BCFW Recursion in Twistor Space

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Outline

1. Motivation
   - A twistor theorist's perspective
   - A field theorist's perspective
2. BCFW recursion relations in twistor space
   - The BCFW shift
   - Transforming BCFW to twistor space (& ambitwistor space)
   - The basic three-point amplitudes in twistor space
3. Amplitudes in twistor space
   - Examples: Tree-level MHV amplitudes & NMHV amplitudes
   - Twistor geometry & triangulations
   - Super-generalized unitarity in twistor space
4. Conclusions & Outlook

Motivation: For twistor theorists

1. Massless free field equation $\Box\phi(x) = 0$ has general solution

$$\phi(x) = \int \mathrm{d}^4 p \; \mathrm{e}^{\mathrm{i}p\cdot x} \, \tilde{\phi}(p) = \int \mathrm{d}^4 p \; \mathrm{e}^{\mathrm{i}p\cdot x} \, \delta(p^2)\Phi(\lambda, \tilde{\lambda})$$

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2. Likewise, on-shell free fields can be represented on twistor space

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- L_x is the line $\mu^{A'} = x^{AA'}\lambda_A$ and so $\partial/\partial x^{AA'}$ acts on $f(W)|_{L_x}$ as $\lambda_A \partial/\partial\mu^{A'}$. Thus

$$\Box\phi(x) = \int_{L_x} \langle \lambda \, \mathrm{d}\lambda \rangle \, \lambda^A \lambda_A \frac{\partial^2 f_{-2}}{\partial\mu^{A'} \partial\mu_{A'}} = 0$$

- In Lorentzian signature correspondence involves cohomology

$$\left\{ \begin{array}{c} \text{Positive energy soln of linearized eom} \\ \text{for massless field, helicity } h \end{array} \right\} \simeq H^1(\mathbb{PT}^+, \mathcal{O}(2h-2))$$

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3. *Off-shell* objects have traditionally been difficult to understand.
   - Requires either $f(Z, W)$ or smooth $(0,1)$-forms in complex twistor space.

Motivation: For field theorists

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$$A^6_{\rm NMHV} = \frac{\delta^{4|8}\left(\sum_i p_i\right)}{\langle 12\rangle\cdots\langle 61\rangle} \times R_1^{46} \quad + \quad \ldots$$

where $R_1^{46} = \dfrac{\langle 34\rangle\langle 56\rangle\ \delta^{0|4}(\langle 61\rangle\langle 45\rangle(\eta_4[56]+\eta_5[64]+\eta_6[45]))}{x_{46}^2\langle 1|x_{16}x_{63}|3]\langle 1|x_{16}x_{64}|4]\langle 1|x_{14}x_{45}|5]\langle 1|x_{14}x_{46}|6]}$ and $x_{ij} = p_i + p_{i+1} + \cdots + p_{j-1}$

- Conformal properties obscure, since $K^{AA'} = \frac{\partial^2}{\partial\lambda_A\partial\tilde{\lambda}_{A'}}$ in mom space. But in twistor space $K^{AA'} = \mu^{A'}\frac{\partial}{\partial\lambda_A}$
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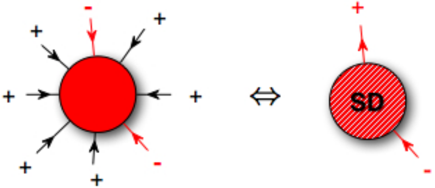
   - Twistor theory provides generating functions for all the n-particle MHV superamplitudes



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$$\sum_{n=3}^{\infty} g^{2n}\begin{pmatrix} n\text{-particle PT} \\ \text{superamplitude}\end{pmatrix} = \int \mathrm{d}^{4|8}x \log\det(\bar\partial + A)\Big|_{L_x}$$

$$\sum_{n=3}^{\infty} \kappa^{2n}\begin{pmatrix} n\text{-particle BGK} \\ \text{superamplitude}\end{pmatrix} = \int \mathrm{d}^{4|8}x \,\langle\lambda \mathrm{d}\lambda\rangle\, I\left(B, \frac{1}{\bar\partial + \mathcal{L}_V}B\right)\Bigg|_{L_x}$$

The BCFW shift in twistor space

Work in $(++--)$ signature spacetime with $\mathcal{N}=4$ susy, so twistor space is $\mathbb{RP}^{3|4}$ & on-shell fields are homogeneous *functions*

$$f(\lambda,\mu,\chi)=\int \mathrm{d}^{2|4}\tilde\lambda\, \mathrm{e}^{\mathrm{i}[\mu\tilde\lambda]}\,\Phi(\lambda,\tilde\lambda,\eta) \qquad \Phi(\lambda,\tilde\lambda,\eta)=\frac{1}{(2\pi)^2}\int \mathrm{d}^{2|4}\mu\, \mathrm{e}^{-\mathrm{i}[\mu\tilde\lambda]} f(\lambda,\mu,\chi)$$

- In Lorentzian signature, twistor theory really requires cohomology.
- Under the Fourier transform, one sees directly that

$$P_{AA'}=\lambda_A\tilde\lambda_{A'}\to \mathrm{i}\lambda_A\frac{\partial}{\partial\mu^{A'}} \quad ; \quad K^{AA'}=-\frac{\partial^2}{\partial\lambda_A\partial\tilde\lambda_{A'}}\to \mathrm{i}\mu^{A'}\frac{\partial}{\partial\lambda_A}$$

and in fact all the (super)conformal generators act geometrically on (super)twistor space as $W_I\partial/\partial W_J$.

- Consider the BCFW supershift $|\hat 1]=|1]+t|n]$, $\hat\eta_1=\eta_1+t\eta_n$, $|\hat n\rangle=|n\rangle-t|1\rangle$. It is generated by

$$\left\{\tilde\lambda_n\frac{\partial}{\partial\tilde\lambda_1}\,,\ \eta_n\frac{\partial}{\partial\eta_1}\,,\ -\lambda_1\frac{\partial}{\partial\lambda_n}\right\} \quad \text{or} \quad \left\{-\mu_1\frac{\partial}{\partial\mu_n}\,,\ -\chi_1\frac{\partial}{\partial\chi_n}\,,\ -\lambda_1\frac{\partial}{\partial\lambda_n}\right\} \quad \text{or} \quad -W_1\frac{\partial}{\partial W_n}$$

- The half Fourier transform of a shifted momentum amplitude is

$$\mathcal{A}(W_1,\ldots,W_n-tW_1)=\int\prod_{i=1}^{n}\mathrm{d}^{2|\mathcal{N}}\tilde\lambda_i\,\mathrm{e}^{\mathrm{i}[\mu_i\tilde\lambda_i]}\,A(\hat 1,\ldots,\hat n)$$

and the BCFW shift is superconformally invariant.

BCFW recursion in twistor space

1. Restore momentum-conserving δ-functions to BCFW relation.

$$A(1,\ldots,n) = \sum \int \frac{d^4p\, d^4\eta}{p^2}\, \delta^4(p_L - p)\tilde{A}_L(\hat{1},\ldots,-\hat{p})\, \delta^4(p + p_R)\tilde{A}_R(\hat{p},\ldots,\hat{n})\Big|_{t=t_*}$$

- Parametrize $p = \ell - t|1\rangle[n|$, for ℓ a variable null vector

$$\delta^4(p_L - p) = \delta^4(p_L + t\lambda_1\tilde{\lambda}_n - \ell) = \delta^4(\hat{p}_L - \ell)$$
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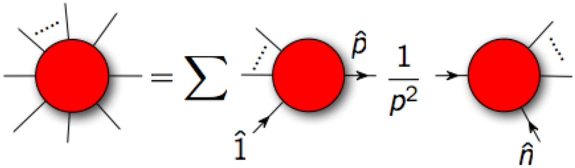

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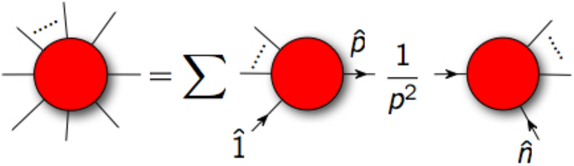

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$$\mathcal{A}(W_1,\ldots,W_n)=\sum\int D^{3|4}W\,\mathcal{A}_L(W_1,\ldots,W)\,\mathrm{sgn}(\langle W_1W\rangle[\partial_W\partial_{W_n}])\int\frac{dt}{t}\mathcal{A}_R(W,\ldots,W_n-tW_1)$$

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3. A (complete) Fourier transform $(W, W_n) \to (Z, Z_n)$ in $\mathcal{A}_R$ leads to [A-HCCK]

$$\mathcal{A}(W_1,\ldots,Z_n) = \sum \int D^{3|4}W\, D^{3|4}Z\ \mathcal{A}_L(W_1,\ldots,W)\ \mathrm{sgn}(\langle W_1 W\rangle\ W\cdot Z\ W_1\cdot Z_n\,[Z\, Z_n])\mathcal{A}_R(Z,\ldots,Z_n)$$

- Provides a systematic split-signature derivation of Hodges' twistor diagrams.

The seed amplitudes

To start the recursion, need 3-point seed amplitudes. In momentum space, these are

$$A_{\mathrm{MHV}}(1,2,3) = \frac{\delta^4(\sum |i\rangle[i|)\,\delta^{0|8}(\sum |i\rangle\eta_i)}{\langle 12\rangle\langle 23\rangle\langle 31\rangle} \qquad A_{\overline{\mathrm{MHV}}}(1,2,3) = \frac{\delta^4(\sum p)\,\delta^{0|4}(\eta_1[23]+\eta_2[31]+\eta_3[12])}{[12][23][31]}$$

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  - Manifestly conformally invariant, homogeneity zero (upto signs).
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$$\mathcal{A}_{\mathrm{MHV}}(1,2,3) = \mathrm{sgn}(\langle 23\rangle)\,\tilde{\delta}^{2|4}(W_1; W_2, W_3)$$

  - Sign factors ensure antisymmetry of kinematic factor (*cf* $\mathrm{tr}(T_1[T_2, T_3])$)

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$$\mathcal{A}_{\mathrm{MHV}}(1,2,3) = \mathrm{sgn}(\langle 23\rangle)\,\tilde{\delta}^{2|4}(W_1; W_2, W_3) \qquad \mathcal{A}_{\overline{\mathrm{MHV}}}(1,2,3) = \mathrm{sgn}([\partial_2\,\partial_3])\tilde{\delta}^{3|4}(W_1, W_2)\,\tilde{\delta}^{3|4}(W_1, W_3)$$

  - Sign factors ensure antisymmetry of kinematic factor (*cf* $\mathrm{tr}(T_1[T_2, T_3])$)

- Can also find concrete twistor formulae for $\mathcal{A}_{\overline{\mathrm{MHV}}}(1,2,3)$

Example: MHV amplitudes

1. Consider the 'homogeneous term'

$$\int D^{3|4}W\, \mathcal{A}_L(W_1, \ldots, W)\, \mathrm{sgn}(\langle W_1 W\rangle\, W_1 \cdot \partial_{W_n}[\partial_W \partial_{W_n}])\, \mathcal{A}_{\overline{\mathrm{MHV}}}(W, W_{n-1}, W_n)$$

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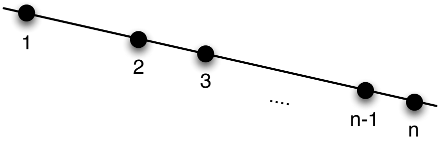
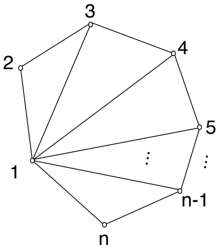
2. For *n*-particle MHV amplitudes, this is the only term

$$\mathcal{A}_{\mathrm{MHV}}(1,\ldots,n) = \mathcal{A}_{\mathrm{MHV}}(1,2,3)\,\mathcal{A}_{\mathrm{MHV}}(1,3,4)\cdots\mathcal{A}_{\mathrm{MHV}}(1,n-1,n)$$

- Amplitude is simply a *product of conformal δ-functions* (possibly with a sign)
- Can also check by direct half Fourier transform





Example: NMHV amplitudes

1. The contribution to an n-point NMHV from all but the homogeneous term is

$$\sum \int D^{3|4}W\, \mathcal{A}_{\mathrm{MHV}}(1,\ldots,i,W)\,\mathrm{sgn}(\langle 1W\rangle W_1\cdot\partial_{W_n}[\partial_W \partial_{W_n}])\,\mathcal{A}_{\mathrm{MHV}}(W,i+1,\ldots,n)$$

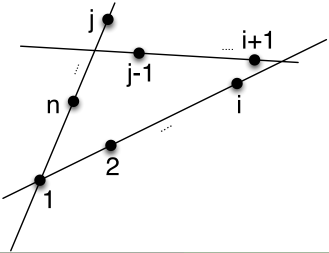
- Integral reduces to 5-point NMHV amplitude using form of MHV amplitudes:

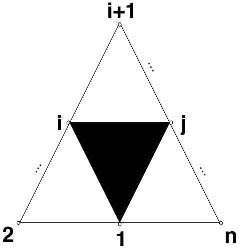
$\mathcal{A}_{\mathrm{MHV}}(1,\ldots,,W) = \mathcal{A}_{\mathrm{MHV}}(1,\ldots,i)\,\mathcal{A}_0(1,i,W)$ $\qquad$ $\mathcal{A}_{\mathrm{MHV}}(W,i{+}1\ldots,n) = \mathcal{A}_{\mathrm{MHV}}(i{+}1,\ldots,n{-}1)\,\mathcal{A}_0(i$

2. Solving the recursion gives

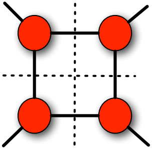
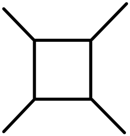
$$\mathcal{A}_{\mathrm{NMHV}}(1,\ldots,n) = \sum_{j=5}^{n}\sum_{i=2}^{j-3} \mathcal{A}_{\mathrm{NMHV}}(1,i,i+1,j-1,j) \;\times\; \mathcal{A}(1,\ldots,i)\mathcal{A}(i+1,\ldots,j-1)\,\mathcal{A}(j,\ldots,n,1)$$

- $\mathcal{A}_{\mathrm{NMHV}}(1,2,3,4,5) = \mathrm{sgn}\left(W_1\cdot\partial_{W_2}\, W_1\cdot\partial_{W_5}\,[\partial_{W_2}\,\partial_{W_5}]\right)\mathcal{A}_{\mathrm{MHV}}(2,3,4,5)$ closely related to dual superconformal invariants R.





Generalized unitarity

$A_{1\,\mathrm{loop}}(1,2,3,4) =$  $\times$  $= \int \prod_{i=1}^{4} \frac{d^4 p_i}{p_i^2} d^4\eta_i \, \delta^{(4)}(p_i + k_i - p_{i+1}) \, \tilde{A}(\hat{p}_i, k_i, -\hat{p}_{i+1})$

Parametrize off-shell loop momenta by $p_1 = \ell_1 + t_1|1\rangle[4|$, $p_2 = \ell_2 + t_2|1\rangle[2|$, $p_3 = \ell_3 + t_3|3\rangle[2|$, $p_4 = \ell_4 + t_4|3\rangle[4|$

$$\frac{d^4 p_1}{p_1^2} = d^3\ell_1 \frac{dt_1}{t_1} \,\mathrm{sgn}(\langle 1|\ell_1|4]) \quad \text{and} \quad p_1 + k_1 - p_2 = \ell_1 - \ell_2 + \hat{k}_1 \quad \text{where} \quad \hat{k}(t_1, t_2) \equiv |1\rangle\,([1| + t_1[4| - t_2[2|)$$

- Can match δ-functions to cut amplitudes (works for n-particle 1-loop MHV).
- Generalized unitarity method $\equiv$ BCFW (with loop connectivity).

$$\mathcal{A}_{1\,\mathrm{loop}} = \int D^{3|4}\{UVXY\} \prod_{i=1}^{4} \frac{dt_i}{t_i} \,\mathrm{sgn}(\langle 1U\rangle\langle 1V\rangle\langle 3X\rangle\langle 3Y\rangle)\,\mathrm{sgn}([\partial_{W_4}\partial_Y][\partial_{W_4}\partial_U])\mathcal{A}(Y, W_4 + t_4 W_3 - t_1 W_1, U)$$
$$\times\ \mathcal{A}(U, W_1, V)\,\mathrm{sgn}([\partial_{W_2}\partial_V][\partial_{W_2}\partial_X])\mathcal{A}(V, W_2 + t_2 W_1 - t_3 W_3, X)\,\mathcal{A}(X, W_3, Y)$$

- Regularization / relation to Lorentzian signature still needs to be understood.

Conclusions

1. BCFW recursion provides tools for QFT in twistor space.
   - Reveals remarkable structure: geometric pictures in twistor space and algebraic 'triangulation' picture
   - Can also study gravity: BCFW relation same. In place of KLT (product vs convolution) one has

$$\mathcal{A}_{\mathrm{MHV}}(1,2,3) = \mathrm{sgn}(\langle 23\rangle\, W_2\cdot\partial_1\, W_3\cdot\partial_1)\,\delta^{4|4}(W_1) \qquad \mathcal{A}_{\overline{\mathrm{MHV}}}(1,2,3) = \mathrm{sgn}([\partial_2\partial_3]\, W_1\cdot\partial_2\, W_1\cdot\partial_3)\,\delta^{4|4}(W_2)\delta^{4|4}(W_3)$$

$$\mathcal{M}_{\mathrm{MHV}}(1,2,3) = |\langle 23\rangle\, W_2\cdot\partial_1\, W_3\cdot\partial_1|\delta^{4|8}(W_1) \qquad \mathcal{M}_{\overline{\mathrm{MHV}}}(1,2,3) = |[\partial_2\partial_3]\, W_1\cdot\partial_2\, W_1\cdot\partial_3|\,\delta^{4|8}(W_2)\,\delta^{4|8}(W_3)$$

2. Generalized unitarity methods naturally tailored to twistor space.
   - Very important to understand regularization / continuation to Lorentzian signature.
3. At present, just imported BCFW & 3-pt amps from momentum space.
   - Would prefer a twistor derivation, perhaps from the twistor actions

$$S_{\mathcal{N}=4\ \mathrm{SYM}} = \int_{\mathbb{CP}^{3|4}} \Omega\wedge hCS(A) + g^2\int d^{4|8}x\ \log\det(\bar\partial + A)\big|_{L_x}$$

$$S_{\mathcal{N}=4\,\mathrm{SG}} = \int_{\mathbb{CP}^{3|4}} \Omega\wedge I\left(B, d(\bar\partial H + \frac{1}{2}\{H,H\})\right) + \kappa^2\int d^{4|8}x\,\langle\lambda d\lambda\rangle\ I\left(B, \frac{1}{\bar\partial + \mathcal{L}_V}B\right)$$

   or ambitwistor actions

$$S_{\mathcal{N}=3\ \mathrm{SYM}} = \int_X \Omega\wedge hCS(A) \qquad S_{\mathcal{N}=7\ \mathrm{SG}} = \int_X \Omega\wedge \widetilde{H}(\bar\partial H + \frac{1}{2}\{H,H\}) + (H\leftrightarrow \widetilde{H}) \quad ?$$

4. Lots still to do, but a cohesive twistor framework is emerging.
   - Not only interesting conceptually, but also a usable and perhaps powerful calculus.

A superconformal puzzle

The sign factors of $\mathrm{sgn}(\langle 12\rangle\langle 23\rangle\langle 31\rangle)$ or $\mathrm{sgn}([\partial_2\,\partial_3])$ in $\mathcal{A}_{\mathrm{MHV}}(1,2,3)$, $\mathcal{A}_{\overline{\mathrm{MHV}}}(1,2,3)$ and the BCFW relations seem to break conf inv. Is this really true?

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- Sign is $\pm$ according to ordering of three points along the line, so if we could orient all the twistor lines, would replace sign factor by prescription $+1$ if ordering agrees with orientation, else -1

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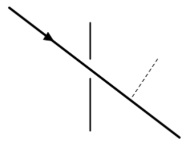

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How is this compatible with fact that amplitudes are annihilated by conformal generators?

- Signs are locally constant on momentum space
- Problem really comes from ∞ in affine spacetime, mom description invalid

Two points of view on the origin of this breaking:

- Violation is inherent in scattering theory
  - Arises at infinity in spacetime, where momentum space not valid
- Violation is artifact of $(++--)$ signature
  - Topological argument relied on real twistors
  - Factors of $\mathrm{sgn}(\langle 12\rangle\langle 23\rangle\langle 31\rangle)$ *etc* originally arose from Jacobians in δ-functions when doing Fourier transform
  - In complex twistor space, exchange properties of amplitude really inherited from antisymmetry of forms, *eg* $\int_{\mathbb{CP}^{3|4}} \Omega \wedge \mathrm{tr}(A \wedge A \wedge A)$
  - Conformal breaking might arise from 'gauge fixing' twistor cohomology classes to real twistor space
- Could a complex / Lorentzian twistor BCFW relation actually be simpler?