Theory of heavy flavor production at hadron colliders
MCNet school 09 Lund

Eric Laenen

UvA, Nikhef, UU

Family values

Six quark flavors in Nature:

$$\begin{pmatrix} \text{up} \\ \text{down} \end{pmatrix}, \quad \begin{pmatrix} \text{charm} \\ \text{strange} \end{pmatrix}, \quad \begin{pmatrix} \text{top} \\ \text{bottom} \end{pmatrix},$$

Part of *families* of elementary fermions

$$\begin{pmatrix} u \\ d \\ \nu_e \\ e \end{pmatrix} \begin{pmatrix} c \\ s \\ \nu_\mu \\ \mu \end{pmatrix} \begin{pmatrix} t \\ b \\ \nu_\tau \\ \tau \end{pmatrix}$$

Why there are three families, we do not know. (There may be more.)
The discovery of the heavy quarks c, b led each time to major leap in our understanding of fundamental physics.

Charm

c quark: first heavy quark discovered (1974), via $|c\bar{c}\rangle$ bound state, the J/ψ.
"Heavy": $m_c \gg m_\pi \simeq 140$ MeV, the lightest meson.

$$m_c \simeq \frac{1}{2} m_{J/\Psi} \simeq \frac{1}{2} 3.1 \,\mathrm{GeV}$$

Its discovery

- completed the *second* fermion family
- made the description of the electro-weak forces as a non-abelian gauge theory based on $SU_{I_W}(2) \times U_Y(1)$ consistent (and led to the name Standard Model)
- cemented belief in QCD

1975: τ lepton ($m_\tau = 1770$ MeV) found, the first member of the 3rd family.
1977: b quark: found at Fermilab via $|b\bar{b}\rangle$ bound state, with mass

$$m_b \simeq 5\,\mathrm{GeV}$$

Its discovery

- pointed to the existence of a complete new family, (and thus to the existence of the top quark)
- which would allow CP violation ($\sim$ matter-antimatter discrepancy in Universe), experimentally observed, to be incorporated in the Standard Model

Many b's produced at Tevatron, instrumental in top discovery, B_s mixing, etc..

Top

t quark, long expected, was finally discovered in 1995 by the D0 and CDF experiments at Tevatron proton-anti-proton collider.
It is surprisingly heavy:

$$m_t = 173.1 \pm 1.3\,\mathrm{GeV}$$

($\sim$ mass of 1 Gold atom)
Its discovery

- completed the 3rd generation
- validated again "pre-discovery" via quantum effects (now used for Higgs search)

To date, it has not *yet* led to fundamental new insights.

Top

Top is particularly interesting because

- it couples to many other particles, through various coupling structures
- large top mass $m_t = y_t\, v$ is signal of strong coupling to Higgs condensate
- large mass leads to rapid decay, before any hadronization. Only "bare" quark. Good for precision studies
- rapid decay preserves spin information

It plays many roles

- trouble maker for SM: quadractic divergences to Higgs mass
- enabler for MSSM, Little Higgs

LHC is a top factory. Top is the new bottom..

It is clearly important to understand the production (and decay) mechanisms of heavy quarks.
In two lectures, I will talk about a number of aspects of producing heavy flavors in hadron collisions. Some of these are generic aspects of hadron colliders, some are peculiar to heavy flavors.

First lecture:

- Lowest order, parton model
- Next-to-leading order, and all the fun that comes with that
- Heavy quark decay

Second lecture:

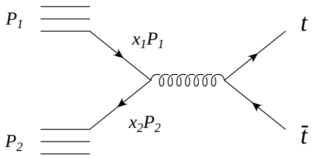
- Initial state heavy quarks
- More final state: fragmentation
- Resummation for heavy quarks: k_T, p_T, threshold

Not discussed:

- Heavy quark mass Andre Hoang
- Matching heavy flavor production to parton showers Paolo Nason
- Quarkonium production
- Small-x production

Factorization is the second major ingredient to make QCD a predictive theory.

Recall the essence of the parton model:



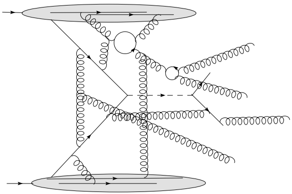
$$\sigma_{AB\to X}(Q) = \sum_{ab}\int d\xi_1 \int d\xi_2 \times \phi_{a/A}(\xi_1)\, \phi_{b/B}(\xi_2) \hat{\sigma}_{ab\to X}(\xi_1,\xi_2,Q)$$

Key notions:
i) Lorentz contraction of protons
ii) Time dilations of dynamics
→ protons are static disks that only "see" eachother when they overlap.

Hadronic cross section is weighted sum ($\int d\xi_1 d\xi_2$, ξ_i are momentum fractions) of partonic cross sections.

The parton distributions $\phi_{a/A}(\xi)$ are *assumed* to be process independent.

With quantum QCD, things look bad at first:



This does not look at all like the parton model. However, one can show, using gauge invariance and unitarity, that for QCD, to all orders in perturbation theory

$$\sigma_{AB\to X}(Q) = \sum_{ab} \int d\xi_1 \int d\xi_2 \phi_{a/A}(\xi_1,\mu)\, \phi_{b/B}(\xi_2,\mu) \times \hat{\sigma}_{ab\to X}(\xi_1,\xi_2,\mu,Q,\alpha_s(\mu)) + \underbrace{\mathcal{O}\left(\frac{1}{Q^p}\right)}_{\text{power corrections}}$$

Collins, Soper, Sterman; Bodwin

Parton model formula survives! Note

- it separates short-distance from long-distance physics
- the introduction of an arbitrary (factorization) scale μ
- the power corrections
- the α_s dependendence of the partonic cross section
- the μ dependence of the parton distributions function $\phi_{a/A}(\xi,\mu)$
- the $\phi_{a/A}(\xi,\mu)$ are *shown* to be process-independent

To make predictions, one must

- compute $\hat{\sigma}_{ab\to X}$ to sufficient high order
- know the $\phi_{a/A}(\xi,\mu)$ as accurately as possible

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- is very interesting for QCD’ers
- is quite interesting for BSM physics seekers
- (perhaps not so interesting for B-physicists)

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Mission

Obtain best possible description of heavy quark production in hadron colliders

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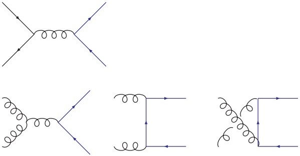
It is important

- to stress-test Standard Model in promising sectors (top physics)
- signals and backgrounds: many SM and BSM signals involve heavy quarks (b-jets)
- whetstone to sharpen our QFT tools (case in point: b-quarks at Tevatron, later)

Two *partonic* subprocesses/channels.

$$q\bar{q}: \qquad q(k_1) + \bar{q}(k_2) \to Q(p_1) + \bar{Q}(p_2)$$
$$gg: \qquad g(k_1) + \bar{g}(k_2) \to Q + \bar{Q}$$
$$s = 2k_1 k_2 \quad t_1 = -2k_1 p_1 \quad u_1 = -2k_2 p_1$$



Square amplitude and sum/average over spin, color:

$$q\bar{q}: \qquad \frac{\pi\alpha_s^2 C_F}{N_c}\left[\frac{t_1^2 + u_1^2}{s^2} + \frac{2m^2}{s}\right]$$
$$gg: \qquad \frac{\pi\alpha_s^2}{N_c(N_c^2-1)}\left[C_F - C_A\frac{t_1 u_1}{s^2}\right]\left[\frac{u_1}{t_1} + \frac{t_1}{u_1} + \frac{4m^2 s}{t_1 u_1}\left(1 - \frac{m^2 s}{t_1 u_1}\right)\right]$$

$$s = 2k_1k_2 \quad t_1 = -2k_1p_1 \quad u_1 = -2k_2p_1$$

Remarks:

- $k_1 = x_1 P_1, \;\; k_2 = x_2 P_2, \quad 0 < x_1, x_2 < 1.$
- Kinematics
  - $p_T^2 = t_1 u_1/s - m^2$
  - $y = \frac{1}{2} \ln(u_1/t_1)$
- Calculated using QCD Feynman rules, but this is not enough at higher orders: quarks/gluons are not asymptotic states, which will lead to divergences.

Cross sections link QFT and experiment

$$\sigma = \frac{1}{2\,s}\overline{\sum_{s_i,c_i}}\int \frac{d^3p_1}{2E_1}\ldots\frac{d^3p_m}{2E_m}\delta^{(4)}(q_1+q_2-\sum_i p_i)\times|M(q_1,q_2,p_i,s_i,\mu,g(\mu))|^2$$

Trends

- Compute $|M|^2$ analytically to higher orders in g^2, for ever higher n
  - Compute M analytically, at fixed spins/helicities s_i; square and sum numerically
- Do phase space integrations numerically if possible
- (Interface with MC at leading and next to leading n, lectures by Paolo Nason)

Best observables: fully differential cross sections

$$\frac{d^{3n}\,\sigma}{d^3p_1\ldots d^3p_n}, \quad 2\rightarrow \mathrm{n\ scattering}$$

in perturbation theory. They allow

- numerical, flexible integration over regions of phase space where experiment X has a detector $\rightarrow$ better comparison, less theoretical uncertainty
- generation different observables by selective integration over different variables $(p_T, y, \ldots)$

E.g., or $pp \rightarrow B + \mathrm{jet} + X$

$$\int_{p^B_{T,min}}^{p^B_{T,max}} d^2p_T^{B-meson} \int_{y_{min}}^{y_{max}} dy^{jet} \int_{p^{jet}_{T,min}}^{p^{jet}_{T,max}} dp_T^{jet} \frac{d^4\sigma}{dy^{jet}\, d^2p_T^B\, dp_T^{jet}}$$

Before going to higher order cross sections, we can already see some interesting aspects. The phase space integral can be written as

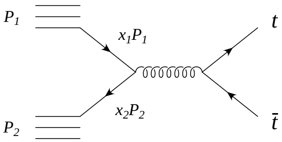
$$\frac{1}{4\pi^2}\int \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\,\delta^{(4)}(k_1+k_2-p_1-p_2) = C\int dt_1 du_1 \delta(s+t_1+u_1)$$

Carrying out this integral one finds for the $q\bar{q}$ channel

$$\sigma^{(0)}_{q\bar{q}} = KC_F^2\alpha_s^2(\mu)\sqrt{1-\frac{4m^2}{s}}\left(1+\frac{2m^2}{s}\right)$$

and something slightly longer for the gg channel. Note

- phase space suppression when $s \to 4m^2$
- scales monotonically with μ, through $\alpha_s(\mu)$
- proportional with C_F^2



Quarks transform in **3** dimensional representation, gluons in **8** dimensional rep of $\mathrm{SU}(3)$, anti-quarks in $\overline{\mathbf{3}}$. $SU(3)$ algebra of generators

$$[T_a, T_b] = if_{abc} T_c, \qquad \{T_a, T_b\} = \frac{1}{3}\delta_{ab} + d_{abc} T_c$$

When producing heavy flavors in pairs, it becomes relevant what their combined representation is (like two spin-1/2 particles).

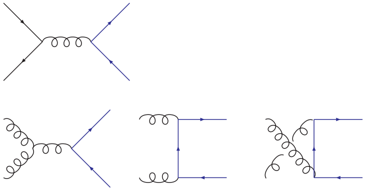
Analogs to $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} + \mathbf{3}$

$$\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} + \mathbf{8}$$
$$\mathbf{8} \otimes \overline{\mathbf{8}} = \mathbf{1} + \mathbf{8_A} + \mathbf{8_S} + \mathbf{10} + \overline{\mathbf{10}} + \mathbf{27}$$

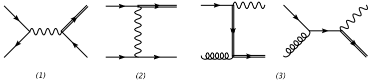
Applied to LO heavy flavor production, we see that for the $q\bar{q}$ channel only the **8** final state is allowed, and for the gg channel only $\mathbf{1}, \mathbf{8_A}, \mathbf{8_S}$. This will play a clear role when we discuss resummation.

We have been quite minimal, in that we asked for 1 heavy quark in the final state, and nothing else.

When using QCD, that requires immediately the heavy anti-quark.



When using EW, that is not true: e.g. single top quark production.

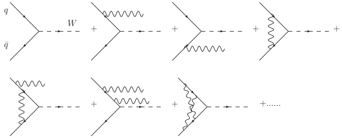


Can also ask for $Q\bar{Q}$ plus n jets at LO. Many, many more diagrams...

The perturbative expansion of a scattering amplitude in the coupling g

$$\mathcal{A}(g) = g^k \mathcal{A}^0 + g^{k+1}\mathcal{A}^1 + g^{k+2}\mathcal{A}^2 + \dots$$

corresponds to



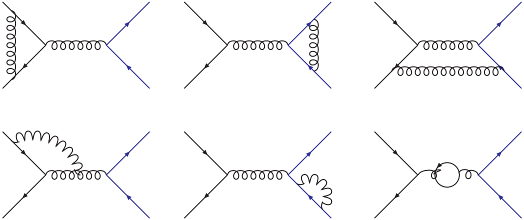
For the cross section one needs the square which reads ($\alpha = g^2$)

$$|\mathcal{A}(g)|^2 = \underbrace{\underbrace{\alpha^k |\mathcal{A}^0|^2}_{LO} + \alpha^{k+1}\left[|\mathcal{A}^1|^2 + (\mathcal{A}^2)^\dagger \mathcal{A}^0 + (\mathcal{A}^0)^\dagger \mathcal{A}^2\right]}_{NLO} + \dots$$

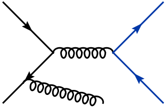
The LO and NLO approximations of $|\mathcal{A}(g)|^2$ are indicated. Note that LO is always positive, but NLO not necessarily so.

NLO diagrams for heavy quark production

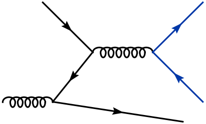
Loop diagrams (there are a good number more):



leading to lots of logs and dilogs..
Real (aka. bremsstrahlung) diagrams, include new initial states



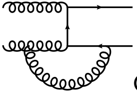
+ 4 more



+ more

The results also feature 3 kinds of divergences: UV, IR, Collinear, of the form

$$\frac{1}{\epsilon^2} \; (\mathrm{IR} \times \mathrm{CO}), \quad \frac{1}{\epsilon} \; (\mathrm{all})$$



Consider the expression for vector integral with 2 massless ($k^2 = 0$), and 1 massive propagator ($p^2 = m^2$)

$$C_\mu = \mu^{2\varepsilon} \int \frac{d^{4-2\varepsilon} l}{(2\pi)^d} \frac{l_\mu}{[l^2 + i\epsilon][(l+p)^2 - m^2 + i\epsilon][(l+k)^2 + i\epsilon]}$$

First, we use Passarino-Veltman reduction. C_μ must be expressable as

$$C_\mu = C_1 p_\mu + C_2 k_\mu$$

We must find $C_{1,2}$. Contract with p^μ and k^μ to get two equations

$$p \cdot C = m^2 C_1 + p \cdot k C_2$$
$$k \cdot C = p \cdot k C_1$$

On left hand side we have e.g.

$$\int \frac{d^{4-2\varepsilon} l}{(2\pi)^d} \frac{p \cdot l}{[l^2 + i\epsilon][(l+p)^2 - m^2 + i\epsilon][(l+k)^2 + i\epsilon]}$$

where we can write $p \cdot l = \frac{1}{2}[[(l+p)^2 - m^2 + i\epsilon] - \frac{1}{2}[l^2 + i\epsilon]$, cancelling denominators.

We are left with computing *scalar* integrals

$$C_0 = \mu^{2\varepsilon} \int \frac{d^{4-2\varepsilon} l}{(2\pi)^d} \frac{1}{[l^2 + i\epsilon][(l+p)^2 - m^2 + i\epsilon][(l+k)^2 + i\epsilon]},$$

$$B_0 = \mu^{2\varepsilon} \int \frac{d^{4-2\varepsilon} l}{(2\pi)^d} \frac{1}{[l^2 + i\epsilon][(l+k)^2 + i\epsilon]},$$

This particular B_0 is *zero* in dimensional regularization. For C_0 use

$$\frac{1}{A^\alpha B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{1}{[xA + (1-x)B]^{\alpha+\beta}}$$

twice. The $\int d^{4-2\varepsilon} l$ integral can then be done using a generic formula, e.g. in the appendix of Peskin and Schroeder. Result

$$\mu^{2\varepsilon} \int_0^1 dx \int_0^1 dy y \frac{(4\pi)^\varepsilon}{16\pi^2} \Gamma(1+\varepsilon/2) [K^2]^{-1-\varepsilon}$$

$$K^2 = xy(1-y) 2p \cdot k + (1-y)^2 m^2$$

Note that there is no UV divergence here. It remains to do the x, y integrals. These will produce IR and collinear divergences.

Such scalar integrals are not so hard, and are anyway nicely standardized Ellis, Zanderighi

$$\mu^{2\varepsilon}\int_0^1 dx\int_0^1 dyy\,\frac{(4\pi)^\varepsilon}{16\pi^2}\Gamma(1+\varepsilon/2)[K^2]^{-1-\varepsilon}$$
$$= \frac{iC_\varepsilon}{t_1}\left[\frac{1}{2\varepsilon^2}-\frac{1}{\varepsilon}\ln\left(\frac{-t_1}{m^2}\right)+\ln^2\left(\frac{-t_1}{m^2}\right)+\mathrm{Li}_2\left(\frac{t}{m^2}\right)+\frac{\pi^2}{24}\right]$$

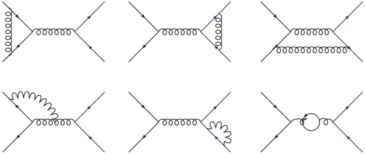
Notice

- double poles in ε: soft + collinear
- Logs, dilogs and $\pi^2 \sim \zeta_2$

For other other integrals there will be UV divergences as well; these can show up in the $\int d^{4-2\varepsilon} l$ integral, when the combined integrand behaves as $(l^2)^{-1}$ or $(l^2)^{-2}$.

Other loops

For the sum of loop corrections in

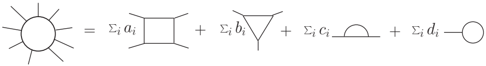


we have to deal with

- Reduction of vector and tensor integrals to scalar integrals
- Scalar integrals from boxes, triangles,bubbles, and tadpoles.

What if we wish to compute $ij \to Q\bar{Q} + g + g$ to NLO? Do we need to know penta/hexagons?

The answer is *no*. It turns out that all loop integrals in the end can be expressed in this basis. There is much recent progress in methods to find the coefficients for a given process.



Once all loop diagrams have been computed, the result takes the general form

$$d\sigma^{(1),V} = \alpha_s^3 \left(\frac{A}{\varepsilon^2} + \frac{B}{\varepsilon} + C \right)$$

- The double poles represent overlapping IR and COL singularities
- Single poles can represent IR, COL or UV singularities

Let us briefly review how the UV singularities can be consistently eliminated.

Re-interpret (renormalize) the coupling and mass parameters in the Lagrangian

$$g \to g(\mu_R)\left[1 + \frac{\alpha_s(\mu_R)}{4\pi}\left(\frac{z_g^{(1)}}{\varepsilon} + f_g\right)\right]$$

$$m \to m\left[1 + \frac{\alpha_s(\mu_R)}{4\pi}\left(\frac{z_m^{(1)}}{\varepsilon} + f_m\right)\right]$$

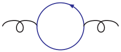
The choice of finite bits f_i determines the renormalization *scheme*. We have

$$z_g^{(1)} = -\frac{1}{2}\beta_0, \quad \beta_0 = \tfrac{1}{3}11C_A - 2n_f\,, z_m^{(1)} = -3$$

UV renormalization, cont'd

One can choose any scheme one wishes, but common is the $\overline{\mathrm{MS}}$ scheme, for which the finite bit f_i are chosen $-\gamma_E + \ln(4\pi)$. In the standard $\overline{\mathrm{MS}}$ scheme, the β-function coefficient reads

$$\beta_0 = \tfrac{1}{3} 11 C_A - 2 n_f \,,$$



where n_f *includes* the heavy quark itself. A more intuititive scheme Collins, Wilczek, Zee in which heavy quarks in loops *decouple* manifestly when momenta flowing into loop are small:

$$g \to g_R \left(1 + \tfrac{g_R^2}{32\pi^2} \left[\beta_0 \left(\tfrac{-1}{\epsilon} + \gamma_E - \ln 4\pi + \ln \frac{\mu_R^2}{\mu^2}\right) - \frac{2}{3} \ln \frac{m^2}{\mu_R^2}\right]\right.,$$

Upshot

$$\beta_0' = \tfrac{1}{3} 11 C_A - 2 n_{lf}$$

Clearly, no the heavy quark no longer contributes to α_s evolution.

Through renormalization of g and m, all UV poles can be cancelled.

Scale logs

Note that when poles cancel $\ln(\mu_R)$ terms remain.

Before continuing, let us imagine we are calculating an observable that has only UV divergences, which are removed via renormalization. The result then takes the schematic form

$$
\begin{aligned}
O(Q,\mu_R) =& \alpha_S(\mu_R)\,[O_0] + \\
& \alpha_S(\mu_R)^2 \left[O_{10} + O_{11} \ln\left(\frac{Q^2}{\mu_R^2}\right)\right] + \\
& \alpha_S(\mu_R)^3 \left[O_{20} + O_{21} \ln\left(\frac{Q^2}{\mu_R^2}\right) + O_{20} \ln^2\left(\frac{Q^2}{\mu_R^2}\right)\right] + \ldots
\end{aligned}
$$

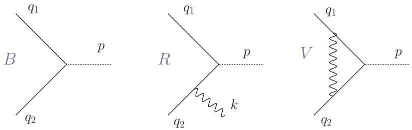
Properties

- both explicit and implicit dependence on μ_R
- O_{ij} such that $\mu_R(d/d\mu_R)O(Q,\mu_R) = \mathcal{O}(\alpha_S^4(\mu_R))$
- (demonstrates how one could determine $\alpha_S(\mu_R)$ function from data)

It remains to get of IR and collinear divergences. I would like to demonstrate how this is done using a toy model. We will see that

- IR divergences cancel when adding virtual and real corrections, due to Kinoshita-Lee-Nauenberg theorem
- one consistently removes collinear divergences by “renormalizing/factorizing” the PDF’s

$s = (q_1 + q_2)^2, \;\; p^2 = m^2, \;\; w \equiv \frac{s-m^2}{s}$



For Born and Virtual: $w = 0$. For Real: $w = 2(q_1 + q_2) \cdot k/s > 0$
Parton level computation in $4 - \epsilon$ dimensions:

$$\frac{d\sigma_B(m^2, w)}{dw} = F_B(m^2)\,\delta(w)$$

$$\frac{d\sigma_V(m^2, w)}{dw} = \alpha\left(\frac{-A}{\epsilon^2} F_B(m^2) + F_V(m^2)\right)\,\delta(w), \quad < 0$$

$$\frac{d\sigma_R(m^2, w)}{dw} = \alpha\left(\frac{-1}{\epsilon}\,\frac{K(w)}{w^{1+\epsilon}} F_B(m^2) + F_R(m^2, w)\right), \quad K(0) = A,, \quad > 0$$

How to combine? For this simple case, use identity

$$\frac{1}{w^{1+\epsilon}} = -\frac{1}{\epsilon}\delta(w) + \frac{1}{w_+} - \epsilon\left(\frac{\ln w}{w}\right)_+ + \ldots$$

Substitute:

$$\frac{d\sigma_B(m^2,w)}{dw} = F_B(m^2)\,\delta(w)$$

$$\frac{d\sigma_V(m^2,w)}{dw} = \alpha\left(\frac{-A}{\epsilon^2}F_B(m^2) + F_V(m^2)\right)\,\delta(w)$$

$$\frac{d\sigma_R(m^2,w)}{dw} = \alpha\Big(\frac{A}{\epsilon^2}F_B(m^2)\delta(w) - \frac{1}{\epsilon}\frac{K(w)}{w_+}F_B(m^2)$$
$$+ K(w)\left(\frac{\ln w}{w}\right)_+ F_B(m^2) + F_R(m^2,w)\Big)$$

Notice

- Cancellation of double poles [KLN], for IR-safe observables!
- Appearance of *double logs*
- Left-over $1/\epsilon$ term

$$\frac{d\sigma_{NLO}(m^2,w)}{dw} = F_B(m^2)\,\delta(w) + \alpha\left(\ln\left(\frac{\mu}{m}\right)\frac{K(w)}{w_+}F_B(m^2) + \right.$$
$$\left. K(w)\left(\frac{\ln w}{w}\right)_+ F_B(m^2) + F_V(m^2)\delta(w) + F_R(m^2,w)\right)$$

Plus distributions: a natural extension of the Dirac distribution. They occur quite a bit in one form or another.

Defined in integrals with smooth test functions

$$\int_y^1 dx \left[\frac{\ln^i(1-x)}{1-x}\right]_+ \phi(x) = \int_y^1 dx \left[\frac{\ln^i(1-x)}{1-x}\right] (\phi(x) - \phi(1)) - \phi(1) \int_0^y dx \left[\frac{\ln^i(1-x)}{1-x}\right]$$

The PDF's function as test functions!

If the final state has many external lines the story is more complicated, but conceptually the same. How to cancel the $1/\epsilon$ terms while keeping all external lines fixed? Integrating over w (now a multidimensional phase space integral):

$$d\sigma = F_B(m^2) + \alpha\Big(\frac{-A}{\epsilon^2}F_B(m^2) + F_V(m^2) + \int_0^{w_{max}} dw \left[\frac{-1}{\epsilon}\frac{K(w)}{w^{1+\epsilon}}F_B(m^2) \right.$$
$$\left. + F_R(m^2, w)\right]\Big)$$

Phase space slicing: Fabricius et al; Baer et al; Harris, Owens; Giele, Glover, Kosower; Keller, EL.

$$\int_0^{w_{max}} dw \frac{K(w)}{w^{1+\epsilon}} = \underbrace{\int_0^{\delta} dw \frac{K(0)}{w^{1+\epsilon}}}_{analytical} + \underbrace{\int_\delta^{w_{max}} dw \frac{K(w)}{w}}_{numerical}$$
$$= A\left(\frac{-1}{\epsilon} + \ln\delta - \epsilon\ln^2\delta\right) + \int_\delta^{w_{max}} dw \frac{K(w)}{w}$$

for small δ. Do w integral numerically!

Subtraction method Ellis, Ross, Terrano; Frixione, Kunszt, Signer; Catani, Seymour:
Find a clever $\tilde{K}(w)$ such that:

$$K(w) - \tilde{K}(w) \stackrel{w \to 0}{\longrightarrow} c\, w$$

$$\int_0^{w_{max}} dw \frac{\tilde{K}(w)}{w^{1+\epsilon}} = \frac{-A}{\epsilon}$$

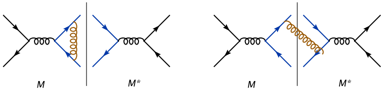
Add and subtract this term so that

$$d\sigma = F_B(m^2) + \alpha\Bigg(F_V(m^2) + \int_0^{w_{max}} dw \Bigg[\left(\frac{-1}{\epsilon}\right) \frac{K(w) - \tilde{K}(w)}{w} F_B(m^2) + F_R(m^2, w)\Bigg]\Bigg)$$

Do w integral (multidimensional phase space integral) again numerically!

A BIT MORE ABOUT IR DIVERGENCES

Occur when soft gluon is emitted, either through bremsstrahlung, or in a loop. They cancel when adding real to virtual contributions, not different from QCD corrections in e^+e^- collisions.



Cancellation automatic, no special action required

Divergences are proportional to the Born cross section, *per color structure*!

$$\sigma_{IR,ij} = \sum_C F^C_{ij}\left(\frac{1}{\epsilon}\right)\sigma^C_{ij,B}$$

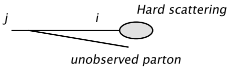
where C counts color structures

$$q\bar{q}[ij \to kl]: \qquad \delta_{ij}\delta_{kl}, \quad [\mathbf{t_a}]_{ij}\cdot[\mathbf{t_a}]_{kl}$$

$$gg[bc \to kl]: \qquad \delta_{bc}\delta_{kl}, \quad [\mathbf{t_a}]_{kl}\cdot f_{abc}, \quad [\mathbf{t_a}]_{kl}\cdot d_{abc}$$

More on this when we discuss threshold resummation.

These arise from (hard) emissions of massless partons by massless partons.



Phase space integration over the angles of the unobserved parton leads to a structure

$$\sigma_j^{\mathrm{NLO}} = \delta_{ij}\sigma_i^{(0)} + \alpha_s \tfrac{1}{\epsilon} P_{ij}^{(1)} \otimes \sigma_i^{(0)} + \alpha_s \sigma_i^{\mathrm{fin},(1)}$$

In heavy quark pair production they only occur in the initial state, and *they do not cancel*!

How to get rid of them and still maintain predictive power?

These initial state divergences can be consistently absorbed by the PDF's. Schematically, at NLO

$$\phi \otimes \left(\sigma_B + \alpha \left[\left\{ \frac{1}{\epsilon} + \ln \frac{\mu}{Q} \right\} P \otimes \sigma_B + F \right] \right) =$$

$$\underbrace{\phi \otimes \left(1 + \alpha \left\{ \frac{1}{\epsilon} + \ln \frac{\mu}{\mu_F} \right\} P \right)}_{\phi(\mu_F)} \otimes \left(\sigma_B + \alpha \left\{ \ln \frac{\mu_F}{Q} \right\} P \otimes \sigma_B + \alpha F \right)$$

This absorption is nothing else than a renormalization. By this absortion the PDF's acquire scale μ_F dependence.

Aside: PDF as operator matrix elements

This renormalization can be made more precise. A PDF is really an operator matrix element

$$\phi_{q/P}(x,\mu) = N_q \int_{-\infty}^{\infty} dy^- e^{ixy^- p^+} \langle P|\bar{\psi}_q(0,y^-,0_T) W(y_0,0)\gamma^+ \psi_q(0,0,O_T)|P\rangle$$

- $W(y_0,0)$ is Wilson line: $P\exp\left[ig_s \int_0^{y^-} dy^- A^+(y^-)\right]$, ensures gauge invariant of expression.
- Renormalization of PDF is really renormalization of non-local operators
- Moments of the splitting functions are their anomalous dimensions

Because

$$\phi(\mu_F) = \phi \otimes \Big(1 + \alpha\Big\{\frac{1}{\epsilon} + \ln\frac{\mu}{\mu_F}\Big\}P\Big)$$

we have in convolution notation

$$\frac{d\phi_i(\mu_F)}{d\ln\mu_F} = P_{ij}(\alpha_s) \otimes \phi_j(\mu_F)$$

or, in integro-differential form

$$\frac{d\phi_i(x,\mu_F)}{d\ln\mu_F} = \int_x^1 \frac{dy}{y} P_{ij}(x/y,\alpha_s)\phi_j(y,\mu_F)$$

the well-known DGLAP equation. The label i runs over *gluon*, and *active flavors*, ie. those that participate in this evolution.

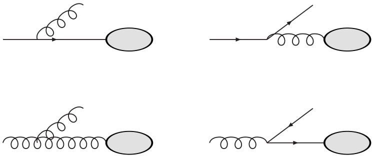
The coefficients of the collinear $1/\epsilon$ poles are the splitting functions

$$P_{qq}(x) = \tfrac{\alpha_s}{2\pi} C_F \left[\frac{1+x^2}{1-x}\right]_+ + \ldots,$$

$$P_{gq}(x) = \tfrac{\alpha_s}{2\pi} C_F \left[\frac{1+(1-x)^2}{x}\right] + \ldots,$$

$$P_{qg}(x) = \tfrac{\alpha_s}{2\pi} T_F \left[x^2 + (1-x)^2\right] + \ldots,$$

$$P_{gg}(x) = \tfrac{\alpha_s}{2\pi} 2C_A \left[\frac{x}{(1-x)_+} + ..\right] + \ldots,$$



In a heroic effort Moch, Vermaseren, Vogt, they have now been calculated to $\mathcal{O}(\alpha_s^3)$.
Necessary for *any* NNLO calculation at LHC.

Worry (?): do I not have to "absorb" a different factor into the PDF's for each different process?

No! The QCD factorization theorem says that the divergences are always the same.

$$\sigma_{AB\to Q\bar{Q}X}(S,m) = \sum_{ij}\int d\xi_1 \int d\xi_2\, \phi_{i/A}(\xi_1,\mu_F) \otimes \phi_{j/B}(\xi_2,\mu_F)$$
$$\otimes\hat{\sigma}_{ij\to Q\bar{Q}X'}(\xi_1,\xi_2,\mu_F,m,\alpha_s(\mu)) + \underbrace{\mathcal{O}\left(\frac{1}{Q^p}\right)}_{\text{power corrections}}$$

"Proven" to all orders by Collins, Soper Sterman; Bodwin.

- it separates short-distance from long-distance physics
- introduces arbitrary factorization scale μ_F
- the $\phi_{i/A}(\xi,\mu_F)$ are *shown* to be process-independent
- if $\hat{\sigma}$ to N^{k}LO, then also use N^{k}LO splitting functions for PDF evolution

Recently some real progress was made towards a possible NNLO calculation for the inclusive heavy quark cross section (Czakon, Mitov, Moch), using a trick to rewrite phase space integrals as the difference of two loop integrals.

.

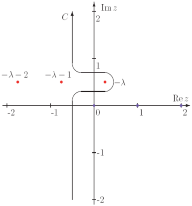
A powerful tool to carry out the difficult loops integrals is the Mellin-Barnes transform

$$\frac{1}{(A+B)^\nu} = \frac{1}{\Gamma(\nu)} \frac{1}{2\pi i} \int_C dz \frac{A^z}{B^{\nu+z}} \Gamma(-z)\Gamma(\nu+z)$$

In a sense, it is an inverse Feynman trick, allowing one to simplify integrals over more complicated denominators. E.g.

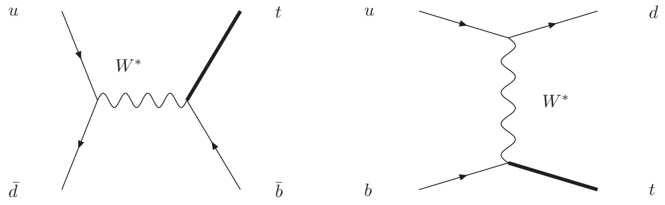
$$\frac{1}{(m^2-k^2)^\lambda} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_C dz \frac{(m^2)^z}{(-k^2)^{\lambda+z}} \Gamma(-z)\Gamma(\lambda+z)$$

One can write massive propagators in terms of massless ones! Contour



Single-top quark production

Production of top via the weak force:



s-channel $\qquad$ t-channel

This process is has recently been observed at the Tevatron, using advanced multivariate methods to separate signal from background.

Study of single-top production dynamics promises:

- direct measurement of V_{tb}, per channel
- check left-handedness of top-charged current coupling
- deduce bottom quark parton distribution function ($\rightarrow$ Higgs production)
- possible more direct manifestation of new physics (FCNC's)

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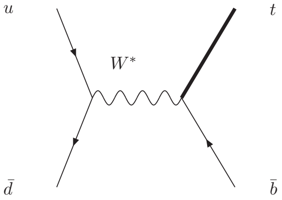
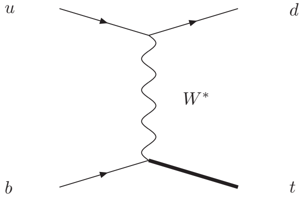


s-channel



t-channel

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Quiz on loop corrections

Are all types (box,triangle,bubble) present?

The top quark decays predominantly as follows

$$t \longrightarrow W^+ b, \qquad W^+ \longrightarrow \begin{cases} l^+ \nu \\ \bar{d} u \end{cases}$$

General formula for decays

$$\Gamma = \frac{1}{2m_t} \int dPS \overline{\sum} |M|^2$$

with $\int dPS$ the phase space integral, and $\overline{\sum}|M|^2$ the decay amplitude squared, summed (averaged) over final (initial) spins.

$$t \rightarrow W^+ b$$

where the top quark is unpolarized. We keep track of the helicity of the W-boson. Let us take the top-spin quantization axis in the z direction, and work in the top rest frame.

Heavy (top) quark decay, cont'd

Parametrize the momenta

$$\begin{aligned} p_t &= (m_t, \vec{0}) \\ p_W &= (E_W, 0, p \sin\chi_W^t, p\cos\chi_W^t), \\ p_b &= (E_b, 0, -p\sin\chi_W^t, -p\cos\chi_W^t) \end{aligned}$$

As a consequence the W boson polarization vectors are

$$\begin{aligned} \epsilon_0 &= \frac{1}{M_W}(p, 0, E_W \sin\chi_W^t, E_W \cos\chi_W^t), \\ \epsilon_{\pm 1} &= \frac{1}{\sqrt{2}}(0, 1, \pm i\cos\chi_W^t, \mp i \sin\chi_W^t) \end{aligned}$$

with the subscript indicating the helicity. Momentum conservation and mass shell conditions lead to

$$E_W = \frac{m_t^2 + M_W^2}{2m_t}, \qquad p = \frac{m_t^2 - M_W^2}{2m_t},$$

The phase space integral for this 2 particle final state can then be written as follows:

$$\begin{aligned} \int dPS_2 &= \frac{1}{4\pi^2}\int \frac{d^3 p_b}{2E_b}\int \frac{d^3 p_W}{2E_W}\,\delta^4(p_t - p_b - p_W) \\ &= \frac{1}{8\pi}\frac{m_t^2 - M_W^2}{2m_t}\int_{-1}^{1} d\cos\chi_W^t \end{aligned}$$

The decay amplitude is

$$M = \frac{-ig}{2\sqrt{2}} V_{tb}\, \bar{u}(p_b, s_b) \not{\epsilon}_\lambda (1-\gamma_5) u(p_t, s_t)$$

so that $(m_b \to 0)$

$$\sum_{s_t, s_b} |M|^2 \quad = \quad \frac{g^2}{8} |V_{tb}|^2 \, \mathrm{Tr}\Big[\not{p}_b \not{\epsilon}_\lambda (1-\gamma_5)(\not{p}_t + m_t) \not{\epsilon}^*_\lambda (1-\gamma_5)\Big]$$

Taking the trace gives

$$g^2 |V_{tb}|^2 \left(p_b^\mu p_t^\nu + p_t^\mu p_b^\nu - \eta^{\mu\nu} p_b \cdot p_t + i\, \epsilon^{\mu\nu\rho\sigma} p_{b,\rho} p_{t,\sigma}\right) \epsilon_{\lambda,\mu} \epsilon^*_{\lambda,\sigma}$$

Now substitute the explicit expression for the momentum and polarization vectors. For $\lambda = 0$ the $\epsilon^{\mu\nu\rho\sigma}$ term does not contribute, and one gets

$$g^2 |V_{tb}|^2 \left(m_t^2 \frac{m_t^2 - M_W^2}{2M_W^2} \right)$$

For $\lambda = -1$ the $\epsilon^{\mu\nu\rho\sigma}$ term *doubles* the result of other 3 terms, and one gets

$$g^2 |V_{tb}|^2 \left(m_t^2 - M_W^2 \right)$$

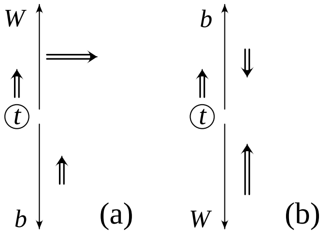
For $\lambda = +1$ the $\epsilon^{\mu\nu\rho\sigma}$ term *cancels* the other terms. So only 2 helicities of the W-boson are allowed! Ratio of helicities

$$\frac{m_t^2}{2M_W^2},$$

so 70% of the W's are longitudinally polarized. CDF has made a measurement of this

$$0.66 \pm 0.16(\mathrm{stat}) \pm 0.05(\mathrm{syst})$$

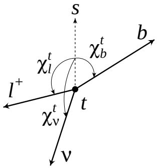
This can be intuitively understood by considering these two cases: (Mahlon)



The requirement that the b quark is left-handed forbids one helicity.
For the total width, sum over both helicities of the W, and obtain

$$\Gamma = \frac{G_F m_t^3}{8\pi\sqrt{2}} |V_{tb}|^2 \left(1 - \frac{M_W^2}{m_t^2}\right)^2 \left(1 + \frac{2M_W^2}{m_t^2}\right)$$
$$\simeq |V_{tb}|^2 \, 1.42 \,\mathrm{GeV}$$

Now let's consider polarized top quark, with the top spin quantized along s^μ (spin $= \lambda_t \hat{s}^\mu, \lambda_t = \pm 1/2$) (use z axis). Consider the angular distribution of the charged lepton from top decays, in the top quark rest frame, with respect to some axis



Use the identity

$$u(p_t, s_t)\bar{u}(p_t, s_t) = (\not{p}_t + m_t)\frac{1}{2}(1 + \gamma_5 \not{s})$$

Then the matrix element squared becomes (summing now over W helicities)

$$2\mathrm{Tr}\left[\not{p}_b \gamma^\mu \tilde{\not{p}}_t \gamma^\nu (1 - \gamma_5)\right], \qquad \tilde{p}_t = p_t - m_t s^\mu \Rightarrow \tilde{p}_t^2 = 0$$

In general, the result is of the form

$$\frac{1}{\Gamma}\frac{d\Gamma}{d\cos\chi_i^t} = \frac{1}{2}\left(1 + \alpha_i \cos\chi_i^t\right)$$

with 100% correlation if $\alpha_i = 1$ and none if 0. We find for polarized top decay to Wb 40% correlation

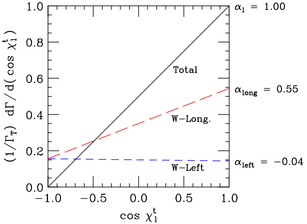
$$\alpha_W = \frac{2M_W^2 - m_t^2}{2M_W^2 + m_t^2} \simeq 0.403$$

Now look at the full decay $t \to W(\to l\nu)b$. By similar methods as above:

$$|M|^2 = g^4|V_{tb}|^2 \frac{1}{[(p_t - p_b)^2 - M_W^2]^2} (\tilde{p}_t \cdot p_l \, p_b \cdot p_\nu)$$
$$\propto (1 + \cos\chi_l^t)$$

Now 100% correlation! In general Jezabek, Kühn; Mahlon, Parke

Decay Product	α_i
W	0.403 ± 0.025
b	-0.403 ± 0.025
ν_ℓ, u, or c	-0.324 ± 0.040
$\bar{\ell}$, $\bar{d}$, or $\bar{s}$	1.000 (exactly)



The second stage decay has more correlation that first stage! This 100% correlation between charged lepton, or d quark can be used to check the top quarks charged current coupling. *Single top processes produce polarized top*

Today

- Motivation for heavy quark studies
- LO production
- NLO production
- Decay

Today

- Motivation for heavy quark studies
- LO production
- NLO production
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Tomorrow

- initial state heavy quarks
- fragmentation
- all-order results

Generic situation

$$\sigma_Q(E,m) = \sigma_0\Big(1 + \sum_n \alpha_S^n \sum_k^n c_{nk} \ln^k\left(\frac{E}{m}\right) + \mathcal{O}(\frac{m}{E})\Big)$$

For large E the mass is more regulator than phase space limitation.

- Resum large logs here through DGLAP equation
- For initial state: heavy quark PDF's (VFNS, ACOT, BSMN, etc);
- For final state: heavy quark fragmentation (FONLL, GM-VFNS)

Result

$$\sigma_Q^{res}(E,m) = \sigma_0\, \underbrace{C(E,\mu)}_{PT}\, \underbrace{E(\mu,\mu_0)}_{Exponent}\, \underbrace{f(\mu_0,m)}_{Init.cond}$$

Main issues:

- Accuracy, matching

Much activity, especially for DIS production of heavy quarks (ie. F_2^{charm}) Aivazis, Collins, Olness, Tung

$$\sigma^{ACOT}(Q,m) = \mathrm{FO} + (\mathrm{RS} - \mathrm{FOM0}) \times G$$

where $\mathrm{FOM0}$ is the zero-mass limit. Subtleties abound

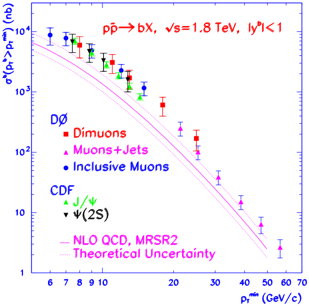
- in choice of matching point
- consistency to higher orders
- matching conditions for α_s and PDF's

To reach the high mass scale by evolution of α_s and the PDF's, one must have matching conditions at $\mu = m_c, m_b$ etc

$$\alpha_s(n_f = 4, \mu = m_c) = f[\alpha_s(n_f = 3, \mu = m_c)]$$
$$\phi_c^{(4)}(x, \mu = m_c) = F[\phi_g^{(3)}, \phi_u^{(3)}, \phi_d^{(3)}, \phi_s^{(3)}]$$

Note: from NNLO onward the transition is *discontinuous*. Bernreuther, Wentzel; Chetyrkin, Kniehl, Steinhauser; Buza, Smith, Migneron, van Neerven, Chuvakin.

A seemingly straightforward confrontation of NLO QCD with Tevatron data for years seemed to fail.



The resolution Cacciari, Nason was very instructive.

A large part was just being consistent with regard to heavy quark *fragmentation*, i.e the transition from *b*-quark to *B*-meson; so far we have only worried about producing heavy quarks.

Fragmentation functions are the “inverse” of the PDF’s, describing the parton $\rightarrow$ hadron transition.

Extend the factorization theorem

$$d\sigma(p\bar{p} \rightarrow H[Q] + X) = \sum_{ij} \int dx_1 dx_2 dz \\ \times \phi_i(x_1)\phi_j(x_2) d\hat{\sigma}^{ij\rightarrow Q+X}(x_1, x_2, z, \ldots) D_Q^{Q\rightarrow H[Q]}(z)$$

Fit FF to e^+e^- data, and use for hadron colliders.

Relatively little effort was put into construction of good FF's for heavy quarks. For most purposes, the Peterson FF

$$D_Q^H(x,\epsilon_P) = N_Q^H \frac{x(1-x)^2}{\left[(1-x)^2+\epsilon_P x\right]^2}$$

Peterson, Schlatter, Schmitt, Zerwas was used to describe $Q \to H(Q)$ transition.

The Peterson FF is the "old shoe" of fragmentation functions: comfortable, and adequate for many, but not all occasions.

A more serious treatment of heavy quark fragmentation at large p_T started in studies of heavy quark production at LEP Mele, Nason; Nason, Oleari, soon extended to hadroproduction Cacciari, Greco

When the heavy quark mass m is much smaller than the hardest scale, e.g. p_T, there are large $\ln(p_T/m)$ terms in the partonic cross section. They can be resummed into a *perturbative* fragmentation function using DGLAP evolution for its moments

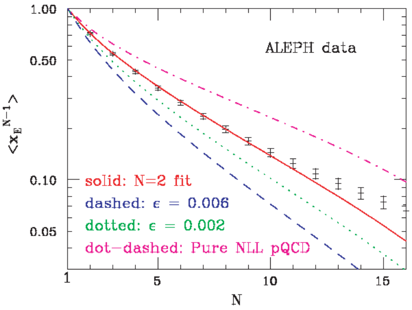
$$D_Q(N,\mu) = \int_0^1 dx\, x^{N-1} D^Q(x,\mu)$$

$$D_Q(N,p_T) \simeq D^Q(N,m) e^{\gamma_1(N)\alpha_s(\mu)\ln\frac{p_T}{m}}$$

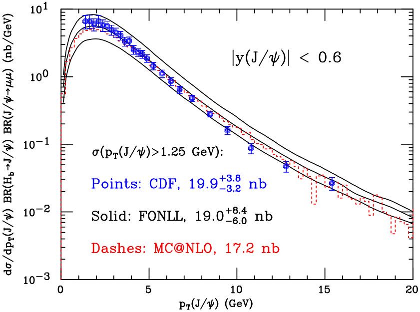
NLL accuracy requires appropiate initial condition $D^Q(N,m)$ Mele, Nason.
To confront data, the PFF must still be convoluted with a function that describes the final hadronization:

$$D_Q^H(x,\mu) = \left(D^H_{Q,\mathrm{pert}} \otimes D^H_{Q,\mathrm{NP}}\right)(x,\mu)$$

In addition, rather than fit ϵ_P, i.e. the whole FF *function*, it is better to fit only dominant moments of $D^H_{Q,\mathrm{NP}}$ to e^+e^- data. This gives already a much better description there.



As a result:



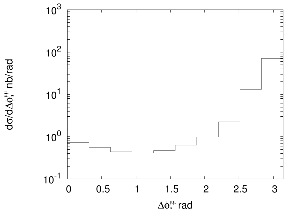
It now seems to be ok! Combination of

- proper FF use
- NLL $\ln(p_T/m)$ resummation

In this model one takes a factorization in with also the transverse momenta of initial partons are convoluted over. Jung, Baranov, Lipatov, Zotov.

$$\sigma = \int dx_1 dx_2 \int dk_{t\,1} dk_{t\,2} \mathcal{A}(x_1, k_{t\,1}, q) \mathcal{A}(x_2, k_{t\,2}, q) \hat{\sigma}(x_1, x_2, k_{t\,1} k_{t\,2}, q)$$

Description is closer to MC description, avoids too constraining kinematics at LO. Generalization to exact higher orders challenging.



$$O = \phi \otimes \hat{O} \otimes F + P_0$$

- $\hat{O}$: partonic version of observable
- ϕ/F: universal parton distribution/fragmentation functions
- P_O: power corrections

This relation can be used in various ways:

- take ϕ, F from WWW, assume $P_O = 0$, calculate $\hat{O}$, compare with O_{exp}.
- calculate $\hat{O}$ to certain approximation (assume $P_O = 0$), measure O, fit ϕ, F.

The game: find the *weakest link*, and update. Need best possible calculation of $\hat{O}$

$$\hat{O} = \alpha_s^p \sum_{n=0}^{N} \alpha_s^n \hat{O}_n + C$$

($N = 0 =$ LO, $N = 1 =$ NLO, etc).
NNLO is state of the art, very difficult.

Ideally, the (asymptotic) series converges rapidly, and $N = 1, 2$ is good enough.

But: what if series is bad? E.g. O_n can contain powers of some numerically large logarithm L.

Two generic situations:

1. Single log:

$$\hat{O}_1 = 1 + \alpha(L+1) + \alpha^2(L^2+L+1) + \alpha^3(L^3+L^2+L+1) + \ldots$$

2. Double log:

$$\hat{O}_2 = 1 + \alpha(L^2+L+1) + \alpha^2(L^4+L^3+L^2+L+1) + \ldots$$

"1": constants ($\pi^2 \ldots$). Effective expansion parameters: O_1: αL; O_2: αL^2

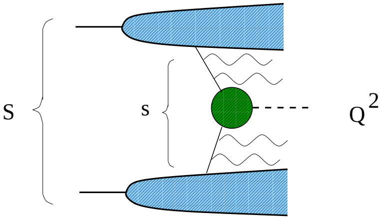
Resummation = organization of large logarithms in perturbative expansions:

$$\begin{aligned}\hat{O} &= 1+\alpha_s(L^2+L+1)+\alpha_s^2(L^4+L^3+L^2+L+1)+\ldots \\ &= \exp\left(\underbrace{\underbrace{Lg_1(\alpha_s L)}_{LL}+g_2(\alpha_s L)}_{NLL}+\alpha_s g_3(\alpha_s L)+\ldots\right)\underbrace{C(\alpha_s)}_{\text{constants}} \\ &\quad + \text{ suppressed terms}\end{aligned}$$

$L = \ln(?)$. Argument differs per observable. Benefits/hopes:

- Restore predictive power
- Increase theoretical accuracy

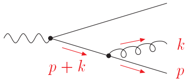
Arguments of recoil log was "visible". Invisible logs can also plague PT, e.g. in Drell-Yan $(p + \bar{p} \to \gamma^*(Q) + X)$



"Threshold" double logs: $L^2 = \ln^2\left(1 - \frac{Q^2}{s}\right)$ with $S > s > Q^2$, then

$$S \gtrsim Q^2 \quad \to \quad \frac{Q^2}{s} \simeq 1 \quad \to \quad \ln^2\left(1 - \frac{Q^2}{s}\right) \gg 1$$

NOTE: argument of L contains s parton cms energy, to be integrated over $[Q^2, S]$. But L can be large in whole integration region.

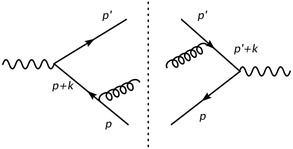


- propagator with $p^2 = k^2 = 0$. Singularities: $E_{\rm g} = 0 \longrightarrow$ *soft*; $\theta_{\rm qg} = 0 \longrightarrow$ *collinear*.

$$\frac{1}{(p+k)^2} = \frac{1}{2p\cdot k} = \frac{1}{2E_{\rm g}E_{\rm q}(1-\cos\theta_{\rm qg})}\,.$$

- phase space integration

$$\alpha_s \int \frac{d^4k}{(2\pi)^4}\,\frac{p\cdot p'}{p\cdot k\; p'\cdot k} \sim \alpha_s \int \frac{dE_{\rm g}}{E_{\rm g}} \int \frac{d\theta_{\rm qg}}{\theta_{\rm qg}} \sim \alpha_s\, \ln^2(\ldots)\,.$$



When dimensionally regulating this integral

$$\alpha_s \int \frac{d^{4-2\epsilon}k}{(2\pi)^4} \, \frac{p\cdot p'}{p\cdot k \, p'\cdot k} \sim \alpha_s \int^K \frac{dE_{\mathrm{g}}\, E_{\mathrm{g}}^{-\epsilon}}{E_{\mathrm{g}}} \int \frac{d\theta_{\mathrm{qg}} \sin^{-\epsilon}\theta_{\mathrm{qg}}}{\theta_{\mathrm{qg}}}$$

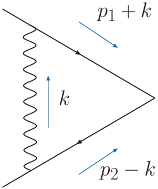
$$\sim \alpha_s \left(\frac{1}{\epsilon^2} + \ln^2(K)\right). \quad (1)$$

we see that double logs are directly linked to IR and COL divergences

How to gain all-order control over IR and COL divergences?

Follow approach of Collins, Soper, Sterman

For a general Feynman diagram, find its infrared and collinear divergences, by solving the *Landau equations*. Triangle diagram



$$\int d^4k \int d\alpha_1 d\alpha_2 \alpha_3 \frac{N\delta(1-\alpha_1-\alpha_2-\alpha_3)}{D^3},$$

$$D = \alpha_1(p_1-k)^2 + \alpha_2(p_2+k)^2 + \alpha_3 k^2 + i\epsilon \quad (2)$$

Integrals are in complex plane. Divergence when $D = 0$, *unavoidably*.

$$0 = D = \alpha_1 (p_1 - k)^2 + \alpha_2 (p_2 + k)^2 + \alpha_3 k^2 + i\epsilon$$

when, for each line, either $\alpha_i = 0$ or the line is on-shell. Quadratic equation in $k^\mu \to 2$ solutions. $D = 0$ can be *avoided* unless 2 solutions *pinch* the k^μ contours.

Landau equations:

$$D = 0, \quad \frac{\partial}{\partial k^\mu} D = 0, \quad \text{and} \quad \alpha_i = 0 \;\text{ or }\; l_i^2 = 0$$

Here

$$-\alpha_1 (p_1 - k)^\mu + \alpha_2 (p_2 + k)^\mu + \alpha_3 k^\mu = 0$$

$$-\alpha_1(p_1 - k)^\mu + \alpha_2(p_2 + k)^\mu + \alpha_3 k^\mu = 0$$

Solutions

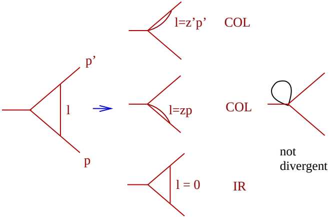
$$k^\mu = zp_1^\mu, \quad \alpha_2 = 0, \quad \alpha_1(1-z) = \alpha_3 z \qquad (\mathrm{COL}-1)$$

$$k^\mu = -yp_2^\mu, \quad \alpha_1 = 0, \quad \alpha_2(1-z) = -\alpha_3 y \qquad (\mathrm{COL}-2)$$

$$k^\mu = 0, \quad \alpha_1/\alpha_3 = 0, \quad \alpha_2/\alpha_3 = 0 \qquad (\mathrm{IR})$$

Do-able for this case, but solving equation for multiloop diagram seems very hard.

Trick: it can be done graphically, the divergences can be represented as reduced diagrams.



Reduced diagrams represent solutions to Landau equations.

S. Coleman, R. Norton '65: they must represent physically possible scatterings, with free propagation between collision points.

The solutions must be still powercounted.

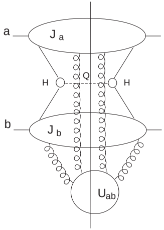
$$\frac{d^4 l}{(l^2 + i\epsilon)^2} \quad \sim \quad \text{log IR divergence}$$

Procedure:

- in multi-dimensional integration variable space, split variables near solution surface into orthogonal and in-surface coordinates
- Expand numerator and denominator to first order in orthogonal variables, and power count.

It is possible to do this explicitly to any order Sterman. (For IR solutions, this is just eikonal approximation.).

For Drell-Yan cross section e.g., all reduced diagrams look like



One can associate an explicit expression with such reduced diagrams:

$$\sum_C \int \frac{dk_A^+}{2\pi}\frac{dk_B^-}{2\pi} H^{(C)}\left(k_A^+, k_B^-\right) \prod_\ell \int \frac{d^4 q_\ell}{(2\pi)^4} \prod_j \int \frac{d^4 \bar{q}_j}{(2\pi)^4}$$

$$J_A^{(C)}\left(k_A^+, q_\ell^\alpha\right)^{\{\mu_1\ldots\mu_n\}} U^{(C)}\left(q_\ell^\alpha, \bar{q}_j^\beta\right)_{\{\mu_1\ldots\nu_1\ldots\}} J_B^{(C)}\left(k_B^-, \bar{q}_j^\beta\right)^{\{\nu_1\ldots\nu_n\}} .$$

Note:

- sum over all allowed cuts
- the integration over soft gluon momenta $\{q^\mu\}$
- the tensor-coupling of "soft function" U to "jet functions" J.
- that the H function already couples simply (some Ward identities already used)

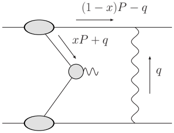
Simple case: coupling of one soft gluon (q^μ) to a fast quark with momentum $p^\mu = (E, 0, 0, E)$ or, in lightcone notation, $(p^+, 0, \mathbf{0_T})$ with $p^+ = (E+E)/\sqrt{2}$.

$$
\begin{aligned}
g\, J^\mu(p,q) \quad \underbrace{G_{\mu\nu}(q)}_{\text{gluon propagator}} \quad &\simeq \quad g\, J^+(p,q)\, G_{+,\nu}(q) \\
&\simeq \quad g\, J(p) \underbrace{\left(\frac{ip^+}{p^+ q^-}\, G_{+,\nu}(q) \right)}_{U}
\end{aligned}
$$

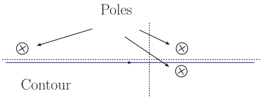
- since $p^+ \gg q^\mu$ the jet sees, both for its polarization and internal momenta, only the q^- component of the soft gluon
- $\rightarrow$ soft gluon factorizes from fast (jetlike) quark

Note: q^- must be large enough that $(p+q)^2 = 2p.q + q^2 \sim 2p^+ q^-$.

This is ok for all process with only final state jets. When there is an initial state jet (PDF), there is a potential problem.



when q^- is much smaller than q_T^2, trapped at 0.



Deform q^- contour anyway, pick up bad residues

Sum over all attachments, be careful about $i\epsilon$'s, do q^- integrals (leave q_T's fixed outside jet), use identity

$$-2\pi i\delta(x) = \frac{1}{x+i\epsilon} - \frac{1}{x-i\epsilon}$$

After summing over cuts, dangerous residues cancel, essentially by unitarity.

Result

$$d\sigma = \int \frac{dk_A^+}{2\pi}\,\frac{dk_B^-}{2\pi}\; H\left(k_A^+, k_B^-\right)\; J_A^{(C)}(k_A^+)\left(\sum_C U^{(C)}\right)\; J_B^{(C)}(k_B^-,)\,.$$

When there is plenty of energy, the sum over cuts leaves $U = 1$.

This is essentially the standard factorization for any semi-inclusive process at a hadron collider Collins, Soper, Sterman; Bodwin.

Near threshold $z = Q^2/s = 1$

$$U = \sum_C U^{(C)} \neq 1$$

because there different final states C are differently weighted. Divergences still cancel, but logs remain. Factorization theorem now reads Sterman

$$\frac{d\sigma_{ab\to V}}{dQ^2} = H(\alpha_s(Q)) \int dx_a\, \psi_{a/a}(x_a, Q) \int dx_b\, \psi_{b/b}(x_b, Q) \int dw_s\, U(w_s, \alpha_s)\delta(1 - Q^2/s - (1 - x_a) - (1 - x_b) - w_s) + Y \quad (3)$$

where ψ is PDF at fixed *energy*

$$\psi_{q/q}(x, \mu) = \int_{-\infty}^{\infty} dy^0 e^{ixy^0 p^0} \langle P|\bar{\psi}_q(y^0, \vec{0})\gamma^+ \psi_q(0, \vec{0})|P\rangle$$

The functions ψ and S can be separately resummed

- ψ via Sudakov evolution equations
- S via RG equation

Or alternatively Catani, Trentadue by observing that near threshold all radiation is soft, so that non-abelian exponentiation theorem Gatheral applies. Upshot

$$\begin{aligned} \frac{d\sigma^{\mathrm{resum}}}{dQ^2}(z) &= \int_C \frac{dN}{2\pi i}\, z^{-N}\, \hat{\sigma}(N) \\ \sigma(N) &= \exp\left[-\int_0^1 dx\, \frac{x^{N-1}-1}{1-x} \left\{ \int_{Q^2}^{Q^2(1-x)^2} \frac{d\mu}{\mu}\, A(\alpha_s(\mu)) \right.\right. \\ &\qquad \left.\left. +D(\alpha_s((1-x)Q)) \right\} \right] \times (1+\alpha_s(Q^2)\frac{C_F}{\pi}+\ldots) \end{aligned}$$

When the hard scattering can proceed in various color configurations

$$q\bar{q}[ij \to kl]: \qquad \delta_{ij}\delta_{kl}, \quad [\mathbf{t_a}]_{ij} \cdot [\mathbf{t_a}]_{kl}$$

$$gg[bc \to kl]: \qquad \delta_{bc}\delta_{kl}, \quad [\mathbf{t_a}]_{kl} \cdot f_{abc}, \quad [\mathbf{t_a}]_{kl} \cdot d_{abc}$$

the soft function is a matrix in this space, and it is resummed via

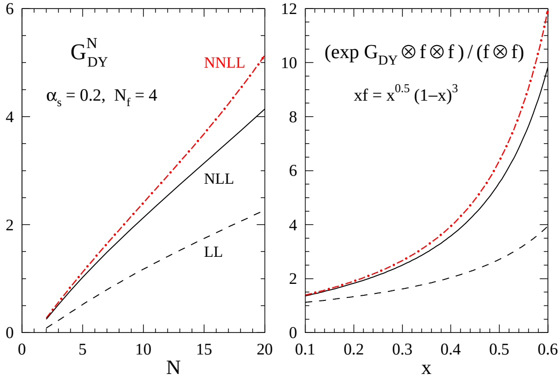
$$S_{IJ}(\frac{Q}{N\mu}, \alpha_s(\mu)) = S_{IJ}(1, \alpha_s(Q/N)) \exp\left[\int_{\mu}^{Q/N} \frac{d\mu}{\mu} 2\mathrm{Re}\,\Gamma_{IJ}(\alpha_s(\mu))\right]$$

The soft radiation is coherently sensitive to all external color states.

$$\hat{\sigma}_{DY}(N, Q^2) = g_0(Q^2)\, \exp\left[G^N_{DY}(Q^2)\right]$$

$$G^N_{DY} = \ln N g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \ldots, \qquad \lambda = \beta_0 \alpha_s \ln N$$

One sees good convergence both in the exponent and for the hadronic cross section Vogt



Status

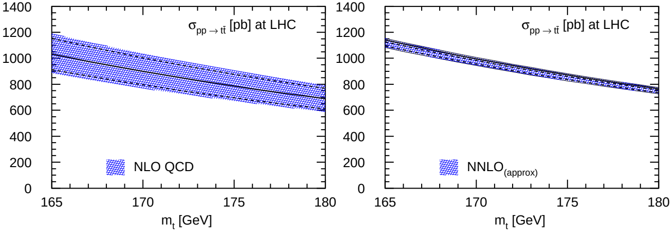
- Inclusive cross section (NLL and color structures)
- Inclusive NNLO cross section (estimates)
- NNLL color effects (Moch, Uwer; Mitov Czakon; Sterman, Aybat, Dixon)

$$\frac{\hat{\sigma}^{N}_{ij,I}(m_t,\mu_F,\mu_R)}{\hat{\sigma}^{(0),N}_{ij,I}(m_t,\mu_F,\mu_R)} = g^{0}_{ij,I}(m_t,\mu_F,\mu_R)\,\exp\left[G^{N+1}_{ij,I}(m_t,\mu_F,\mu_R)\right]$$

Benefits

- Control over higher orders, exponent functions known to second order
- Reduced scale dependence
- Sensitivity to color structures
- Can expand in α_s and generate estimates for uncalculated orders

Moch, Uwer



Using threshold resummation and RG invariance they constructed the c_i in

$$\sigma_{tt}^{(2)} = \alpha_s^2 \sum_{n=0}^{4} c_n \ln^n \sqrt{1 - \frac{4m^2}{s}}$$

and observe much less scale dependence.

Equivalent, resummed results by Cacciari, Frixione, Mangano, Nason, Ridolfi but they vary μ_F and μ_R independently, and find scale reduction not so pronounced.

Summary

- Consistent treatment of full heavy quark production mechanism important
- A high energy heavy quarks start acting like partons, needs careful matching
- Large heavy quark threshold corrections can be resummed, with good benefits