Simplicity in One-Loop QED and Gravity Amplitudes

Simon Badger (DESY, Zeuthen)

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Workshop on Gauge and String Amplitudes

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Work in collaboration with Emil Bjerrum-Bohr and Pierre Vanhove

Overview

- Hidden simplicities in on-shell gauge theory scattering amplitudes
  - Tree-Level MHV amplitudes [Parke-Taylor (1986)]
  - One-Loop $\mathcal{N}=4$ MHV amplitudes [Bern,Dixon,Dunbar,Kosower (1994)]
  - Simplifications exploited by on-shell constructions $\Rightarrow$ NLO QCD
- Simplicity in $\mathcal{N}=8$ Super-gravity:
  - Superspace/string based power-counting
    $\Rightarrow \mathcal{N}=8$ finite up to $5<L<9$ in $D=4$ [Howe,Stelle;Kallosh]
    [Green,Russo,Vanhove]
  - Explicitly finite up to 3-loops [Kosower,Carrasco,Dixon,Roiban,Johansson]

$$D_c \leq 4 + \frac{6}{L}$$

- Origin of simplifications at one-loop $\Rightarrow$ Unordered gauge groups
  - Cancellations also in non-supersymmetric theories:
    photon amplitudes [Bjerrum-Bohr,Vanhove,SB]

No-Triangle Hypothesis @ One-Loop

- KLT relations $\Rightarrow \mathcal{N}=8 \sim (\mathcal{N}=4)^2$ [Kawai,Lewellyn,Tye (1986)]
- Explicit calculations for ≤ 6 gravitons [Green,Brink,Schwarz (1982)]

  [Grisaru,Seigel (1982)]

  [all-n MHV:Bern,Dixon,Perelstein,Rosowsky (1998)]

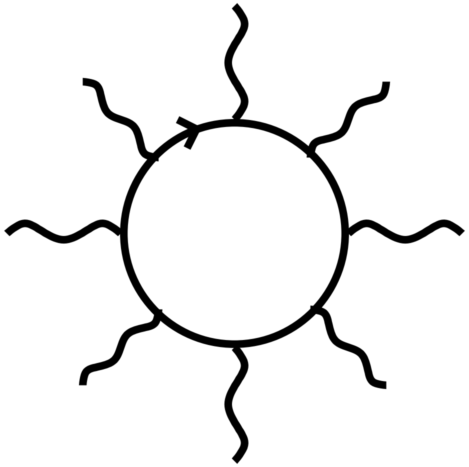
  [Bjerrum-Bohr,Dunbar,Ita (2005)]
- Postulate the no-triangle property (c.f. $\mathcal{N}=4$ SYM)

  [Bern,Dunbar,Bjerrum-Bohr (2005)]

  [Bjerrum-Bohr,Dunbar,Ita,Perkins,Risager (2006)]
- Proof from string inspired construction [Bjerrum-Bohr,Vanhove (2008)]
- Alternative construction from field theory super-amplitudes

  [Arkani-Hamed,Cachazo,Kaplan (2008)]

Simplifications in photon amplitudes



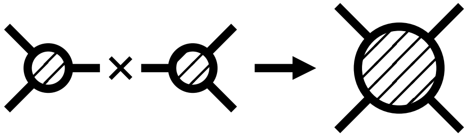
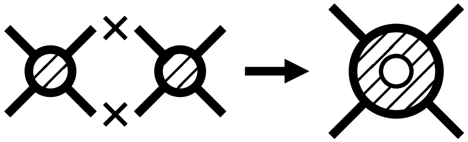
- Leading order photon scattering
- IR/UV finite
- Massless internal fermion loop
- Furry's theorem $\Rightarrow A_n^{(1)} = 0$ for n odd

- Explicit computations of one-loop photon amplitudes:

$0 \to 4\gamma$	$0 \to 6\gamma$	$0 \to n \geq 8\gamma$ MHV
Karplus, Neuman (1950)	Ossola,Papadopoulos,Pittau (2006) Gehrmann,Heinrich,Mastrolia,Binoth (2006)	Mahlon (1994)
I_4, I_3, I_2, R	I_4, I_3^{3m}	I_4

Cancellations originate from permutation sums

Motivations for On-Shell Constructions

- On-shell quantities are simpler than Feynman representations
- Construct S-matrix elements using analytic properties
- New insight from Twistor Space [Witten (2003)]
  - Use amplitudes as fundamental building blocks
  - Make use of complex momenta
  - Recursion:  [Britto,Cachazo,Feng,Witten (2004)]
  - Unitarity:  [Bern,Dixon,Dunbar,Kosower (1994)] [Bern,Dixon,Kosower (1997)] [Britto,Cachazo,Feng (2004)]
- Recent advances are developing into practical tools for calculations

Integral Reduction At One-Loop

- General expression for one-loop amplitude:

$$I_n[\{q_i \cdot l\}^n] = \int \frac{d^D l}{(2\pi)^D} \frac{\prod_{i=1}^n q_i \cdot l}{\prod_{i=1}^n (l-k_i)^2 - m_i^2}$$

- Using Passarino-Veltman reduction: cancel one propagator per tensor power

$$I_n[\{q_i \cdot l\}^m] \to \sum_{k \in i} c_k I_{n-1}[\{q_k \cdot l\}^{m-1}]$$

$\Rightarrow$ basis of scalar integrals with $n \leq 4$

- Two-derivative coupling in Gravity $\Rightarrow$

$$I_n[\{q_i \cdot l\}^{2n}] = \int \frac{d^D l}{(2\pi)^D} \frac{\prod_{i=1}^{2n} q_i \cdot l}{\prod_{i=1}^n (l-k_i)^2}$$

World Line Approach

[Bern,Kosower (1991);Strassler (1992);Bern,Dunbar,Shimada (1993); Dunbar,Norridge (1994),Schubert (1994)]

- Colourless gauge theories have additional on-shell cancellations
- Natural in string-based constructions: [Bjerrum-Bohr,Vanhove (2008)]

$$\mathcal{I}_4 \propto \int_0^\infty \frac{dT}{T} T^{-D/2+4} \int_0^1 \prod_{i=1}^{3} \nu_i \delta(\nu_4 - 1) \exp(-TQ_4)$$
$$= I_4(s,t) + I_4(s,u) + I_4(t,u)$$

- Alternative reduction: two powers per propagator

$$\mathcal{I}_n[(q_i \cdot l)^m] \sim \mathcal{I}_{n-1}[(q_i \cdot l)^{m-2}] + \mathcal{I}_n^{[D+2]}[(q_i \cdot l)^{m-2}]$$

- Sufficient cancellations prove "no-triangle" property of $\mathcal{N} = 8$ SUGRA

[Bjerrum-Bohr,Vanhove (2008)]

[Arkani-Hamed,Cachazo,Kaplan (2008)]

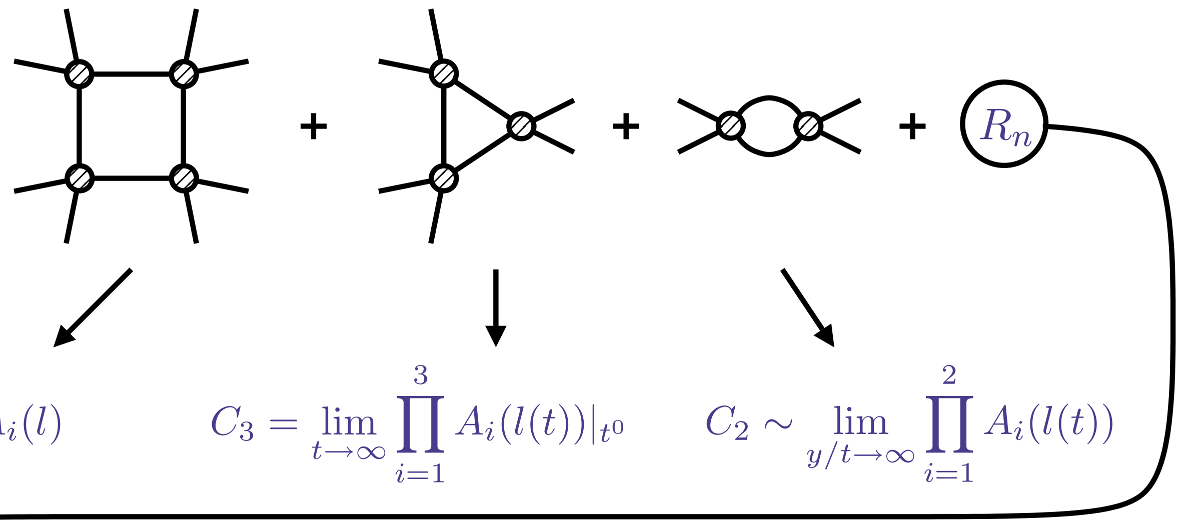
- Also explains n-photon amplitudes [SB,Bjerrum-Bohr,Vanhove (2008)]
- Let’s do it with the on-shell tools. . .

Generalised Unitarity for One-Loop Amplitudes

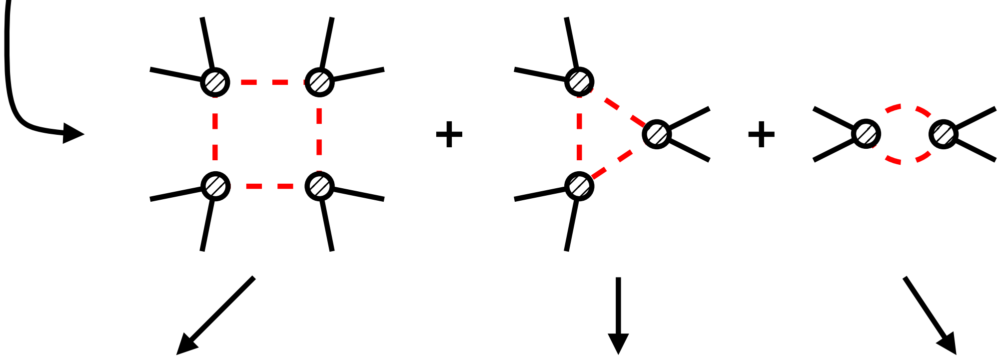
Generalised cuts isolate rational integral coefficients:

[Bern,Dixon,Dunbar,Kosower;Britto,Cachazo,Feng,Mastrolia,Yang;Ossola,Papadopoulos,Pittau;Forde]

$A_n^{(1)} =$



$$C_4 = \prod_{i=1}^{4} A_i(l) \qquad C_3 = \lim_{t\to\infty} \prod_{i=1}^{3} A_i(l(t))|_{t^0} \qquad C_2 \sim \lim_{y/t\to\infty} \prod_{i=1}^{2} A_i(l(t))$$



D-dimensional cuts:

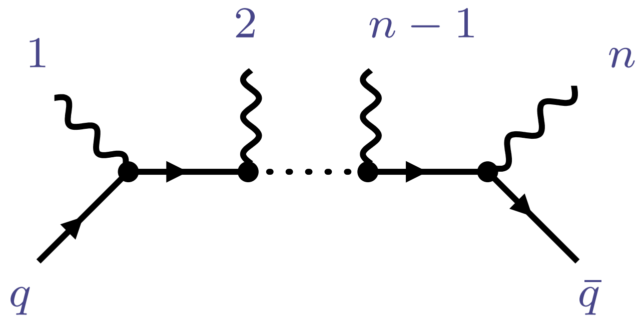
[Ossola,Papadopoulos,Pittau;

Giele,Kunszt,Melnikov;

Britto,Feng,Mastrolia;SB]

$$C_4^{[4]} = \lim_{\mu^2\to\infty} \prod_{i=1}^{4} A_i(l)|_{\mu^4} \qquad C_3^{[2]} = \lim_{\mu^2,t\to\infty} \prod_{i=1}^{3} A_i(l(t))|_{t^0,\mu^2} \quad C_2^{[2]} \sim \lim_{\mu^2,y/t\to\infty} \prod_{i=1}^{2} A_i(l(t))$$

Tree-level $q\bar{q} + n$-Photon



- Simple, arbitrary helicity, representation:

$$A_{n+2}(q^-,\bar{q}^+;1,\ldots,n) = \frac{1}{\prod_{\gamma^+}\langle \xi_k k\rangle \prod_{\gamma^-}[\xi_k k]}$$

$$\sum_{\sigma\in S_n} \langle a(1) q\rangle [\bar{q} b(n)] \prod_{i=1}^{n-1} \frac{\langle a(i+1)|q+K_{1,i}|b(i)]}{\langle q|K_{1,i}|q]}$$

- Eikonal identity:

$$\sum_{\sigma\in S_n} \frac{1}{\langle q1\rangle \left(\prod_{k=1}^{n-1}\langle k\, k+1\rangle\right)\langle n\bar{q}\rangle} = \frac{\langle q\bar{q}\rangle^{n-2}}{\prod_{k=1}^{n}\langle qk\rangle\langle \bar{q}k\rangle}$$

- Compact MHV tree level formulae [Kleiss,Stirling (1986)]

$$A_n^{(0)}(q^-,\bar{q}^+;1^-,2^+\ldots,n^+) = \frac{i\langle q1\rangle^3\langle \bar{q}1\rangle\langle q\bar{q}\rangle^{n-2}}{\prod_{\alpha=1}^{n}\langle q\alpha\rangle\langle \bar{q}\alpha\rangle}$$

Large z BCFW Scaling

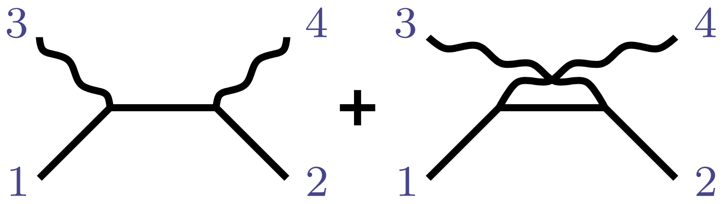
- Consider large momentum behaviour after $q, \bar{q}$ shift: [Britto,Cachazo,Feng,Witten]

$$q \to q + z|q\rangle[\bar{q}|, \qquad \bar{q} \to \bar{q} - z|q\rangle[\bar{q}|$$

$$A_n(q^h(z), \bar{q}^{(-h)}(z); 1, \ldots, n) \stackrel{z\to\infty}{\to} \frac{\mathbf{1}}{\boldsymbol{z^{n-2}}} z^{2h} C_\infty$$

[SB,Bjerrum-Bohr,Vanhove]

- Improved boundary behaviour after summation over non-planar graphs



- Scaling property is helicity independent
- QCD comparison: Boundary terms with adjacent fermion shifts

Large z BCFW Scaling

- Helicity independent: MHV example with $\langle q, \bar{q}]$ shift,

$$A_4(\widehat{q}^-, \widehat{\bar{q}}^+, 1^-, 2^+) = \langle \widehat{q}1\rangle^3 \langle \widehat{\bar{q}}1\rangle \left(\frac{1}{\langle \widehat{q}1\rangle\langle 12\rangle\langle 2\widehat{\bar{q}}\rangle\langle \bar{q}q\rangle} + \frac{1}{\langle \widehat{q}2\rangle\langle 21\rangle\langle 1\widehat{\bar{q}}\rangle\langle \bar{q}q\rangle} \right)$$

$$= \langle q1\rangle^3 (\langle \bar{q}1\rangle - z\langle q1\rangle) \left(\frac{1}{\langle q1\rangle\langle 12\rangle(\langle 2\bar{q}\rangle - z\langle 2q\rangle)\langle \bar{q}q\rangle} + \frac{1}{\langle q2\rangle\langle 31\rangle(\langle 1\bar{q}\rangle - z\langle 1q\rangle)\langle \bar{q}q\rangle} \right)$$

$$\stackrel{z\to\infty}{\rightarrow} z\langle q1\rangle^4 \left(\frac{1}{z\langle q1\rangle\langle 12\rangle\langle 2q\rangle\langle \bar{q}q\rangle} + \frac{1}{z\langle q2\rangle\langle 21\rangle\langle 1q\rangle\langle \bar{q}q\rangle} \right) + \mathcal{O}(\frac{1}{z})$$

$$= \mathcal{O}(\frac{1}{z})$$

Tree-Level Scaling to One-Loop Coefficients

- Large z scaling at tree-level relates to large momentum extraction of integral coefficients [Bern,Carrasco,Forde,Ita,Johnasson]

$$A^{(0)}_{q\bar{q}}(z) \rightarrow C_3(t), C_2(\tfrac{y^2}{t}),$$

- Tree level cancellations give direct relations for triangles:

$$\Rightarrow A_n(t) \stackrel{t\rightarrow\infty}{\longrightarrow} \frac{C_\infty}{t^{n-2}}$$

$$\Rightarrow \mathrm{Inf}_t[A_{n_1} A_{n_2} A_{n_3}] \sim \frac{1}{t^{n-6}} \qquad \Rightarrow C_3 = 0 \;\; n > 6$$

- Bubbles follow from similar analysis $\Rightarrow C_2 = 0 \;\; n > 4$

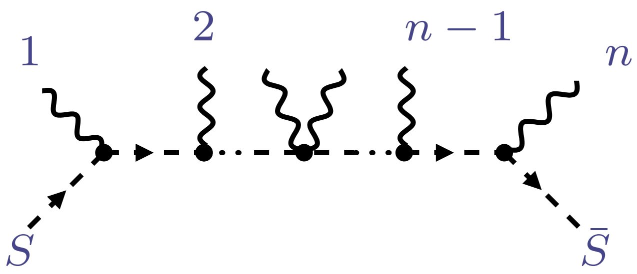
Rational Contributions

- Use D-dimensional cutting techniques
- Could use D-dimensional fermions [Ellis,Giele,Kunszt,Melnikov]
- Simpler to use heavy scalars:

$$A^{(1)}_{n(\gamma)} = A^{(1),\mathcal{N}=1}_{n(\gamma)} - A^{(1),\mathcal{N}=0}_{n(\gamma)}$$
$$\Rightarrow R_{n(\gamma)} = -R^{\mathcal{N}=0}_{n(\gamma)}$$

- Direct evaluation from large mass limit [Ossola,Papadopoulos,Pittau;SB]

Tree-Level Massive Scalar Amplitudes



- More complicated structure: 4-point interactions

- Shift for pair of massive particles [Schwinn,Weinzierl]

$$S \to S + z\left(S - \frac{\mu^2}{\gamma}\right)|S^\flat\rangle[\bar{S}^\flat|, \qquad \bar{S} \to \bar{S} - z\left(1 - \frac{\mu^2}{\gamma}\right)|S^\flat\rangle[\bar{S}^\flat|$$

$$A_n(S(z), \bar{S}(z); 1, \ldots, n) \stackrel{z\to\infty}{\to} \frac{\mathbf{1}}{\boldsymbol{z^{n-2}}} C_\infty$$

- Numerical checks for $n \leq 8$

Vanishing Of Rational Terms

- Large z behaviour

$$A^{(0)}_{S\bar{S}}(z) \to C_4^{[4]}(\mu^2), C_3^{[2]}(t,\mu^2), C_2^{[2]}(\tfrac{y^2}{t},\mu^2)$$

- Boxes follow directly from tree level scaling:

$$\Rightarrow A_n(\mu^2) \overset{\mu^2\to\infty}{\to} \frac{C_\infty}{|\mu|^{n-2}}$$

$$\Rightarrow \mathrm{Inf}_{\mu^2}[A_{n_1}A_{n_2}A_{n_3}A_{n_4}]|_{\mu^4} \sim \frac{1}{\mu^{n-4}} \qquad \Rightarrow C_4^{[4]} = 0 \ n > 4$$

- Triangles and Bubbles also shown to vanish since:

$$l^\nu \sim a_1^\nu + t a_2^\nu + \frac{a_3^\nu}{t} + \frac{\mu^2 a_4^\nu}{t} \qquad \Rightarrow C_3^{[2]}, C_2^{[2]} = 0 \ n > 4$$

No-Triangle Property of $0 \to n(\gamma)$

$$\boxed{A_n^{(1)} = \sum_{K_4} c_{4;K_4} I_{4;K_4} \quad n \geq 8}$$

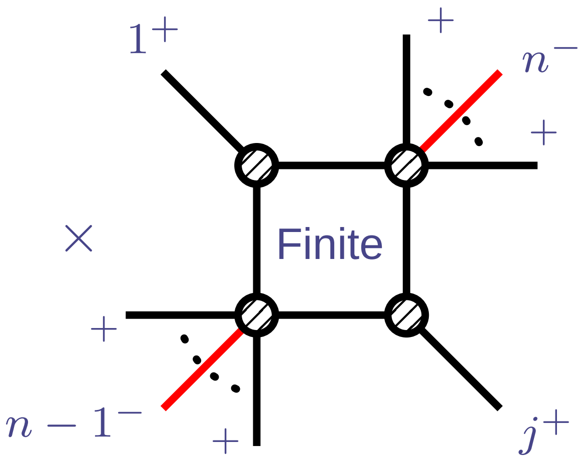
- UV/IR Finiteness $\Rightarrow$ further constraints on coefficients
- $n(\gamma)$: $\#k$-point functions $= \left.\frac{(n-1)!z^{k-1}}{2\prod_{j=1}^{k}(1-jz)}\right|_{z^{n+1}}$
- Example MHV amplitude ($n \geq 6$):

n	$\square$	$\triangle$	$\bigcirc$
4	3	6	3
6	195	90	-
8	5103	-	-
10	102315	-	-

[Mahlon (1993)]

$$A_n^{(1)}(1^+, \ldots, n-2^+, n-1^-, n^-) =$$

$$\sum_{\mathcal{P}_{n-2}} \sum_{j=1}^{n-2} c_{4;1|k_n+(2,j-1)|j|k_{n-1}+(j+1,n-2)} \quad \times$$



$$\boxed{\Rightarrow \sum_{k=1}^{n-2} \widehat{c}_{4;1|k_n+(2,j-1)|j|k_{n-1}+(j+1,n-2)} = 0}$$

Summary of Cancellations

- QCD

$$n(g) : l^n \overset{\text{P-V reduction}}{\longrightarrow} \text{boxes, triangles, bubbles and rational terms}$$

- $\mathcal{N} = 4$ SYM

$$n(g) : l^n \overset{\text{SUSY}}{\longrightarrow} l^{n-4} \overset{\text{P-V reduction}}{\longrightarrow} \text{boxes}$$

- QED

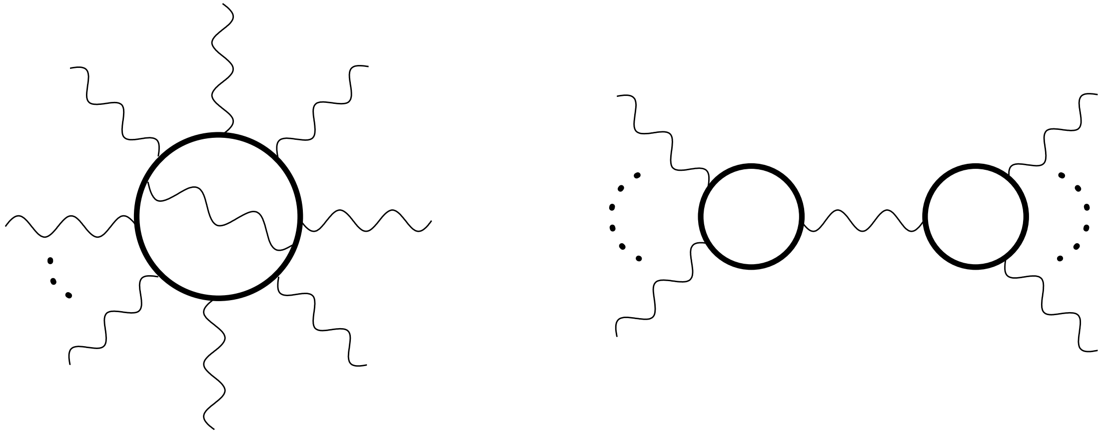
$$n(\gamma) : l^n \overset{\text{improved reduction}}{\longrightarrow} \text{boxes for } n > 6$$

- $\mathcal{N} = 8$ SUGRA

$$n(\text{grav.}) : l^{2n} \overset{\text{SUSY}}{\longrightarrow} l^{2(n-4)} \overset{\text{improved reduction}}{\longrightarrow} \text{boxes}$$

Higher Loop Amplitudes

- Expect simplifications to persist at higher loops
- No-Triangle property applies to sub-graphs at higher loops [Bern,Dixon,Roiban]
- Two-loop Amplitude is also IR finite: [Bern,De Freitas,Dixon,Ghinculov,Wong]
- Six-point two-loop amplitude



Outlook

- Uncovering hidden structures in gauge theories
- Simplicity at one-loop in unordered gauge theories
- No-triangle structure from tree-level scaling properties
- Cancellations occur faster in $\mathcal{N} = 2$ SQED
  - No-triangle property for $n = 6$ [Bernicot]
- Compact representations for NMHV tree amplitudes
  - BCFW formula do not have manifest scaling properties