D_0, D_K, D_i]_{\widetilde{b}_{00}} = -\frac{(K_i^2 + M_0^2 - M_i^2)^2 - 4K_i^2(M_0^2 + \mu^2)}{12}.$$

Three-point spurious terms

Not so clean: get nonzero contribution from

$$I_{32} \quad = \quad \frac{(2\tilde{\ell}\cdot R)^2}{D_0 D_i D_j}, \qquad I_{32}^{D_K} = \frac{(2\tilde{\ell}\cdot R)^2}{D_0 D_K D_i D_j}.$$

After decoupling,

$$C[D_0, D_K, D_i] = -\frac{(K\cdot R)^2((K_i\cdot K_j)(K_i^2 + M_0^2 - M_i^2) - K_i^2(K_j^2 + M_0^2 - M_j^2))}{K^2((K_i\cdot K_j)^2 - K_i^2 K_j^2)}$$

Good news: no dependence on μ^2!

$$c_K(K,0,i)|_{\mu^2} \quad = \quad b(0,i)|_{\mu^2} + \frac{K_i^2}{3}\tilde{b}_{00}(0,i)$$

Four-point spurious term

No contribution.

Summary

- Analytic approach to one-loop amplitudes from (ordinary) unitarity cuts
- Tadpole coefficients from double cuts with auxiliary propagator D_K
- Decoupling conditions $K \cdot K_i = 0$, $M_K^2 = M_0^2 + K^2$.
- Consistency can be checked analytically in small examples.