

Jared's Durham Talk

Note Title

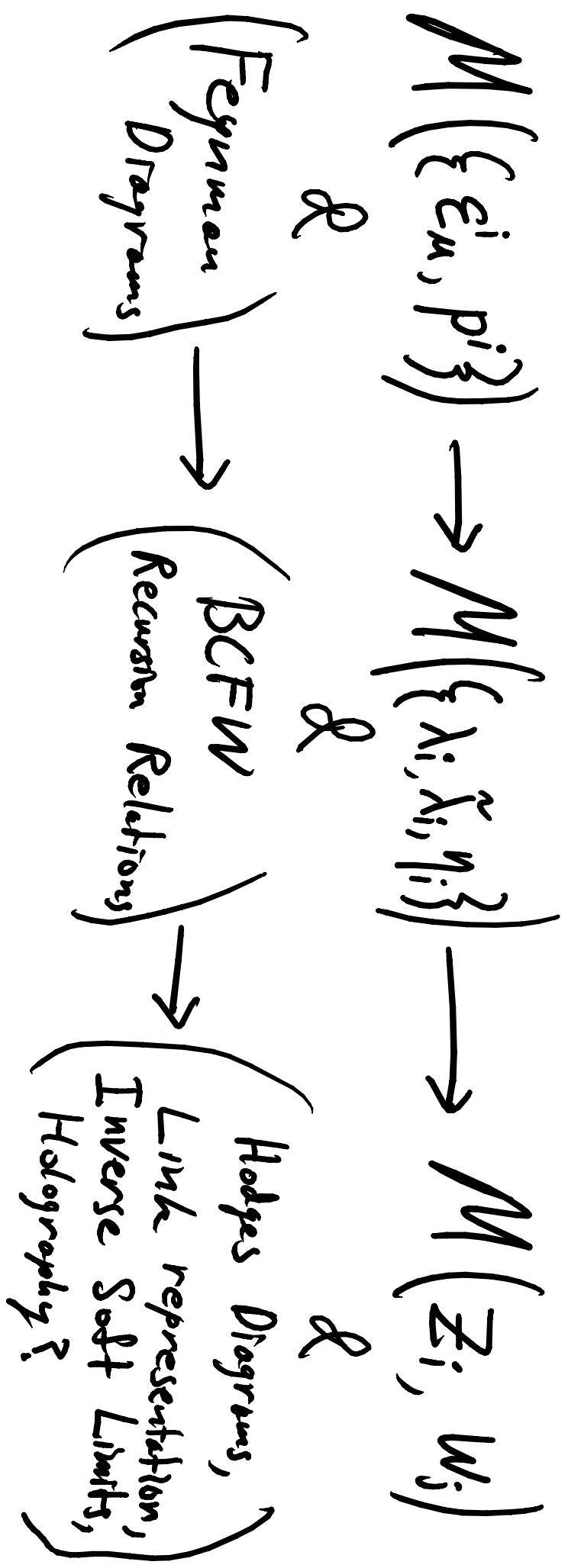
3/27/2009

Computing in Twistor Space

Jared Kaplan, Harvard/IAS

based on work with Arkani-Hamed, Cachazo, & Cheung

A New perspective...



Provides new tools, a new theory?

Outline

I. Twistor Space, Twistor Amplitudes

II. \mathcal{BCFW} in Twistor Space—Hodges Diagrams

- How to compute

III. "Triangulations" & Inverse Soft Factors

Twistor Space in $(2,2)$

In $(2,2)$ signature, $\lambda, \bar{\lambda}$ are real:

$$\rho_{\mu a\dot{a}} = \begin{pmatrix} \rho_1 + \rho_3 & \rho_2 + \rho_4 \\ \rho_2 - \rho_4 & \rho_1 - \rho_3 \end{pmatrix}$$

$$\Rightarrow \det(\rho_{a\dot{a}}) = \rho_1^2 + \rho_2^2 - \rho_3^2 - \rho_4^2 = \rho^2$$

If $\rho^2 = 0$, $\rho_{a\dot{a}} = \lambda_a \bar{\lambda}^{\dot{a}}$

Use γ or $\tilde{\gamma}$ to label $N=4$

One-particle States:

$$e^{iQ_a \gamma_I} | - \rangle$$

$$e^{iQ_a \tilde{\gamma}_I} | + \rangle$$

One smooth $M_n(\{\gamma_i\}, \{\tilde{\gamma}_j\})$.

Thus we can (well)-define:

$$F(\Xi) = \int d^2\lambda e^{i[\mu\lambda]} F(\lambda, \lambda, \eta)$$

with $\Xi = (\lambda, \mu, \eta)$. Also

$$G(w) = \int d^2\lambda e^{i[\tilde{\mu}\lambda]} G(\lambda, \lambda, \tilde{\eta})$$

with $W = (\lambda, \tilde{\mu}, \tilde{\eta})$.

What do Amplitudes depend on?

We can make an $SL(4|4)$ invariant:

$$Z \cdot W = \langle \lambda \tilde{\alpha} \rangle - [\tilde{\lambda} \mu] + \gamma \tilde{\gamma}$$

Also, there are

$$Z_1 T Z_2 = \langle 12 \rangle, \quad W_1 T W_2 = [12]$$

Not Conformally invariant.

Let's begin with the $N=4$ SYM 3-pt. amplitude:

$$M_3(z_1, z_2, W_3) = \text{Sgn}(z_1 \cdot W_3) \text{sgn}(z_2 \cdot W_3) \text{sgn}(z_1 \cdot z_2) + \int D^{4/4}W e^{i z_1 \cdot W} \text{sgn}(z_2 \cdot W) \text{sgn}(z_2 \cdot W_3) \text{sgn}(W_3 \cdot W)$$

These are the MHV & $\overline{\text{MHV}}$ terms.

Two lessons...

① Many signs! Why? Amps

Projective:

$$M(z_i, w_j) = M(t_i z_i, t_j w_j)$$

So $\text{sgn}(z \cdot w)$ is very natural. Determining
 η with Z , $\bar{\eta}$ with W .

②

The choice of Z or W is arbitrary!

Just twistor transform

$$\int e^{iz \cdot w} D^m w$$

to switch.

Motivates "Hodges Diagrams" with:

$$\bullet \quad Z = (\lambda, \mu, \eta)$$

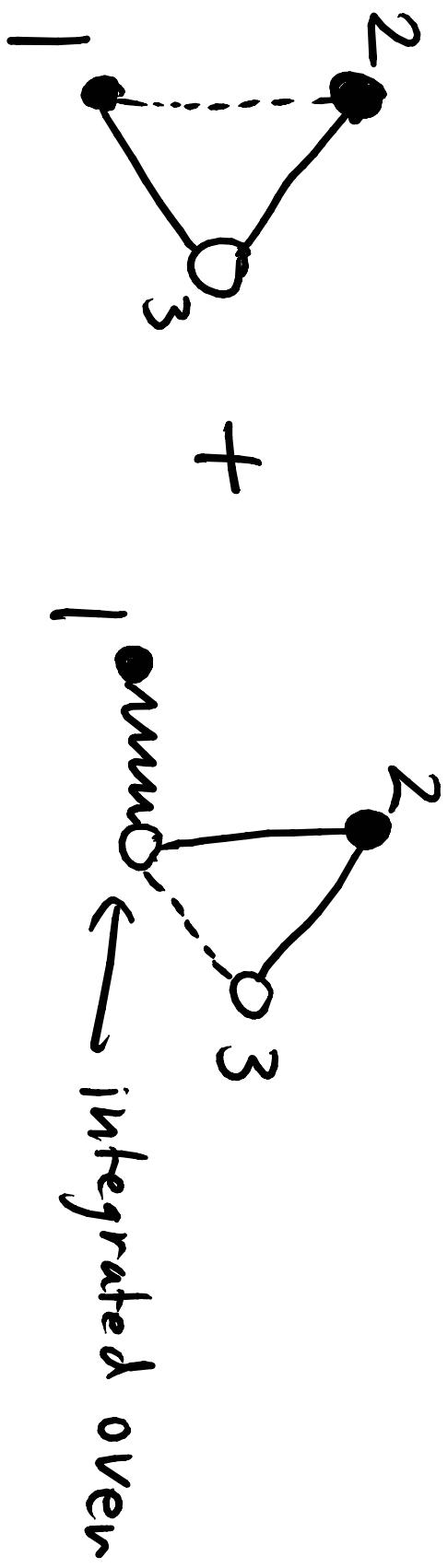
$$Ow = (\tilde{\lambda}, \tilde{\mu}, \tilde{\eta})$$

$$\text{Sgn}(Z \cdot w)$$

$$\text{anno } e^{iz \cdot w}$$

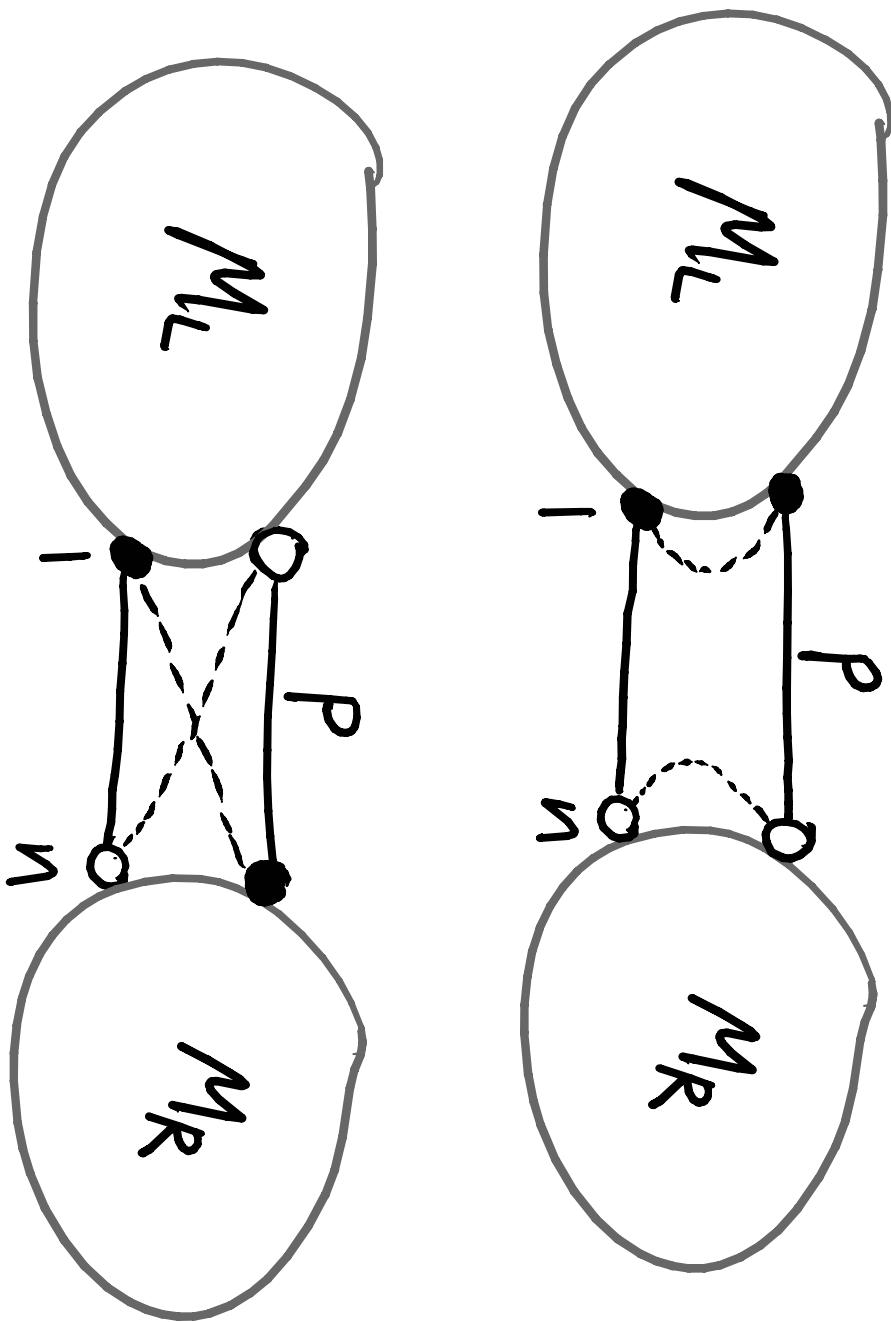
$$1 \bullet \dots \bullet^2 \text{ sgn}(z_1, Iz_2) = \text{sgn}(\langle 12 \rangle)$$

$$M_3(z_1, z_2, W_3) =$$



We have (almost) all the ingredients we need to express any amplitude!

BCFW Bridges



The Z & w
in the " p' "
line are integrated
over **projectively**.
These are
equivalent.

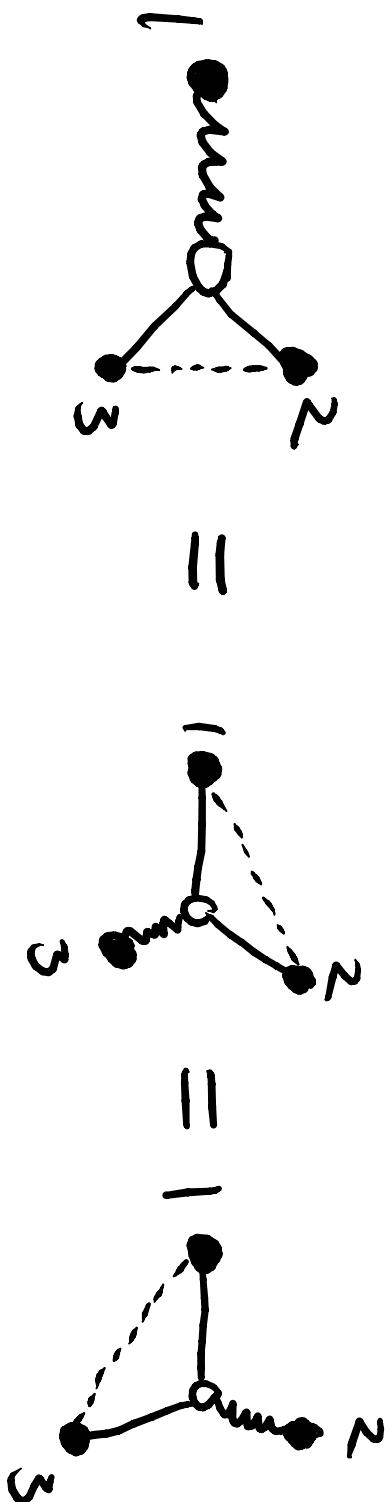
It's extremely easy to compute due
to various identities. **Why?**

- $N=4$ Amplitudes *Cyclically* Symmetric
& have hidden *Parity Symmetry*
- Different BCFW bridges equivalent
in Non-trivial ways

A first Example

$$M_3(z_1, z_2, z_3) = \text{Diagram}$$

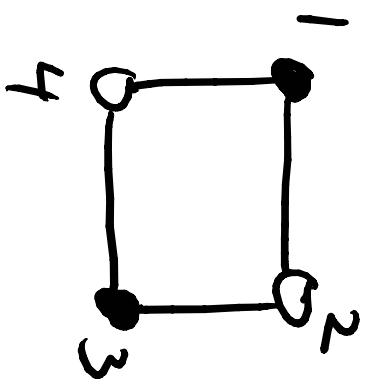

But this is cyclicly symmetric!

$$\begin{matrix} \text{Diagram} & = & \text{Diagram} \\ \text{Diagram} & = & \text{Diagram} \end{matrix}$$


"Triangle Identity"

By direct Fourier Transform:

$$M_4(z_1, w_2, z_3, w_4) =$$



It's trivial to use all z 's:

$$M_4(z_1, z_2, z_3, z_4) = \begin{matrix} 1 & \text{---} & 2 \\ | & \text{---} & | \\ 3 & \text{---} & 4 \end{matrix}^{\text{omega}} = \begin{matrix} 1 & \text{---} & 2 \\ | & \text{---} & | \\ 4 & \text{---} & 3 \end{matrix}^{\text{omega}}$$

Cyclic
"Square
Identities"

Now we know

$$\mathcal{M}_4^{\text{BCFW}} = \mathcal{M}_4^{\text{Fourier Transform}}$$

In any basis, this generates identities.

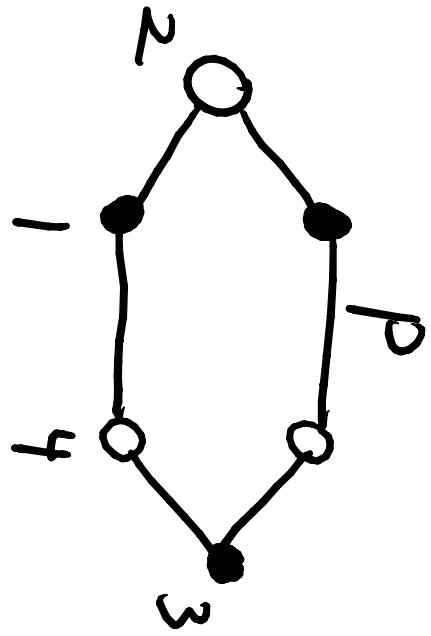
Let's actually compute $\mathcal{M}_4^{\text{BCFW}} \dots$

In momentum space, M_4 is

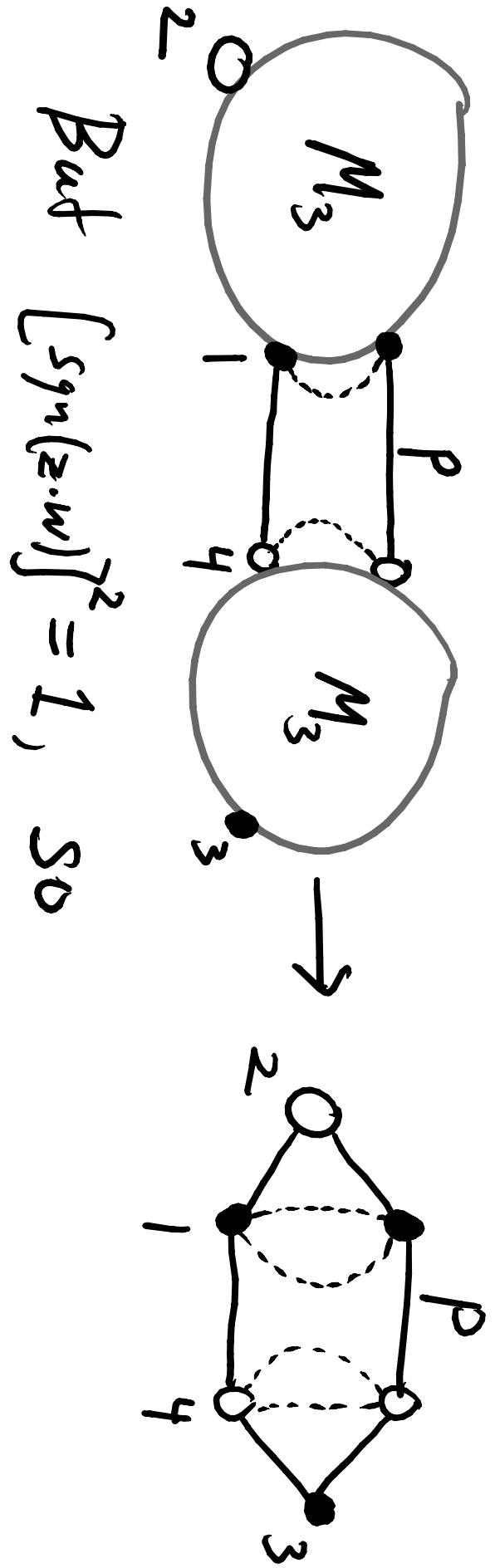
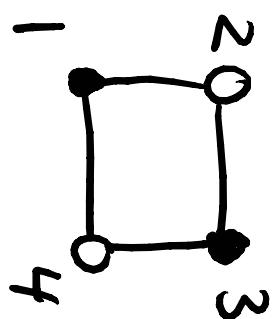
$$\begin{array}{c} \text{Diagram: Four external lines meeting at a central point labeled } = \\ \text{Below it: } = \int d^4 q \\ \text{Diagram: Two lines labeled 1 and 2 meet at a point, which then splits into two lines labeled 1' and 2'. These further split into lines labeled 3 and 4. An arrow points upwards from the top vertex. Labels 1, 2, 3, 4 are placed around the vertices. } \end{array}$$

In twistor space, we can choose a
BCFW bridge & Z or W for the states.

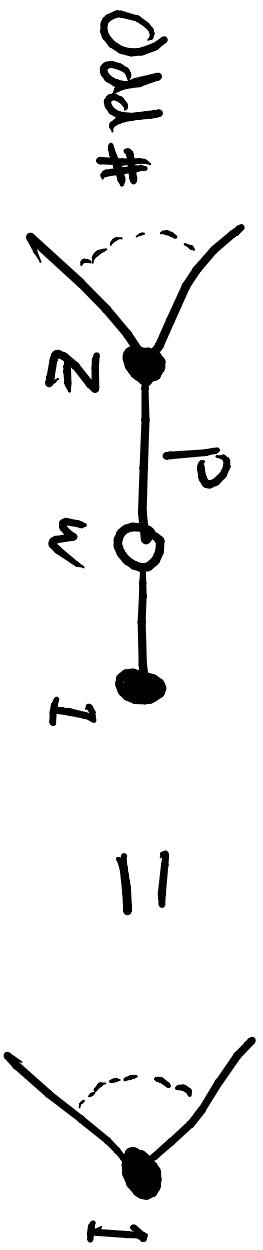
Implies "Scrunched Identity" ...



=



"Scrunched Identity"

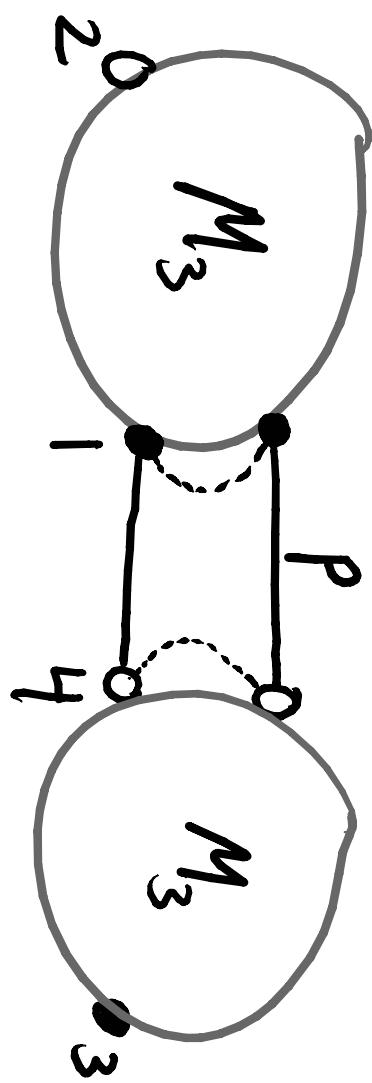


Can prove directly:

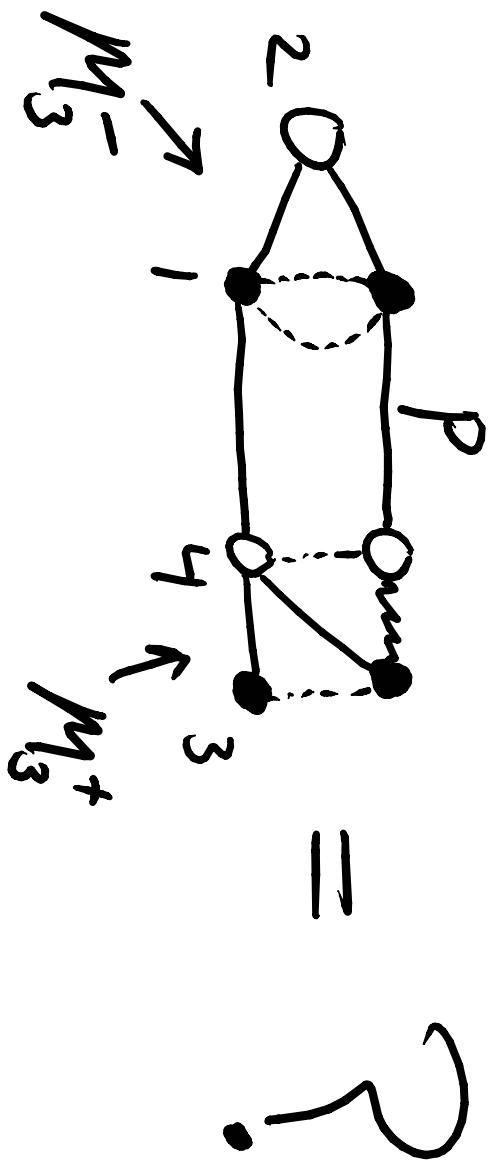
$$\int D^3z D^3w \frac{da}{a} \frac{db}{b} f(z) e^{iW(a z + b z_1)} = f(z_1)$$

Reprojectivize z & w integrals by absorbing a & b .

We've ignored terms!



also includes M_3^- with M_3^+ :



In momentum space this = 0.

In Twistor Space:

$$C_{\text{only}} = \int_D^{4/4} \mathcal{D}^{4/4} Z \frac{d\alpha}{\alpha} \frac{dp}{p} e^{iZ \cdot (W_c + \alpha W_a + \beta I Z_b)} \text{Sgn}(W_a I W_c)$$
$$= \int \frac{d\alpha}{\alpha} \frac{dp}{p} \delta^{4/4} (W_c + \alpha W_a + \beta I Z_b) \text{sgn}(W_a I W_c)$$

$$\text{On } \delta^{4/4}, \text{ Sgn}(W_a I (\alpha W_a + \beta I Z_b)) = \text{sgn}(\alpha) = 0 !$$

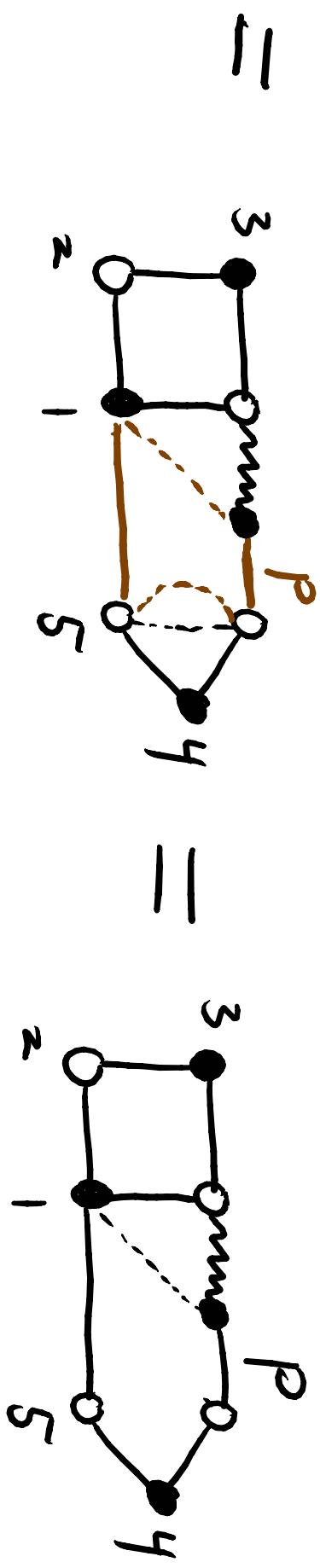
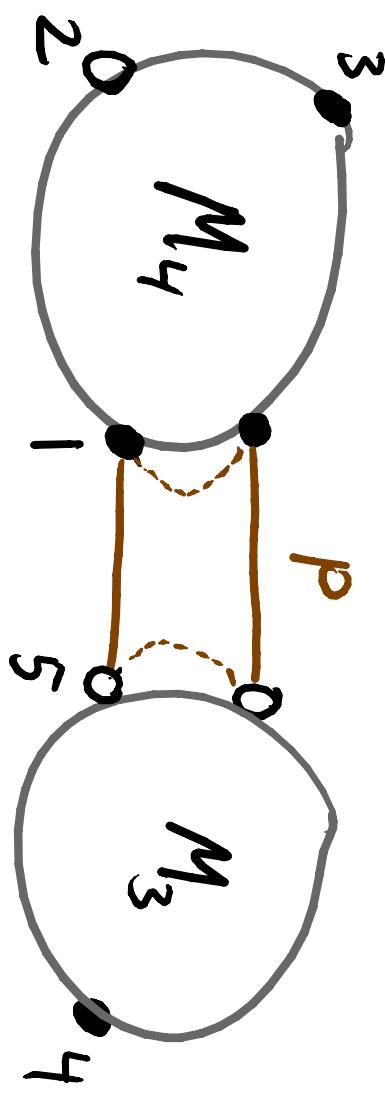
"Vanishing Identity"

$$\begin{array}{c} b \\ \text{---} \\ a \end{array} \quad \begin{array}{c} c \\ \text{---} \\ = \end{array} \quad \circ$$

Kills many terms, including $t t \dots t -$
and $t t \dots +$ type amplitudes.

We have the technology we need
to compute anything efficiently!

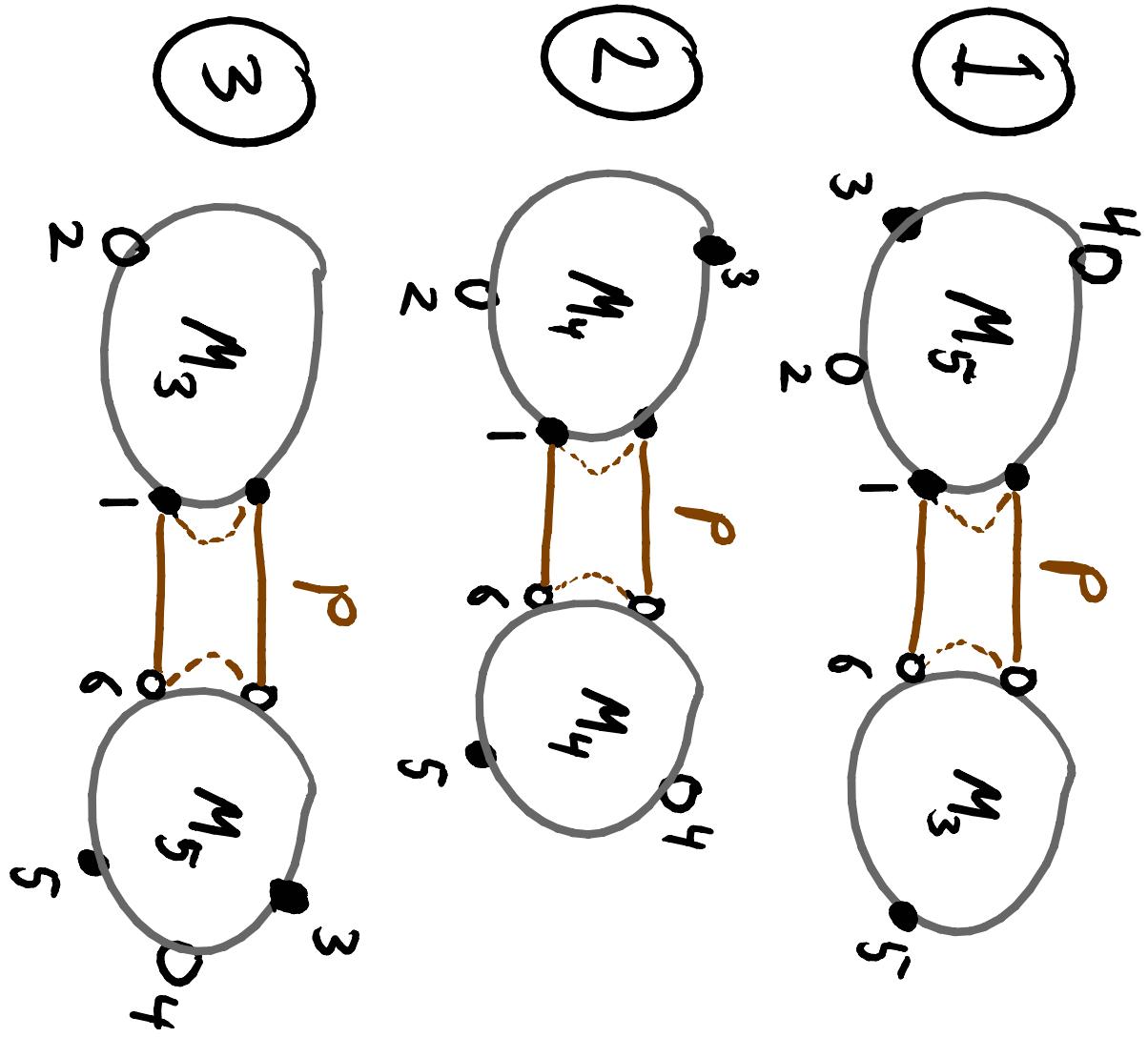
5-pt. from

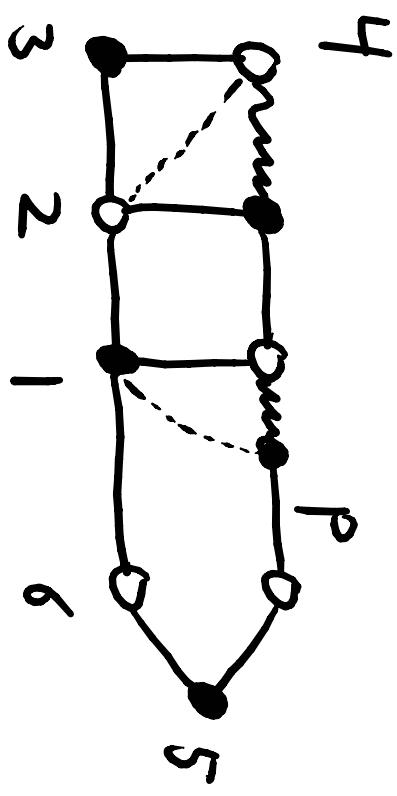
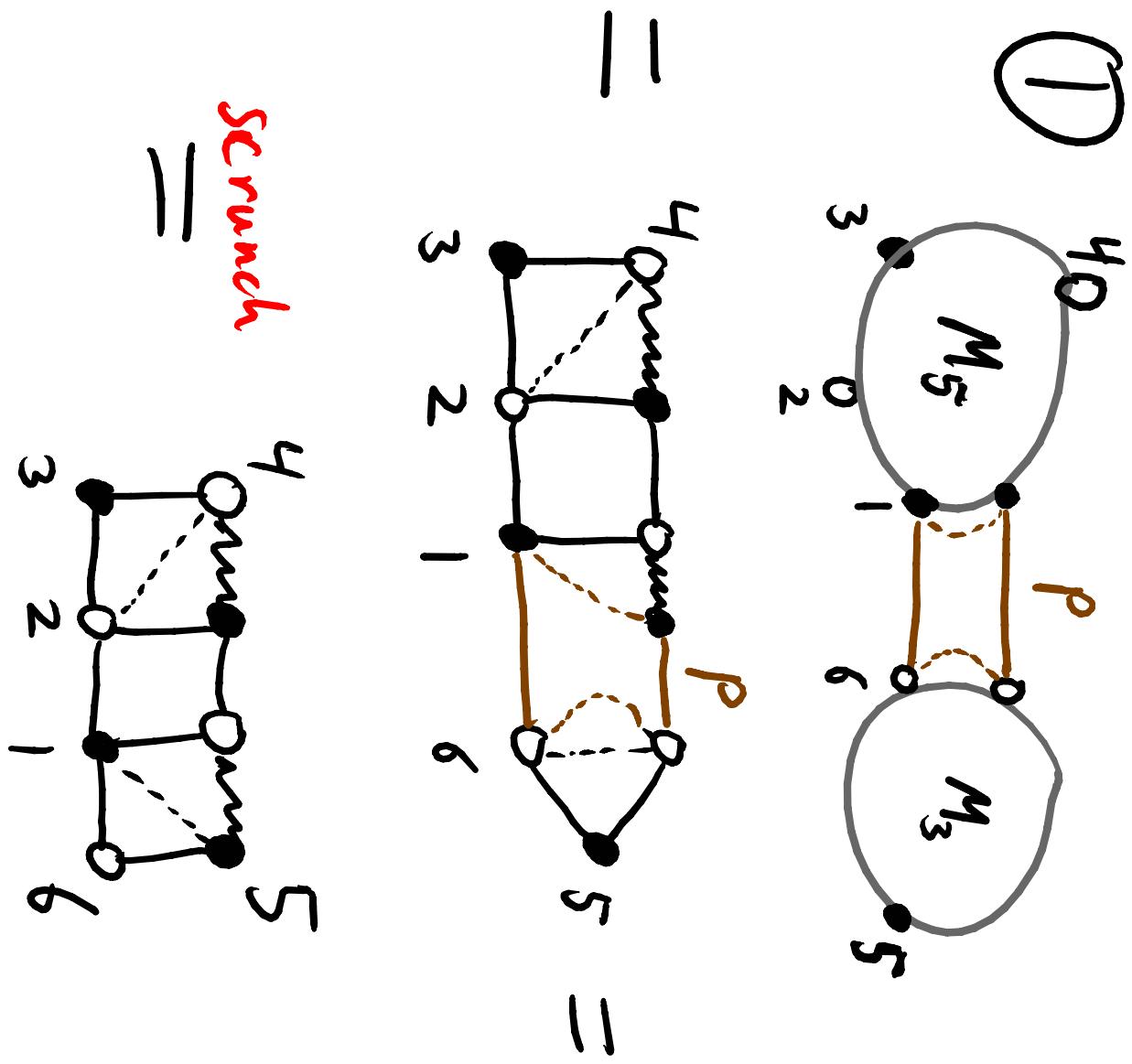


$$\text{Scrunch} = \begin{array}{c} 3 \\ \square \\ 2 \\ | \\ 1 \end{array} = 5\text{-pt. } \overline{\mathcal{MHV}}$$

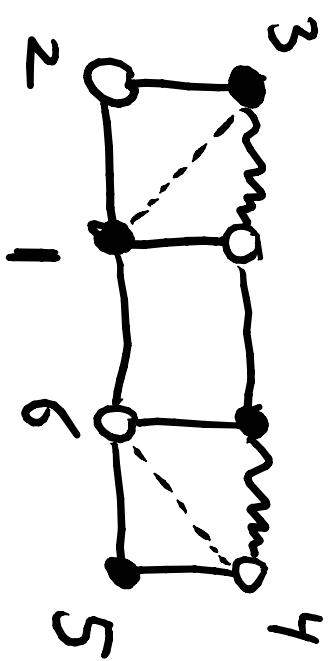
Now let's compute the 6-pt.
 \mathcal{MHV} amplitude . . .

$$S_{ph.} = \textcircled{1} + \textcircled{2} + \textcircled{3}$$

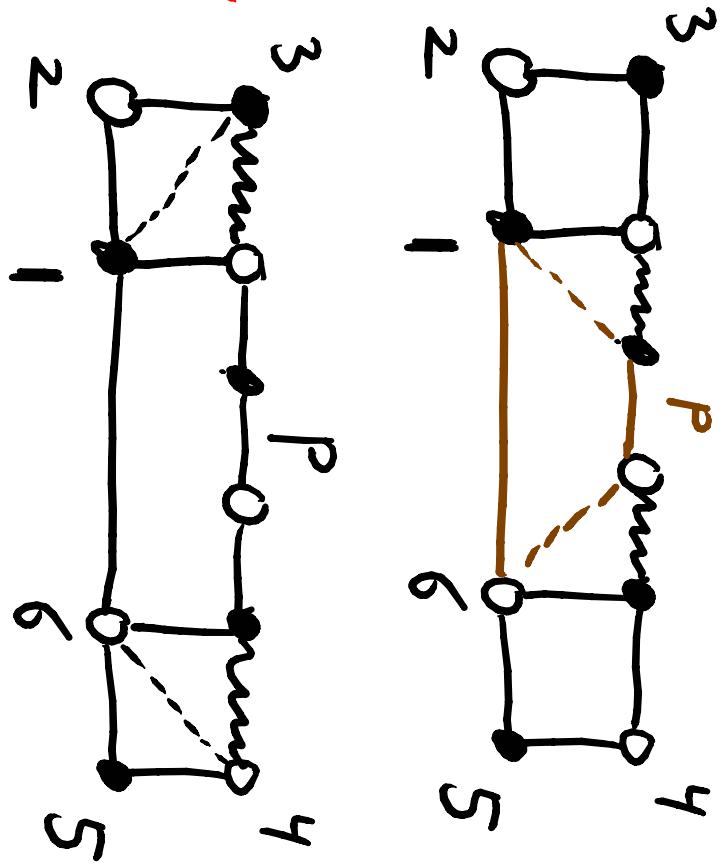
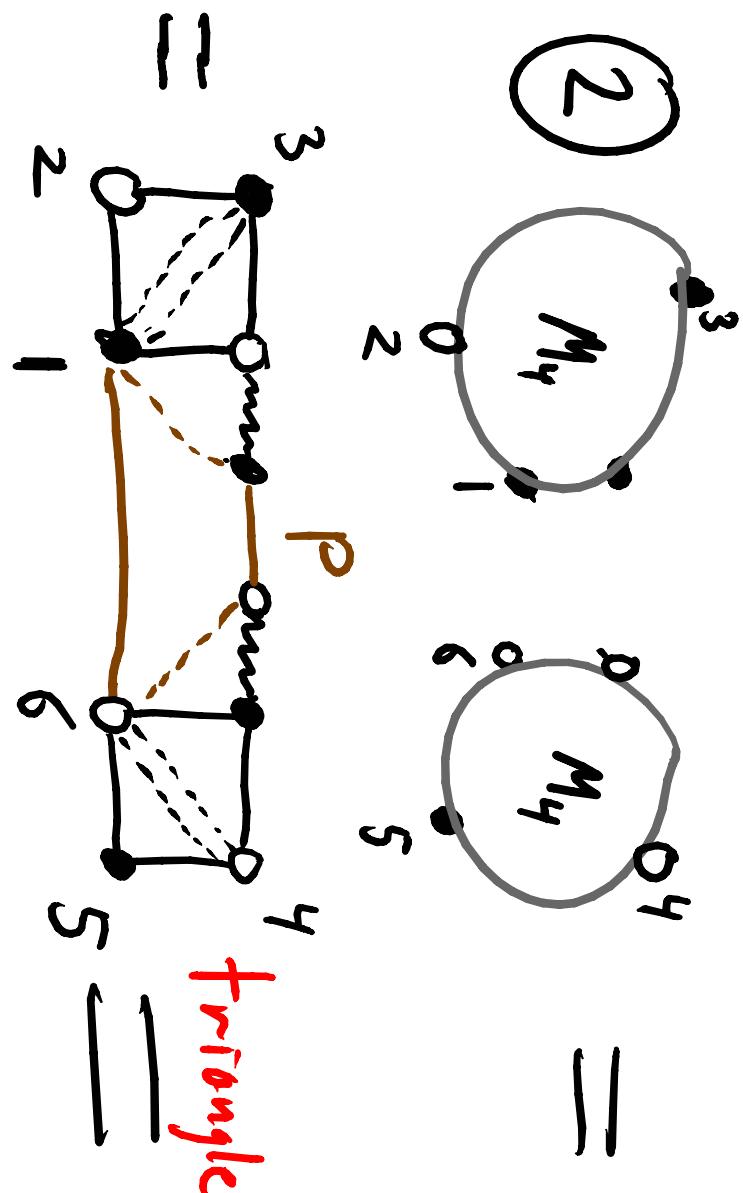




Scrunch

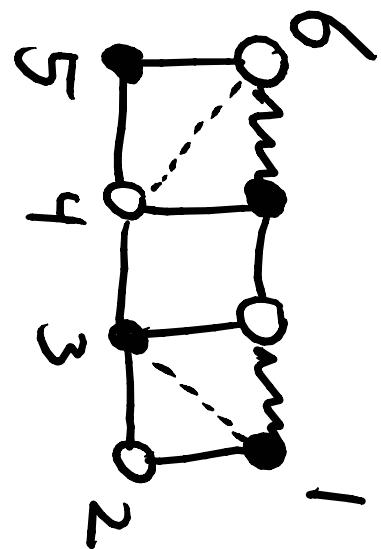
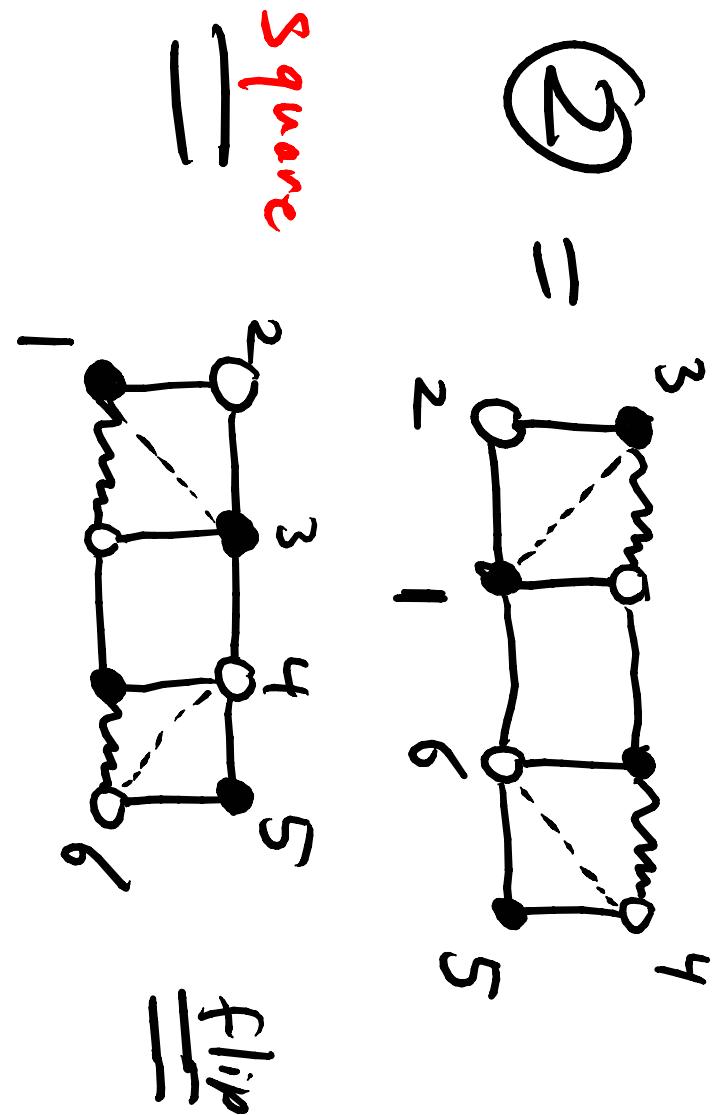


Can easily relate to O...



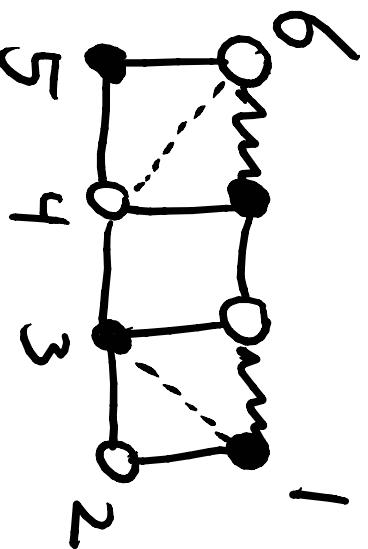
Thus

$$\begin{aligned} ① &= \\ (②) &= \\ | & \quad i \rightarrow i+2 \\ ! & \end{aligned}$$



Turns out (3) is similar so

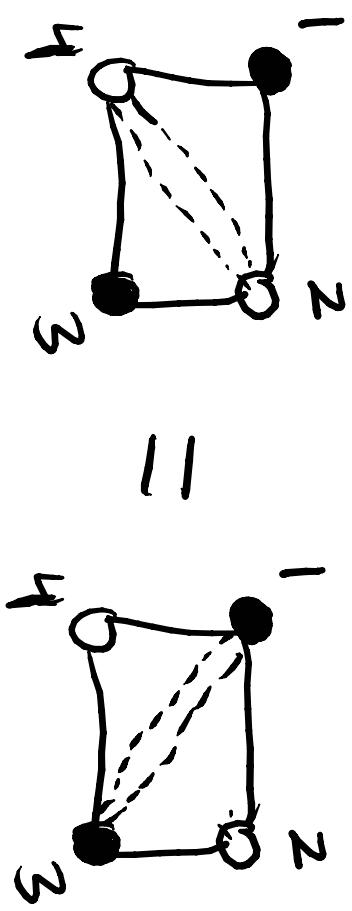
$$\mathcal{M}_6^{NMHV}(z_1, w_2, z_3, w_4, z_5, w_6)$$

$$= \text{Diagram 1} + \left(i \rightarrow i+2 \right) + \left(i \rightarrow i+4 \right)$$


Could easily go on to 7-pt, 8-pt, ...

Instead, let's reflect:

$$\begin{aligned}
 & \mathcal{M}_4(Z_1, W_2, Z_3, W_4) = \\
 &= \mathcal{M}_3^+(Z_1, W_2, W_4) \times \mathcal{M}_3^+(W_2, W_4, Z_3) \\
 &= \mathcal{M}_3^-(Z_1, W_4, Z_3) \mathcal{M}_3^-(Z_1, W_2, Z_3)
 \end{aligned}$$

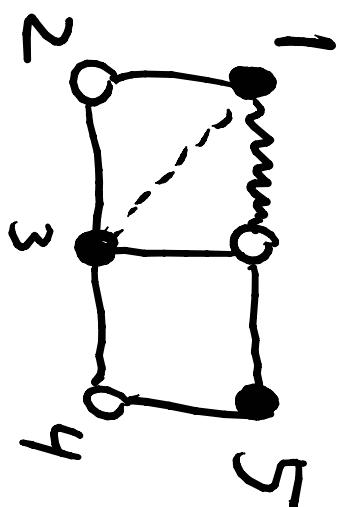


The Square Identity Says:

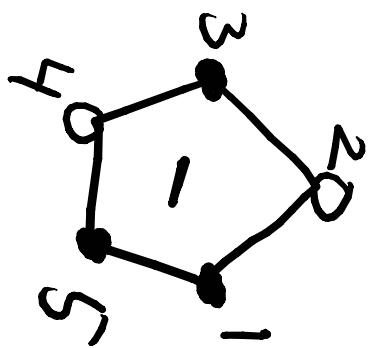
$$\begin{array}{c} \text{Diagram 1: A square with vertices labeled 1 (top), 2 (right), 3 (bottom), and 4 (left). The top edge is solid, while the other three edges are dashed.} \\ \downarrow \\ \begin{array}{c} \text{Diagram 2: A square with vertices labeled 1 (top), 2 (right), 3 (bottom), and 4 (left). The top edge is dashed, while the other three edges are solid.} \\ = \\ \begin{array}{c} \text{Diagram 3: A square with vertices labeled 1 (top), 2 (right), 3 (bottom), and 4 (left). All four edges are solid.} \\ = \\ \begin{array}{c} \text{Diagram 4: A square with vertices labeled 1 (top), 2 (right), 3 (bottom), and 4 (left). All four edges are dashed.} \end{array} \end{array} \end{array}$$

MHV & $\overline{\text{MHV}}$ Domains Well-Defined

$$M_5^- =$$



$$=$$

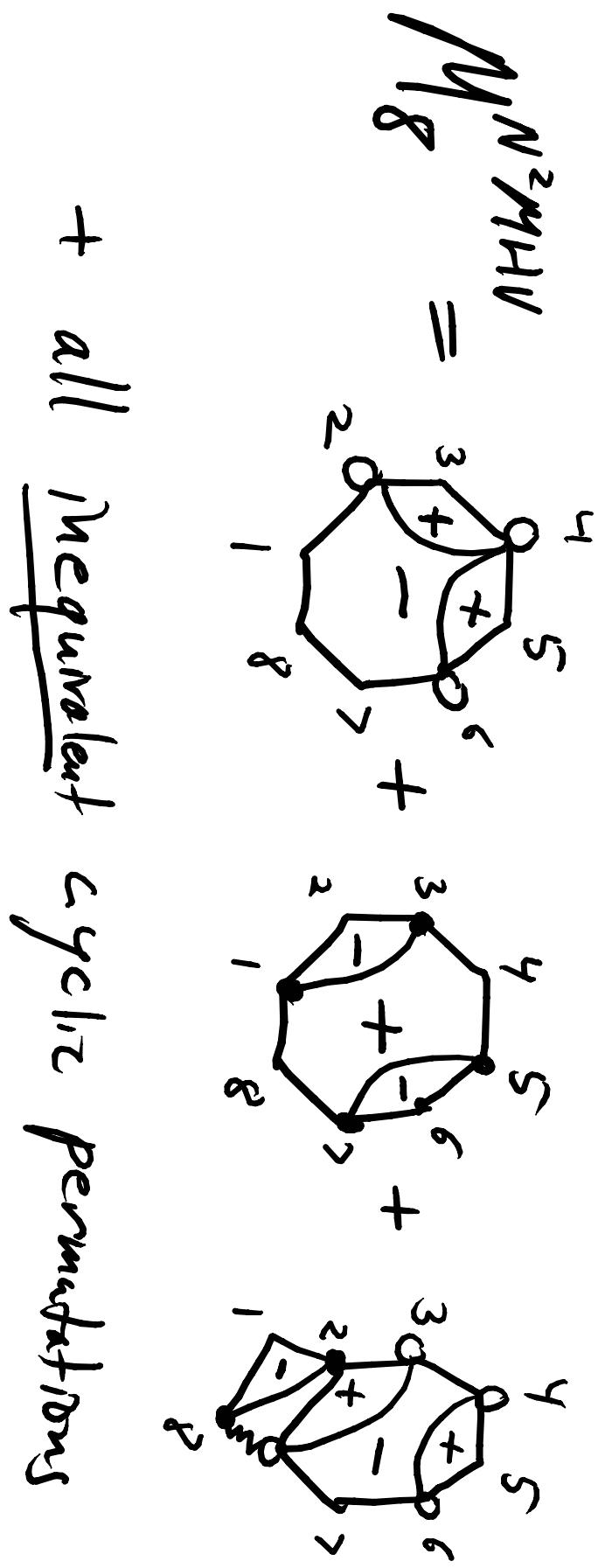


$$= M_3^-(z_1, w_2, z_3) M_4(z_1, z_3, w_4, z_5) \text{ and}$$

$$M_5^+ = M_3^+(w_4, w_5, z_5) M_5^-(z_1, w_2, z_3, w_4, w_5)$$

Just adding triangles!

To show off:

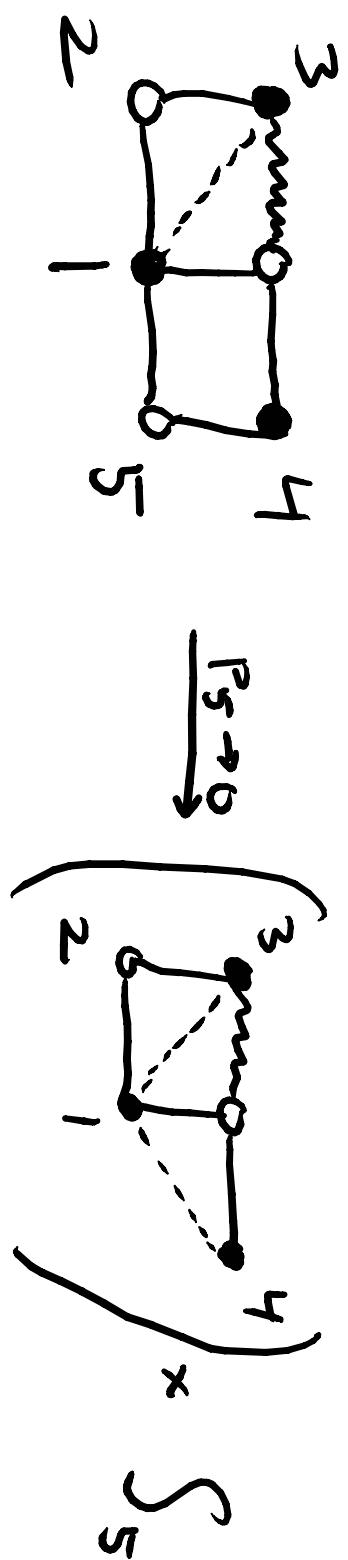


Two Questions:

- What does it mean!?
- Does "if" continue?
 - Yes, up to at least 15-pt.

(Inverse) Soft Links

$$M_{n+1} \left(P_{n+1} \rightarrow 0 \right) \rightarrow \sum_{i=1}^n \frac{\epsilon_{n+1} \cdot P_i}{P_{n+1} \cdot P_i} M_n$$



$$= S_5 \cdot M_4$$

Triangles have nice
soft links!

Objects built from triangles are
constructed (in momentum space) from
inverse soft limits,

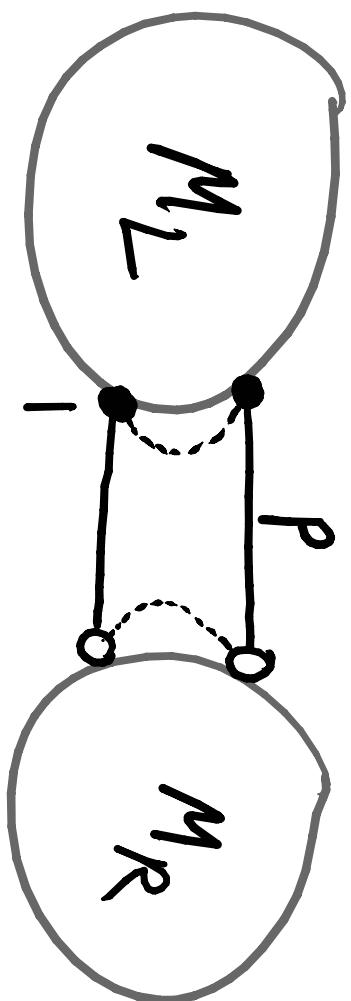
MHV Amplitudes entirely composed of

$$\Delta^- = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ - \end{array} = M_3^-$$

$$\widehat{\text{MHV}} \text{ from } \Delta^+ = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ + \end{array}$$

True for all terms in all $M=4$ Amps?

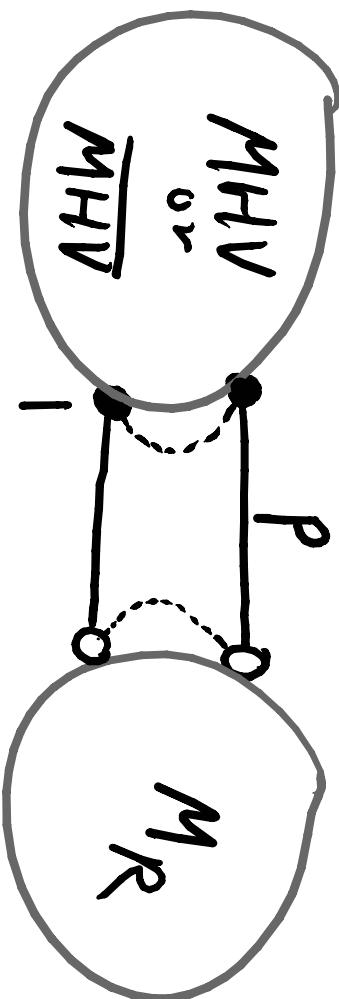
If all



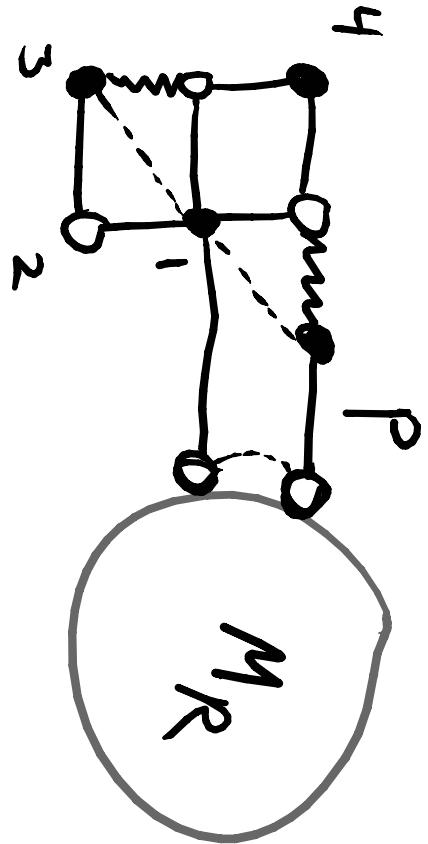
turn into objects made by attaching triangles — “triangulable”.

Can we collapse the “ ϕ ” into triangles?

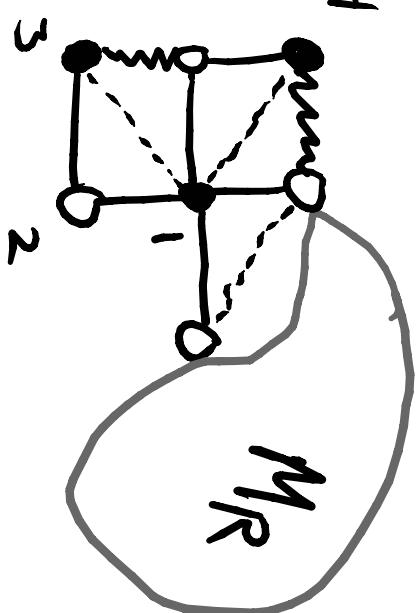
For



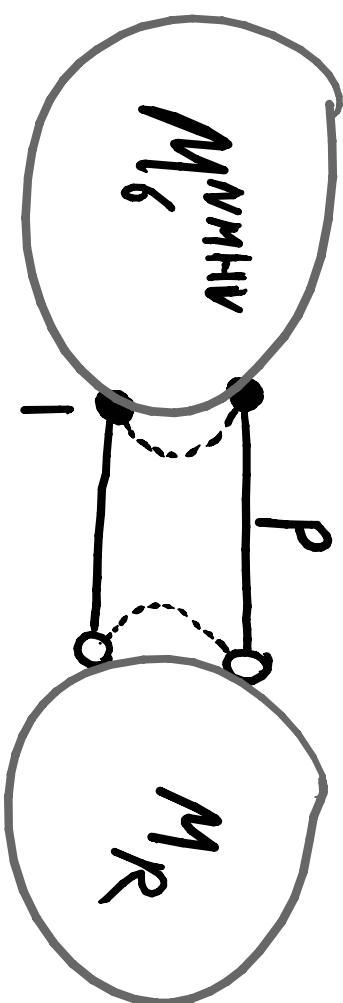
triangularity \Rightarrow Easy. μ_S^- Example:
Exampk:



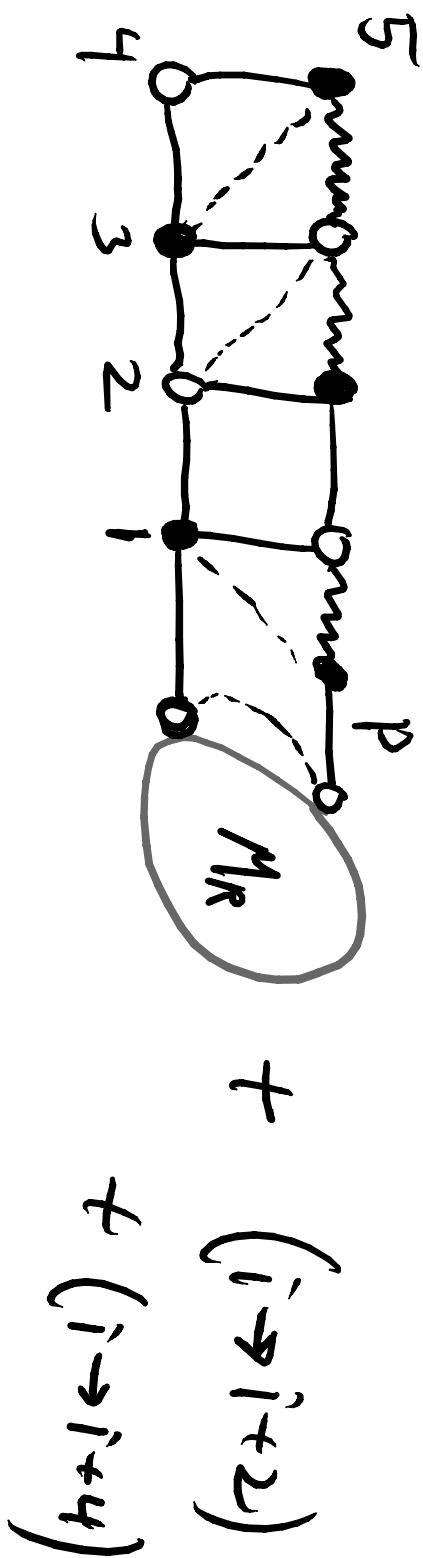
firstyle
=
second



For



There are 3 terms:



But they all work for one of the
two possible orientations of M_6 .

$$M_6 = M_{VVV} + (i \rightarrow i+2) + (i \rightarrow i+4)$$

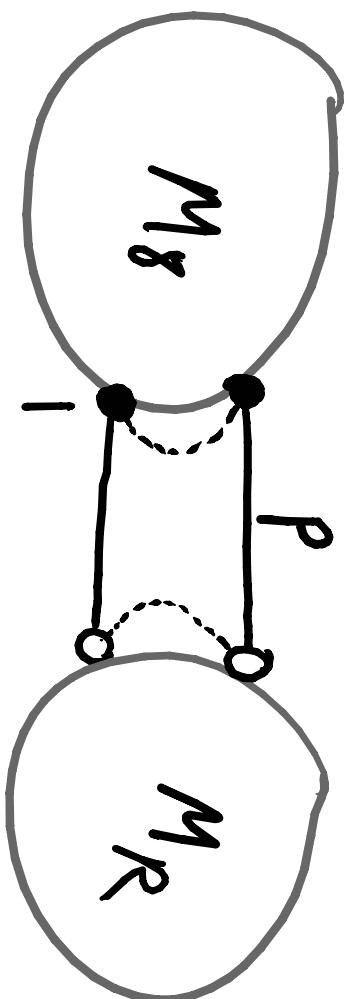
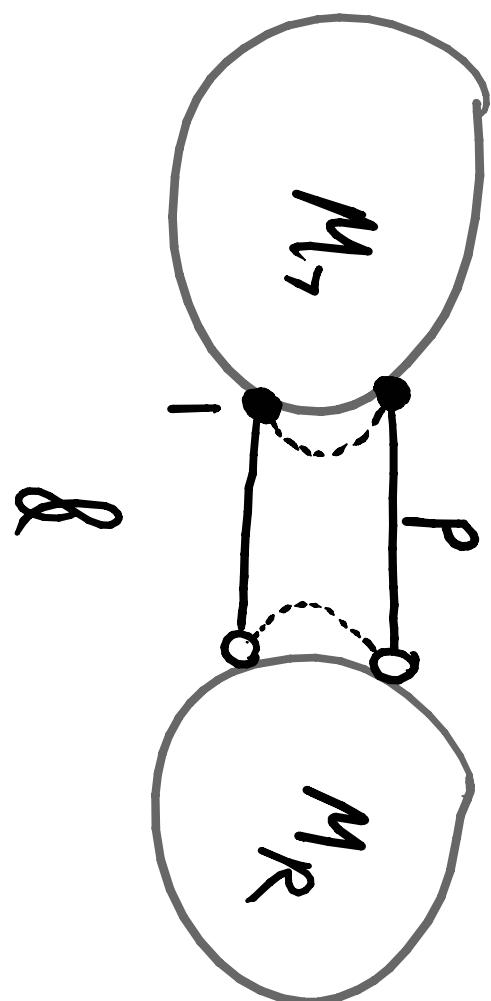
or

$$=$$

$$+ (i \rightarrow i+2) + (i \rightarrow i+4)$$

The equivalence **property** is a non-trivial identity which is useful to manipulate M_7, M_8, \dots

We have thus shown that



Are triangulable. Thus M_{15} is too!

Terms in Amplitudes constructed
from Inverse Soft factors...

What are the rules for putting
them together?

What is the theory?