Exploring scattering amplitudes

Amplitudes 09, Durham

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Work in collaboration with Lance Dixon, David Kosower, Radu Roiban.

Part I. Six-point NMHV amplitudes[1]

[1]Work in progress, D. Kosower, R. Roiban, and C. V.

NMHV at one loop

The six-point, one-loop NMHV gluon amplitudes were computed by Bern, Dixon, Dunbar and Kosower. There are three inequivalent helicity distributions $+++---$, $++-+--$ and $+-+-+-$.
Unlike the MHV case, for each helicity distribution there are three spin factors.
The tree level amplitudes can be written as[2]

$$A^{(0)}_{+++---} = \frac{1}{2}(B_1 + B_2 + B_3), \quad (1)$$

$$A^{(0)}_{++-+--} = \frac{1}{2}(D_1 + D_2 + D_3), \quad (2)$$

$$A^{(0)}_{+-+-+-} = \frac{1}{2}(G_1 + G_2 + G_3). \quad (3)$$

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[2] There are many ways to write the six-point NMHV tree amplitudes, but this is the most natural from the point of view of going to loop level.

The B coefficients on the previous slide are given by

$$B_1 = \frac{i s_{123}^3}{\langle 12\rangle \langle 23\rangle [4(1+2+3)1\rangle [6(1+2+3)3\rangle [45] [56]}, \tag{4}$$

$$B_2 = \frac{i [1(1+2+3)4\rangle^4}{\langle 23\rangle \langle 34\rangle [1(2+3+4)4\rangle [5(2+3+4)2\rangle [56] [61] s_{234}} + \\ + \frac{i \langle 56\rangle^3 [23]^3}{\langle 61\rangle \langle 1(2+3+4)4] \langle 5(2+3+4)2] [34] s_{234}}, \tag{5}$$

$$B_3 = \frac{i [3(1+2+3)6\rangle^4}{\langle 12\rangle \langle 61\rangle [3(1+2+6)6\rangle [5(1+2+6)2\rangle [34] [45] s_{345}} + \\ + \frac{i \langle 45\rangle^3 [12]^3}{\langle 34\rangle \langle 3(1+2+6)6] \langle 5(1+2+6)2] [61] s_{345}}. \tag{6}$$

Similar formulas hold for the D and G coefficients.

The NMHV at one loop

Bern, Dixon, Dunbar & Kosower

$$A^{(1)}_{+++---} = \frac{a}{2}(B_1 W_1^{(1)} + B_2 W_2^{(1)} + B_3 W_3^{(1)}), \qquad (7)$$

$$A^{(1)}_{++-+--} = \frac{a}{2}(D_1 W_1^{(1)} + D_2 W_2^{(1)} + D_3 W_3^{(1)}), \qquad (8)$$

$$A^{(1)}_{+-+-+-} = \frac{a}{2}(G_1 W_1^{(1)} + G_2 W_2^{(1)} + G_3 W_3^{(1)}), \qquad (9)$$

where $a = \frac{g^2 N_c}{8\pi^2}$ and W^i are combinations of one-mass boxes and two-mass hard boxes (these box integrals are dual conformal). Note that the same functions $W_i^{(1)}$ appear for all the helicity distributions. Also, $W_i^{(1)}$ are invariant under the cyclic permutation $1 \to 4,\ 2 \to 5,\ 3 \to 6,\ 4 \to 1,\ 5 \to 2,\ 6 \to 3$.

Structure of the two-loop result

D. Kosower, R. Roiban, and C. V., work in progress

$$A^{(2)}_{+++---} = \frac{a^2}{2}\Big(B_1 W_1^{(2)} + B_2 W_2^{(2)} + B_3 W_3^{(2)} + \widetilde{B}_1 \widetilde{W}_1^{(2)} + \widetilde{B}_2 \widetilde{W}_2^{(2)} + \widetilde{B}_3 \widetilde{W}_3^{(2)}\Big), \quad (10)$$

$$A^{(2)}_{++-+--} = \frac{a^2}{2}\Big(D_1 W_1^{(2)} + D_2 W_2^{(2)} + D_3 W_3^{(2)} + \widetilde{D}_1 \widetilde{W}_1^{(2)} + \widetilde{D}_2 \widetilde{W}_2^{(2)} + \widetilde{D}_3 \widetilde{W}_3^{(2)}\Big), \quad (11)$$

$$A^{(2)}_{+-+-+-} = \frac{a^2}{2}\Big(G_1 W_1^{(2)} + G_2 W_2^{(2)} + G_3 W_3^{(2)} + \widetilde{G}_1 \widetilde{W}_1^{(2)} + \widetilde{G}_2 \widetilde{W}_2^{(2)} + \widetilde{G}_3 \widetilde{W}_3^{(2)}\Big). \quad (12)$$

The quantities $W_i^{(2)}$ are scalar functions of the kinematic invariants and are expressed in terms of two-loop dual conformal integrals, while the quantities $\widetilde{W}_i^{(2)}$ are pseudoscalar (odd) functions.

The tilded spin coefficients on the previous slide are

$$\widetilde{B}_1 = \frac{is_{123}^3}{\langle 12\rangle \langle 23\rangle [4(1+2+3)1\rangle [6(1+2+3)3\rangle [45][56]}, \tag{13}$$

$$\widetilde{B}_2 = -\frac{i[1(1+2+3)4\rangle^4}{\langle 23\rangle \langle 34\rangle [1(2+3+4)4\rangle [5(2+3+4)2\rangle [56][61] s_{234}} + \\ + \frac{i\langle 56\rangle^3 [23]^3}{\langle 61\rangle \langle 1(2+3+4)4] \langle 5(2+3+4)2] [34] s_{234}}, \tag{14}$$

$$\widetilde{B}_3 = -\frac{i[3(1+2+3)6\rangle^4}{\langle 12\rangle \langle 61\rangle [3(1+2+6)6\rangle [5(1+2+6)2\rangle [34][45] s_{345}} + \\ + \frac{i\langle 45\rangle^3 [12]^3}{\langle 34\rangle \langle 3(1+2+6)6] \langle 5(1+2+6)2] [61] s_{345}}. \tag{15}$$

The coefficients $\widetilde{D}$ and $\widetilde{G}$ also differ by signs from the coefficients D and G respectively.

Recall that we had

$$A^{(0)}_{+++---} = \frac{1}{2}(B_1 + B_2 + B_3), \tag{16}$$

$$A^{(0)}_{++-+--} = \frac{1}{2}(D_1 + D_2 + D_3), \tag{17}$$

$$A^{(0)}_{+-+-+-} = \frac{1}{2}(G_1 + G_2 + G_3). \tag{18}$$

We also found some relations between the tilded coefficients which don't seem to appear in the literature

$$\widetilde{B}_1 + \widetilde{B}_2 + \widetilde{B}_3 = 0, \tag{19}$$

$$\widetilde{D}_1 + \widetilde{D}_2 + \widetilde{D}_3 = 0, \tag{20}$$

$$\widetilde{G}_1 + \widetilde{G}_2 + \widetilde{G}_3 = 0. \tag{21}$$

It can be shown that these relations hold exactly in the collinear limits $1 \parallel 2$, $2 \parallel 3$, etc. and they can also be checked numerically.

Superspace structure

Following Drummond, Henn, Korchemsky and Sokatchev, we can try to write a superspace expression for the two-loop NMHV amplitudes.

$$\mathscr{A}^{(0)}_{6,\text{MHV}} = i(2\pi)^4 \delta^4\Big(\sum_i p_i\Big) \frac{\delta^8(\sum_i \lambda_i \eta_i)}{\langle 12\rangle \cdots \langle 61\rangle},$$

$$\mathscr{A}^{(0)}_{6,\text{NMHV}} = \mathscr{A}^{(0)}_{6,\text{MHV}} \Big(\frac{\langle 34\rangle\langle 56\rangle\langle 61\rangle\langle 45\rangle \delta^4(\eta_4[56] + \eta_5[64] + \eta_6[45])}{2x_{14}^2 \langle 1|x_{14}|4]\langle 3|x_{36}|6][45][56]} + \\ + \text{cyclic}\Big),$$

$$\mathscr{A}^{(1)}_{6,\text{NMHV}} = a\mathscr{A}^{(0)}_{6,\text{MHV}} \Big(\frac{\langle 34\rangle\langle 56\rangle\langle 61\rangle\langle 45\rangle \delta^4(\eta_4[56] + \eta_5[64] + \eta_6[45])}{2x_{14}^2 \langle 1|x_{14}|4]\langle 3|x_{36}|6][45][56]} W_1^{(1)} \\ + \text{cyclic}\Big).$$

Superspace structure

Introduce a notation $\mathbb{P}$ ($\mathbb{P}^{-1}$) for the operator of cyclic permutation by one unit to the right (left). Then, we have $\mathbb{P}W_1^{(1)} = W_2^{(1)}$, $\mathbb{P}^2 W_1^{(1)} = W_3^{(1)}$, $\mathbb{P}^3 W_i^{(1)} = W_i^{(1)}$, for $i = 1, 2, 3$.
The two loop structure is

$$\mathscr{A}^{(2)}_{6,\text{NMHV}} = a^2 \mathscr{A}^{(0)}_{6,\text{MHV}} \Bigg(\frac{\langle 34\rangle\langle 56\rangle\langle 61\rangle\langle 45\rangle\,\delta^4(\eta_4[56] + \eta_5[64] + \eta_6[45])}{2x_{14}^2 \langle 1|x_{14}|4]\langle 3|x_{36}|6][45][56]} \times \times (W_1^{(2)} + \widetilde{W}_1^{(2)}) + \text{cyclic} \Bigg). \quad (22)$$

By cyclic permutations we can define the quantities $W_i^{(2)}$ and $\widetilde{W}_i^{(2)}$. They have similar properties to the corresponding one-loop coefficients except for $\mathbb{P}^3 \widetilde{W}_i^{(2)} = -\widetilde{W}_i^{(2)}$.
The quantities $\widetilde{W}_i^{(2)}$ are the hardest to compute. Compute them numerically? Stay tuned!

DHKS form

DHKS proposed to pull out the full MHV amplitude, not only the tree level

$$\mathscr{A}_{6,\mathrm{MHV}} = \mathscr{A}_{6,\mathrm{MHV}}^{(0)}(1 + aM^{(1)} + a^2(M^{(2)} + \widetilde{M}^{(2)}) + \cdots). \tag{23}$$

$$\mathscr{A}_{6,\mathrm{NMHV}} = \mathscr{A}_{6,\mathrm{MHV}}\Big(\frac{\langle 34\rangle\langle 56\rangle\langle 61\rangle\langle 45\rangle\delta^4(\eta_4[56] + \eta_5[64] + \eta_6[45])}{2x_{14}^2\langle 1|x_{14}|4]\langle 3|x_{36}|6][45][56]} \times \\ \times (1 + aV_1^{(1)} + a^2(V_1^{(2)} + \widetilde{V}_1^{(2)}) + \cdots) + \mathrm{cyclic}\Big). \tag{24}$$

At one loop, $W_i^{(1)} = M^{(1)} + V_i^{(1)}$, where $V_i^{(1)}$ is finite

$$V_i^{(1)} = -\log u_i \log u_{i+1} + \frac{1}{2}\sum_{k=1}^{3}\mathrm{Li}_2(1 - u_k), \tag{25}$$

where $u_1 = \frac{s_{12}s_{45}}{s_{123}s_{345}}$, $u_2 = \frac{s_{23}s_{56}}{s_{234}s_{123}}$ and $u_3 = \frac{s_{34}s_{61}}{s_{345}s_{234}}$.

DHKS form (two loops)

At two loops, we have

$$W_i^{(2)} = M^{(2)} + M^{(1)} V_i^{(1)} + V_i^{(2)}, \tag{26}$$

$$\widetilde{W}_i^{(2)} = \widetilde{M}^{(2)} + \widetilde{V}_i^{(2)}, \tag{27}$$

where we used colors for the known and unknown parts.
Note that under the permutation $\mathbb{P}^3$ the properties of $\widetilde{W}_i^{(2)}$ are simpler that those of $\widetilde{V}_i^{(2)}$.
We can confirm that $W_i^{(2)}$ and $M^{(2)}$ are identical at $\mathscr{O}(\varepsilon^{-4})$ and $\mathscr{O}(\varepsilon^{-3})$.

μ integrals

For the two-loop six-point MHV amplitude there are integrals which can not be detected by the 4D cuts but still contribute to the divergent and finite parts of the amplitude.

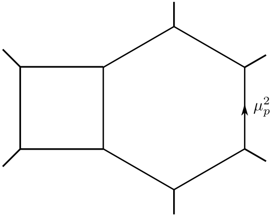


Figure: The hexabox μ integral.

The integral contains

$$\int d^{4-2\epsilon} p \, \mu_p^2 \cdots = \int d^4 p d^{(-2\epsilon)} \mu_p \, \mu_p^2 \cdots . \tag{28}$$

Its expansion in ϵ starts at $\frac{1}{\epsilon}$.

μ integrals

For the NMHV amplitude there are seven independent coefficients for the hexabox μ integral.

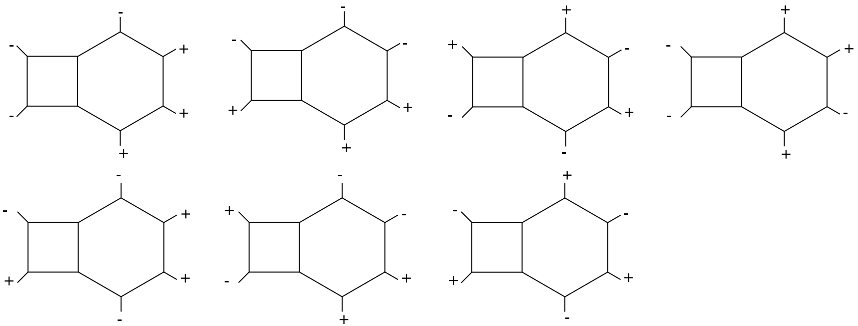


Table: Independent helicity distributions for the hexabox μ integral.

μ integrals

- We have analytic expressions for all these hexabox coefficients.
- The hexabox coefficients look very different than the B, D and G coefficients presented above.
- Is there a link with the $\mathcal{O}(\varepsilon)$ one-loop NMHV integrals? The result for NMHV at $\mathcal{O}(\varepsilon)$ is not known yet... A counting of the number of coefficients suggests that the best representation for this $\mathcal{O}(\varepsilon)$ one-loop NMHV amplitude could be in terms of pentagon μ integrals obtained by collapsing the boxes in the hexabox integrals.
- Is there a supersymmetrisation of these hexabox coefficients?

Part II. Higher-point MHV amplitudes[3]

[3] Based on arXiv:0903.3526

History and future

- four-point one-loop, [Green, Schwarz, Brink, 1982]
- *n*-point one-loop MHV, [Bern, Dixon, Dunbar, Kosower, 1994]
- six-point one-loop NMHV, [Bern, Dixon, Dunbar, Kosower, 1994]
- four-point two-loop, [Bern, Rozowsky, Yan, 1997]
- five-point two-loop, [Bern, Rozowsky, Yan, 1997] [Bern, Czakon, Kosower, Roiban, Smirnov, 2006] [Cachazo, Spradlin, Volovich, 2006]
- six-point two-loop MHV, [Bern, Dixon, Kosower, Roiban, Spradlin, Vergu, Volovich, 2008] [Cachazo, Spradlin, Volovich, 2008]
- five-point three-loop [Spradlin, Volovich, Wen, 2008]
- seven-point two-loop MHV (even part), [Vergu, 2009]
- *n*-point two-loop MHV, [?]
- six-point two-loop NMHV, [?]

Motivation

- Alday and Maldacena hinted that there could be a link between scattering amplitudes and Wilson loops in $\mathcal{N} = 4$ SYM. Subsequently this was shown to hold by explicit perturbative computations.
- Need to catch up with the Wilson loop computations (Anastasiou, Brandhuber, Heslop, Khoze, Spence, Travaglini).
- An understanding of the relations between the master integrals appearing in the Wilson loop and scattering amplitudes computations could have significant practical consequences for the evaluation of higher loop integrals.

Cuts

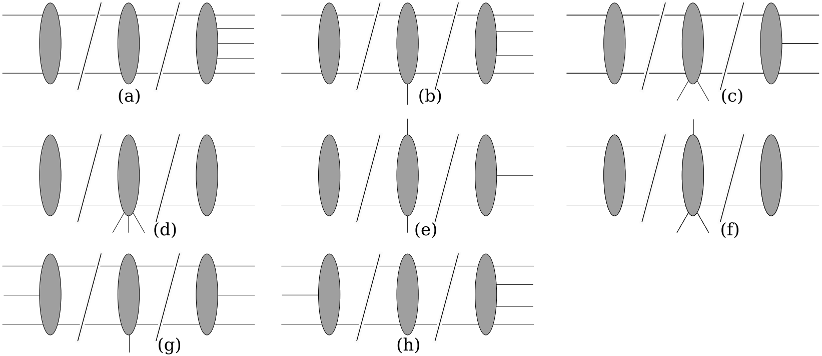
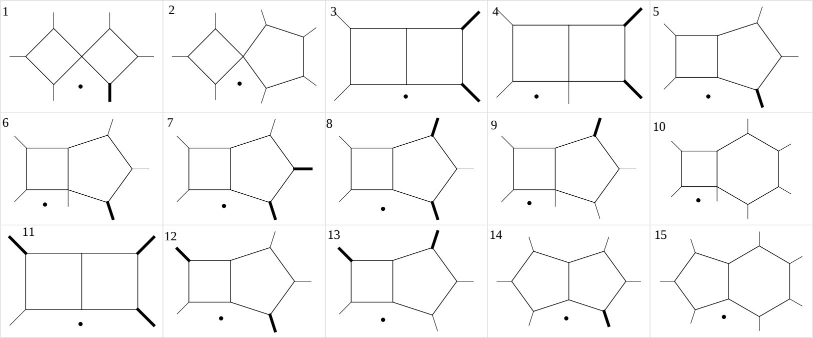
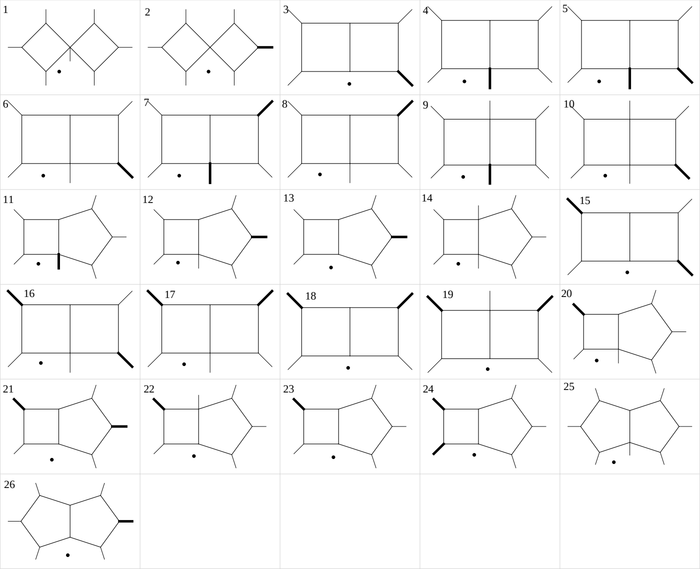


Figure: Cuts used to compute the seven-point two-loop amplitude.

Integrals with zero coefficient



Integrals with nonzero coefficient



n-point cuts

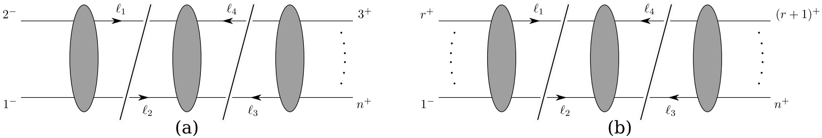
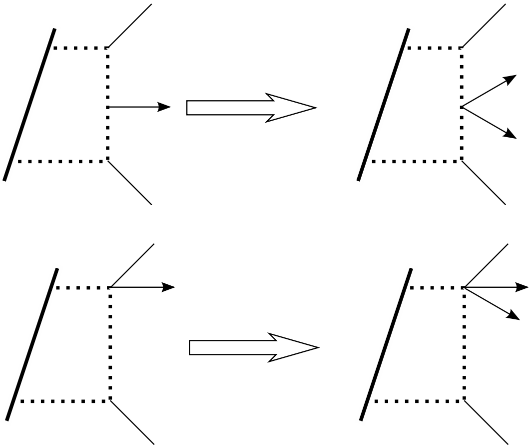


Figure: Two n-point cuts.

By computing the even part of these cuts we can show that only a restricted set of integrals appear.

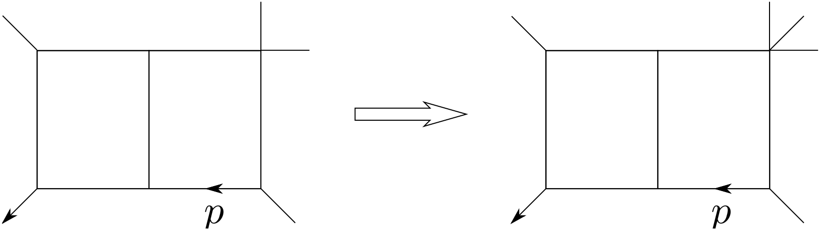
A leg addition rule

When an MHV tree amplitude can be separated by a cut, then we can use a leg addition rule.



A leg addition rule

Let us illustrate this by an example



$$C_{\text{5-pt}} = \frac{4s_{12}^2 s_{51}(p^{(1)}+k_{45})^2}{(p^{(1)}+k_{45})^2-(p^{(2)}+k_{45})^2}, \quad C_{\text{6-pt}} = \frac{4s_{12}^2 s_{61}(p^{(1)}+k_{456})^2}{(p^{(1)}+k_{456})^2-(p^{(2)}+k_{456})^2}.$$

Part III. Other conformal theories[4]

[4] Work in progress, L. Dixon, D. Kosower, and C. V.

All-order ansätze in other theories?

Study a $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM, which preserves $\mathcal{N}=2$ SUSY. This theory is obtained by an orbifold projection of the $SU(2N_c)$, $\mathcal{N}=4$ SYM.

$$\begin{pmatrix} \text{Adj} & (\mathbf{N}_c, \overline{\mathbf{N}}_c) \\ (\mathbf{N}_c, \overline{\mathbf{N}}_c) & \text{Adj} \end{pmatrix} \tag{29}$$

$$\Phi_1 \to \begin{pmatrix} \Phi_1^{(1)} & 0 \\ 0 & \Phi_1^{(1)} \end{pmatrix}, \qquad \Phi_i \to \begin{pmatrix} 0 & \Phi_i^{(12)} \\ \Phi_i^{(21)} & 0 \end{pmatrix}, \quad \text{for } i = 2, 3. \tag{30}$$

Then we deform the orbifolded theory by taking the couplings for the two gauge groups to be different $g \neq g'$.

$$
\begin{aligned}
& g\,\mathrm{tr}\left(\Phi_1\left[\Phi_2,\Phi_3\right]\right) \rightarrow \\
& \qquad \Big\{ g\,\mathrm{tr}\left(\Phi_3^{(21)}\Phi_1^{(1)}\Phi_2^{(12)} - \Phi_2^{(21)}\Phi_1^{(1)}\Phi_3^{(12)}\right) + \\
& \qquad\; g'\,\mathrm{tr}\left(\Phi_3^{(12)}\Phi_1^{(2)}\Phi_2^{(21)} - \Phi_2^{(12)}\Phi_1^{(2)}\Phi_3^{(21)}\right) \Big\}.
\end{aligned} \tag{31}
$$

This is a marginal deformation preserving $\mathcal{N}=2$ SUSY. When $g = g'$ we get back the $\mathbb{Z}_2$ orbifold of $\mathcal{N}=4$ SYM and when $g' = 0$ we get the $\mathcal{N}=2$ $SU(N_c)$ gauge theory coupled to $N_f = 2N_c$ hypermultiplets (which is also conformal).

Color flow

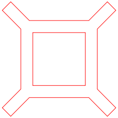


Figure: One-loop color flow

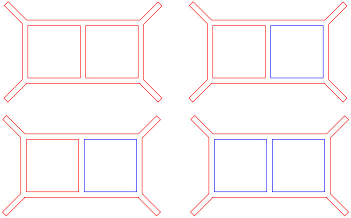


Figure: Two-loop color flow

Preliminary findings for $\mathcal{N} = 2$

- Different expressions for $--++$ and $-+-+$
- 4-dimensional cuts don't capture all the contributions; D-dimensional cuts are necessary
- The integrals appearing are not pseudo-conformal
- Dual conformal symmetry is violated
- Uniform transcendentality is violated (appearance of $\zeta(3)$)
- The cusp anomalous dimension and collinear anomalous dimension are the same as in $\mathcal{N} = 4$ SYM (at two-loop order). In other words, at two loops the IR divergences are identical to the ones in $\mathcal{N} = 4$ SYM.

Thank you!