Cuspygons and minimal surfaces in AdS

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Motivations

We will be interested in gluon scattering amplitudes of planar $\mathcal{N} = 4$ super Yang-Mills.

Motivation: It can give non trivial information about more realistic theories but is more tractable.

- Weak coupling: Perturbative computations are easier than in QCD. In the last years a huge technology was developed.
- The strong coupling regime can be studied, by means of the gauge/string duality, through a weakly coupled string sigma model.

Aim of this project

Learn about scattering amplitudes of planar $\mathcal{N}=4$ super Yang-Mills by means of the AdS/CFT correspondence.

Gauge theory amplitudes (Bern, Dixon and Smirnov)

$$A_n^{L,Full} \sim \sum_\rho Tr(T^{a_{\rho(1)}} \ldots T^{a_{\rho(n)}}) A_n^{(L)}(\rho(1), ..., \rho(2))$$

- Leading N_c color ordered $n-$points amplitude at L loops: $A_n^{(L)}$
- The amplitudes are IR divergent.
- Dimensional regularization $D = 4 - 2\epsilon \rightarrow A_n^{(L)}(\epsilon) = 1/\epsilon^{2L} + \ldots$
- Focus on MHV amplitudes and scale out the tree amplitude $M_n^{(L)}(\epsilon) = A_n^{(L)}/A_n^{(0)}$.

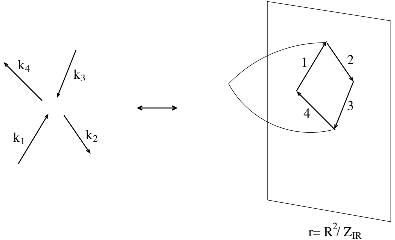
BDS proposal for all loops MHV amplitudes

$$\log \mathcal{M}_n = \sum_{i=1}^{n} \left(-\frac{1}{8\epsilon^2} f^{(-2)} \left(\frac{\lambda\mu^{2\epsilon}}{s_{i,i+1}^{\epsilon}} \right) - \frac{1}{\epsilon} g^{(-1)} \left(\frac{\lambda\mu^{2\epsilon}}{s_{i,i+1}^{\epsilon}} \right) \right) + f(\lambda) Fin_n^{(1)}(k)$$

- $f(\lambda)$, $g(\lambda) \rightarrow$ cusp/collinear anomalous dimension.
- Fine for $n = 4, 5$, not fine for $n > 5$.

String theory set up

- Such amplitudes can be computed at strong coupling by using AdS/CFT.
- Minimal surfaces in AdS, $ds^2 = (dy_{3+1}^2 + dr^2)/r^2$



- For each particle with momentum k^μ draw a segment $\Delta y^\mu = 2\pi k^\mu$
- Concatenate the segments according to the particular color ordering.

- As in the gauge theory, we need to introduce a regulator.
- As $z_{IR} \to \infty$ the boundary of the world-sheet moves to $r = 0$.
- Vev of a Wilson-Loop given by a sequence of light-like segments!

Prescription

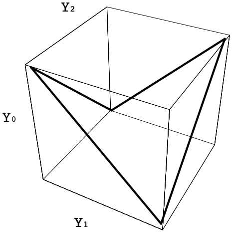
$$\mathcal{A}_n \sim e^{-\frac{\sqrt{\lambda}}{2\pi} A_{min}}$$

- $\mathcal{A}_n$: Leading exponential behavior of the $n-$point scattering amplitude.
- $A_{min}(k_1^\mu, k_2^\mu, ..., k_n^\mu)$: Area of a minimal surface that ends on a sequence of light-like segments on the boundary.

Four point amplitude at strong coupling

Consider $k_1 + k_3 \to k_2 + k_4$

- The simplest case $s = t$.



Need to find the minimal surface ending on such sequence of light-like segments

$$r(y_1, y_2) = \sqrt{(1-y_1^2)(1-y_2^2)}$$
$$y_0 = y_1 y_2$$

In embedding coordinates $(-Y_{-1}^2 - Y_0^2 + Y_1^2 + ... + Y_4^2 = -1)$

$$Y_0 Y_{-1} = Y_1 Y_2, \quad Y_3 = Y_4 = 0$$

- "Dual" $SO(2,4)$ isometries $\to$ most general solution ($s \neq t$)

Let's compute the area...

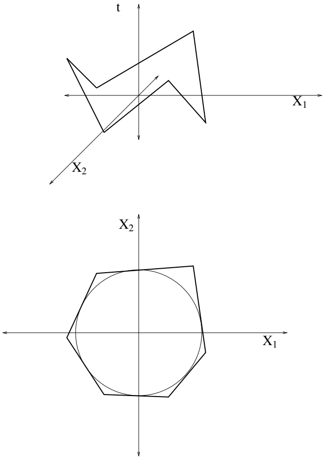
- In order for the area to converge we need to introduce a regulator.
- Dimensional reduction scheme: Start with $\mathcal{N}=1$ in D=10 and go down to $D=4-2\epsilon$.
- For integer D this is exactly the low energy theory living on $Dp-$branes ($p=D-1$)

Regularized supergravity background

$$ds^2=\sqrt{\lambda_D c_D}\left(\frac{dy_D^2+dr^2}{r^{2+\epsilon}}\right)\rightarrow S_\epsilon=\frac{\sqrt{\lambda_D c_D}}{2\pi}\int\frac{\mathcal{L}_{\epsilon=0}}{r^\epsilon}$$

- The regularized area can be computed and it agrees precisely with the BDS ansatz!
- What about other cases with $n>4$?
- for all n $SO(2,4)\rightarrow A_{strong}=A_{1-loop}+F(\frac{x_{ij}x_{kl}}{x_{ik}x_{jl}})$ (Drummond et. al.)

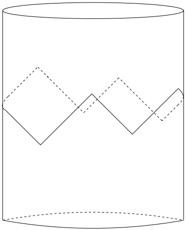
- Consider a special kinematical configuration



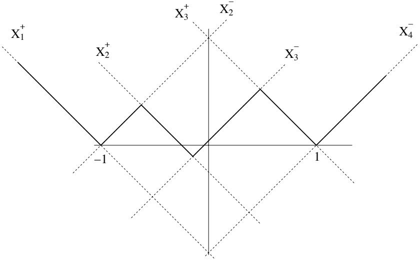
- Projection of the world-sheet to the (y_1, y_2) plane is a polygon which circumscribes the unit circle.
- Eom's and boundary conditions are consistent with $Y_3 = Y_4 = 0$.

The surface lives effectively in a AdS_3 subspace!

The scattering is equivalent to a $2D$ scattering, *e.g.* in the cylinder.



- Consider a zig-zagged Wilson loop of $2n$ sides
- Parametrized by n X_i^+ coordinates and n X_i^- coordinates.
- We can build $2n-6$ invariant cross ratios.



- Consider classical strings on AdS_3.

Strings on AdS_3

Strings on AdS_3 : $\vec{Y}.\vec{Y} = -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 = -1$

$$Eoms : \partial\bar{\partial}\vec{Y} - (\partial\vec{Y}.\bar{\partial}\vec{Y})\vec{Y} = 0, \qquad Virasoro : \partial\vec{Y}.\partial\vec{Y} = \bar{\partial}\vec{Y}.\bar{\partial}\vec{Y} = 0$$

Polhmeyer kind of reduction $\longrightarrow$ generalized Sinh-Gordon

$$\alpha(z,\bar{z}) = \log(\partial\vec{Y}.\bar{\partial}\vec{Y}), \qquad p = -e^{-\alpha}\epsilon_{abcd}\partial^2 Y^a Y^b \partial Y^c \bar{\partial} Y^d$$

$$\downarrow$$

$$p = p(z), \qquad \partial\bar{\partial}\alpha - e^{2\alpha} + |p(z)|^2 e^{-2\alpha} = 0$$

$$\mathcal{A} = \int e^{2\alpha} d^2 z$$

Generalized Sinh-Gordon → Strings on AdS_3?

$$(\partial + B^L)\psi^L_a = 0 \qquad B^L_z = \begin{pmatrix} \partial\alpha & e^{\alpha} \\ e^{-\alpha}p(z) & -\partial\alpha \end{pmatrix}$$
$$(\partial + B^R)\psi^R_{\dot a} = 0$$

Space-time coordinates

$$Y_{a,\dot a} = \begin{pmatrix} Y_{-1} + Y_2 & Y_1 - Y_0 \\ Y_1 + Y_0 & Y_{-1} - Y_2 \end{pmatrix} = \psi^L_a M \psi^R_{\dot a}$$

One can check that Y constructed that way has all the correct properties.

Relation to Hitchin equations

Consider self-dual YM in 4d reduced to 2d

- $A_{1,2} \to A_{1,2}$: 2d gauge field, $A_{3,4} \to \Phi, \Phi^*$: Higgs field.

Hitchin equations

$$F^{(4)} = *F^{(4)} \qquad \to \qquad \begin{array}{l} D_{\bar{z}}\Phi = D_z\Phi^* = 0 \\ F_{z\bar{z}} + [\Phi, \Phi^*] = 0 \end{array}$$

- We can decompose $B = A + \Phi$.
- $dB + B \wedge B = 0$ implies the Hitchin equations.
- We have a particular solution of the $SU(2)$ Hitchin system.

- Classical solutions on $AdS_3 \quad \rightarrow \quad p(z), \alpha(z, \bar{z})$

$$dw = \sqrt{p(z)}dz, \quad \hat{\alpha} = \alpha - \frac{1}{4}\log p\bar{p} \rightarrow \partial_w \bar{\partial}_{\bar{w}} \hat{\alpha} = sinh2\hat{\alpha}$$

$n = 2$ "square" solution: $p(z) = 1, \quad \hat{\alpha} = 0$

- For the solutions relevant to scattering amplitudes we require $p(z)$ to be a polynomial and $\hat{\alpha}$ to decay at infinity.

Consider a generic polynomial of degree $n-2$

$$p(z) = z^{n-2} + c_{n-4}z^{n-4} + \ldots + c_1 z + c_0$$

- We have used translations and re-scalings in order to fix the first two coefficients to one and zero.
- For a polynomial of degree $n-2$ we are left with $2n-6$ (real) variables.
- This is exactly the number of invariant cross-ratios in two dimensions for the scattering of $2n$ gluons!

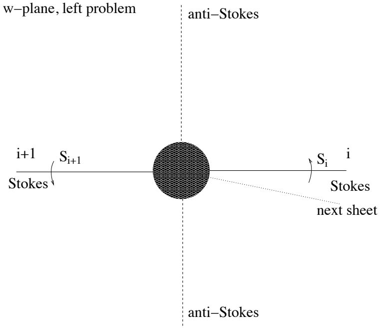
$$\text{Null Wilsons loops of } 2n \text{ sides} \Leftrightarrow P^{n-2}(z) \text{ and } \alpha(z,\bar{z})$$

Regular polygons

- Simplest case: $p(z) = z^{n-2} \rightarrow w = \frac{n}{2} z^{n/2} \quad \alpha(z,\bar{z}) = \alpha(\rho)$
- In the w plane we go around $n/2$ times.

Boundary: $|w| \gg 1 \rightarrow \hat{\alpha} \approx 0$. Gral solution of the linear problem:

$$\psi_a^L = c_a^+ \eta^+ + c_a^- \eta^-, \qquad \eta^+ = \begin{pmatrix} e^{w+\bar{w}} \\ 0 \end{pmatrix}, \quad \eta^- = \begin{pmatrix} 0 \\ e^{-(w+\bar{w})} \end{pmatrix}$$



- Focused in the left problem.
- $w-$plane divided into two regions (anti-Stokes sectors), $\pm Re(w) > 0$
- In each sector, one of the two solutions dominates.
- In the anti-Stokes lines, both are of the same order.

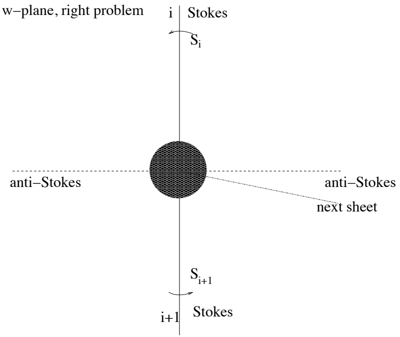
There is a Stokes phenomenon going on...

- The large solution is only defined up to a multiple of the small solution.
- Actually, as we cross the (*e.g.* the first) Stokes line...

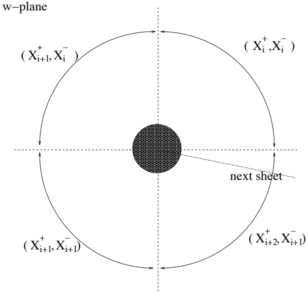
$$\begin{pmatrix} e^{w+\bar{w}} \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} e^{w+\bar{w}} \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 0 \\ e^{-(w+\bar{w})} \end{pmatrix}$$

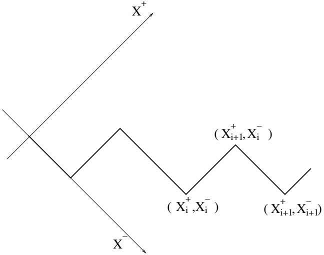
- This jump in the small component of the large solution is characterized by the Stokes matrix $S_a^b = \left(\begin{smallmatrix} 1 & \gamma \\ 0 & 1 \end{smallmatrix}\right)$
- This small component becomes important as we cross to the other anti-Stokes region.

The right-problem is similar: $\psi_a^R = c_a^+ \begin{pmatrix} e^{\frac{w+\bar{w}}{i}} \\ 0 \end{pmatrix} + c_a^- \begin{pmatrix} 0 \\ e^{-\frac{w+\bar{w}}{i}} \end{pmatrix}$



But now the Stokes and anti-Stokes lines are rotated.





- The $w-$plane is divided into quadrants.
- At each quadrant, a pair of solutions (η^L and η^R) is dominant.
- The whole region corresponds to a single point in space-time, a cusp.
- As we cross one of the anti-Stokes lines, the dominant solution L or R changes and we jump to the next cusp.
- At each step only one changes $\rightarrow$ in $R^{1,1}$ only the X^+ or X^- coordinate changes
- As we go around the $w-$plane $n/2$ times, we get the $2n$ cusps!

- The inverse map can be solved exactly.
- The solution has a Z_n symmetry and indeed corresponds to the regular polygon!
- Question: How do we compute the area?

$$A=\int e^{2\hat{\alpha}}d^2w=\int(e^{2\hat{\alpha}}-1)d^2w+\int 1d^2w=A_{sinh}+A_{div}$$

- A_{sinh} is finite, we don't need to introduce any regulator.
- A_{div} is divergent, we need to regularize it

In order to compute A_{div} use dimensional regularization...

$$A_{div}(\epsilon) = \int \frac{1}{r^\epsilon} d^2 w \approx \frac{n}{(\sin \frac{\pi}{2n})^\epsilon \epsilon^2}$$

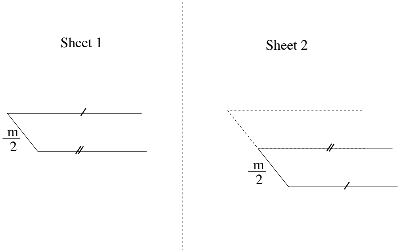
- It has the expected IR behavior!
- Since $\alpha = \alpha(|z|)$, the Sinh-Gordon equation reduces to Painleve III.
- The integrand of A_{sinh} was studied by Zamolodchikov!

$$A_{sinh} = \frac{\pi}{4n}(3n^2 - 8n + 4),$$

- When $n \to \infty$, the regular Wilson loop approaches the circular Wilson loop.
- $A_{sinh} \to \frac{3}{4}\pi n - 2\pi + \mathcal{O}(1/n)$ in agreement with the known result!

First non trivial case: $p(z) = z^2 - m$, the "octagon"

- We can split the area again into A_{sinh} and A_{div}, but...
- We don't know explicitly the solution for α...
- We cannot perform the inverse map...
- The $w-$plane is complicated...



- The information of m survives at large distances and we can compute cross-ratios vs. m

$$e^{m_r} = \frac{(x_4^+ - x_1^+)(x_3^+ - x_2^+)}{(x_4^+ - x_3^+)(x_2^+ - x_1^+)}, \quad e^{m_i} = \frac{(x_4^- - x_1^-)(x_3^- - x_2^-)}{(x_4^- - x_3^-)(x_2^- - x_1^-)}$$

- A_{div} depends on the details of the solution at infinity.
- Also the locations of the cusps depend on these details.
- Q: Could we write A_{div} in terms of the position of the cusps, bypassing these details? almost!

$$A_{div} = A_{div}^{easy}(x_i) + A(\gamma^L, \gamma^R)$$

- $A_{div}^{easy}(x_i)$ is pretty simple and fixed by conformal Ward identities.
- $A(\gamma^L, \gamma^R)$ depends on the Stokes parameters in a simple way. But, its very hard to compute the Stokes parameters in terms of $m, \bar{m}$.

$A_{sinh} = A_{sinh}(m, \bar{m})$

- Connection to Hitchin equations: A_{sinh} is related to the Kahler potential of the moduli space of solutions.
- Moduli space $(\frac{solutions}{gauge\ \ transf.})$ param. by $p(z) = \prod_{i=1}^{n-2}(z - z_i)$
- In that space we can define a natural metric $g_{z_i \bar{z}_i} = \partial_{z_i}\partial_{\bar{z}_j} K$

$$A_{\text{sinh}} \approx \sum (z_i \partial_{z_i} + \bar{z}_i \partial_{\bar{z}_i}) K$$

But...How do we compute the $\gamma(m)$'s and the metric of the moduli space?!

Gaiotto, Moore and Neitzke found a very similar problem (the same Hitchin equations) in a very different context!

- Theories which arise from wrapping $D4$-branes on Riemann surfaces.
- These are some 3d theories with 8 supercharges (N=4).
- The moduli space of solutions is the same as the moduli space of the Hitchin equations in the corresponding Riemann surfaces.

For the simplest case (corresponding to our octagon) they found exact answers! for the Stokes parameters and for the metric of the moduli space!

Gathering all the terms and working a little bit...

Eight sided Wilson loop at strong coupling

$$A_{sinh} + A(\gamma^L, \gamma^R) = \frac{1}{2} \int dt \frac{\bar{m}e^t - me^{-t}}{\tanh 2t} \log\left(1 + e^{-\pi(\bar{m}e^t + me^{-t})}\right)$$

- This is the remainder function for the scattering of eight gluons (for this particular configuration)
- Correct limits as $|m| \to 0$ and $|m| \to \infty$.
- Correct analytic structure.

What have we done?

- We have given a further small step towards the computation of classical solutions relevant to scattering amplitudes at strong coupling.
- Explicit solutions for regular polygons.
- We could compute the area for the octagon, even without knowing (fully) the classical solution.

What things need to be done?

- Try to use as much as possible the technology developed by GMN.
- Physical construction connecting GMN to our problem?
- Could we compute these amplitudes at all values of the coupling?!
- What about other kind of solutions? *e.g.* correlations functions?
- Can we extend what we did to the full AdS_5?

Eight sided Wilson loop at strong coupling

$$A_{sinh}+A(\gamma^L,\gamma^R)=|m|\int_{-\infty}^{\infty}dt\frac{\sinh t}{\tanh(2t+2i\phi)}\log\left(1+e^{-2\pi|m|\cosh t}\right)$$