

*Parton Model and perturbative QCD*  
*Lecture I: QCD, asymptotic freedom and infrared safety*

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Keith Ellis

ellis@fnal.gov

Fermilab

# *QCD, asymptotic freedom and infrared safety*

## ■ Plan for lecture 1

- ★ Motivation for Colour SU(3)
- ★ QCD Lagrangian
- ★ Gauge Invariance
- ★ Feynman rules
- ★ Running coupling
- ★  $\beta$ -function of QCD
- ★ Measurements of  $\alpha_S$

## ■ Bibliography,

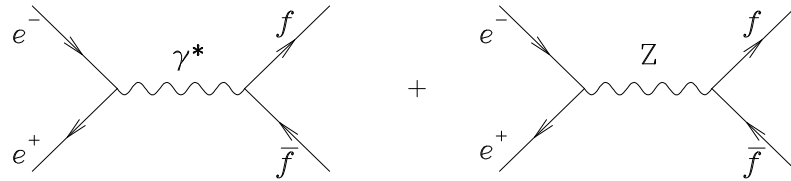
R. K. Ellis, W.J. Stirling and B.R. Webber,

*QCD and Collider Physics*

(Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology)

# Motivation for Colour $SU(3)$

- Consider the ratio  $R$  of the  $e^+e^-$  total hadronic cross section to the cross section for the production of a pair of point-like, charge-one objects such as muons.
- The virtual photon excites all electrically charged constituent-anticonstituent pairs from the vacuum.



- At low energy the virtual photon excites only the  $u$ ,  $d$  and  $s$  quarks, each of which occurs in three colours.

$$\begin{aligned} R &= N_c \sum_i Q_i^2 \\ &= 3 \left[ \left( \frac{2}{3} \right)^2 + \left( -\frac{1}{3} \right)^2 + \left( -\frac{1}{3} \right)^2 \right] = 2 . \end{aligned}$$

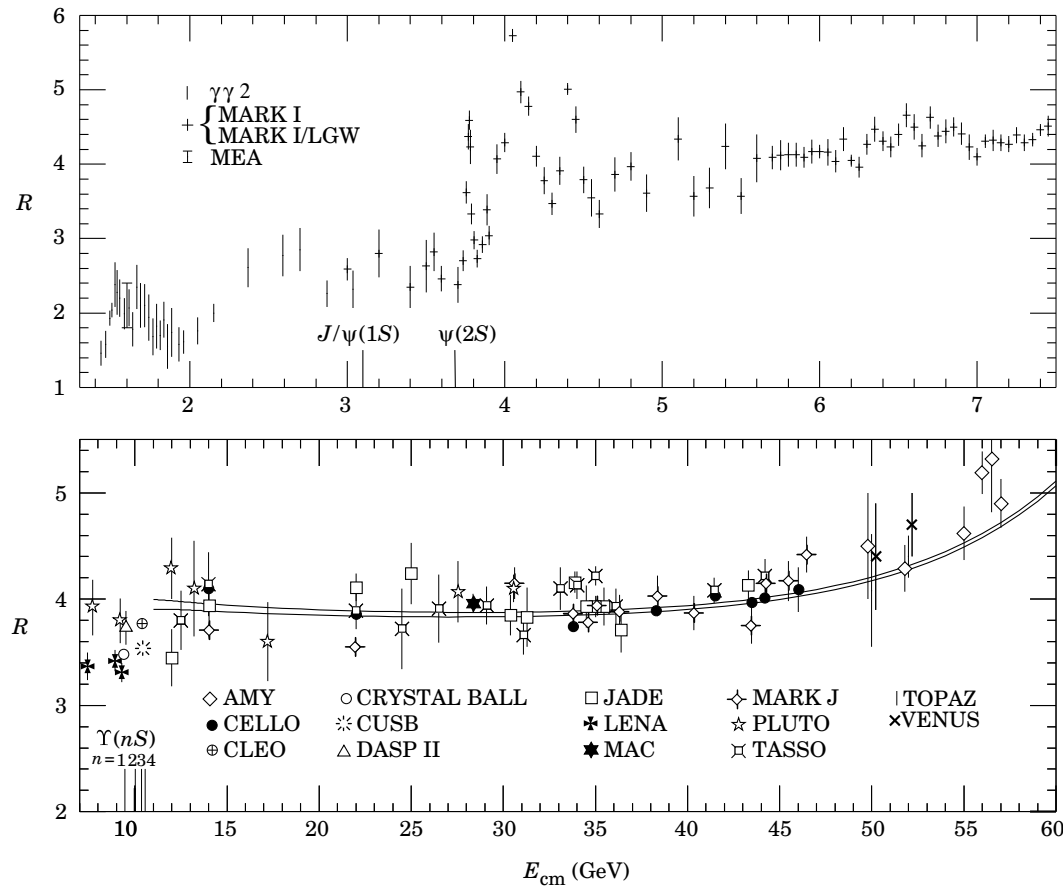
- For centre-of-mass energies  $E_{\text{cm}} \geq 10$  GeV, one is above the threshold for the production of pairs of  $c$  and  $b$  quarks, and so

$$R = 3 \left[ 2 \times \left( \frac{2}{3} \right)^2 + 3 \times \left( -\frac{1}{3} \right)^2 \right] = \frac{11}{3} .$$

# Data

The data on  $R$  are in reasonable agreement with the prediction of the three colour model.

$$R_{e^+e^-} = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$$



# Colour $SU(3)$ and spectroscopy

- The observed baryons are interpreted as three-quark states.
- The quark constituents of the baryons are forced to have half-integral spin in order to account for the spins of the low-mass baryons.
- The quarks in the spin- $\frac{3}{2}$  baryons are then in a symmetrical state of space, spin and  $SU(3)_f$  degrees of freedom.
- However the requirements of Fermi-Dirac statistics imply the total antisymmetry of the wave function.
- We introduce the colour degree of freedom: a colour index  $a$  with three possible values (usually called red, green, blue for  $a = 1, 2, 3$ ) is carried by each quark.
- The baryon wave functions are totally antisymmetric in this new index.

Quark	Charge	Mass	Baryon Number	Isospin
$u$	$+\frac{2}{3}$	$\sim 4 \text{ MeV}$	$\frac{1}{3}$	$+\frac{1}{2}$
$d$	$-\frac{1}{3}$	$\sim 7 \text{ MeV}$	$\frac{1}{3}$	$-\frac{1}{2}$
$c$	$+\frac{2}{3}$	$\sim 1.5 \text{ GeV}$	$\frac{1}{3}$	0
$s$	$-\frac{1}{3}$	$\sim 135 \text{ MeV}$	$\frac{1}{3}$	0
$t$	$+\frac{2}{3}$	$\sim 172 \text{ GeV}$	$\frac{1}{3}$	0
$b$	$-\frac{1}{3}$	$\sim 5 \text{ GeV}$	$\frac{1}{3}$	0

- The group of colour transformations is SU(3), with the quarks  $q_a$  transforming according to the fundamental representation
- Why does this new degree of freedom not lead to a proliferation of states?
- We hypothesize that only colour singlet states can exist in nature.
- For a baryon the colour singlet state is totally antisymmetric  
 $(|a\rangle|b\rangle|c\rangle + |b\rangle|c\rangle|a\rangle + |c\rangle|a\rangle|b\rangle - |b\rangle|a\rangle|c\rangle - |a\rangle|c\rangle|b\rangle - |c\rangle|b\rangle|a\rangle)/\sqrt{6}$

# Argument for $SU(3)$ singlet ground states

- Consider the force between quarks using an (over)simplified model of one gluon exchange.
- In QED:  $V \sim \frac{e^2}{r}$     In QCD:  $V \sim \frac{\lambda^{(1)} \lambda^{(2)}}{r}$
- $\lambda$  are the eight Gell-Mann matrices, the hermitean and traceless generators of  $SU(3)$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

# Interquark forces

- We will calculate the weights given by the products of the  $\lambda$  matrices, but using a physical basis

- The eight gluons couple to the colors of the quarks and can be written as

$$\bar{a}b, \bar{a}c, \bar{b}a, \bar{b}c, \bar{c}a, \bar{c}b \text{ and } (\bar{a}a - \bar{b}b)/\sqrt{2}, (\bar{a}a + \bar{b}b - 2\bar{c}c)/\sqrt{6},$$

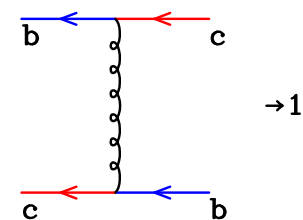
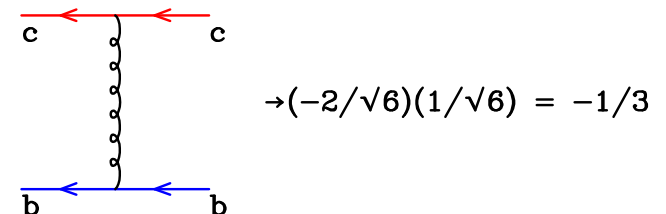
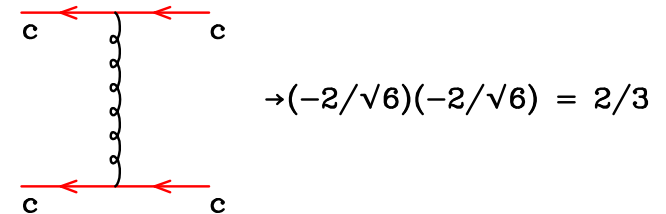
- The last two gluons are orthogonal to the SU(3) singlet gluon,  $((\bar{a}a + \bar{b}b + \bar{c}c))/\sqrt{3}$ , which is not included.

- We only have to consider two cases, forces between quarks of the same colour and of different colours.

$$\sum_A \lambda_{ab}^A \lambda_{cd}^A = \mathcal{N} \left[ \delta_{ad} \delta_{cb} - \frac{1}{N_c} \delta_{ab} \delta_{cd} \right],$$

(Normalization,  $\mathcal{N}$ )

- Introduce the colour exchange operator  $P$  which has eigenvalues,  $p=+1(-1)$  for a symmetric (antisymmetric state). Interaction energy can be written as  $E \sim (p - \frac{1}{3})$



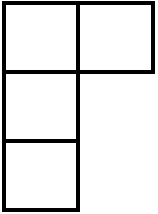
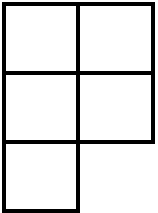
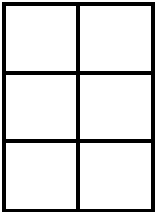


# Interaction energies

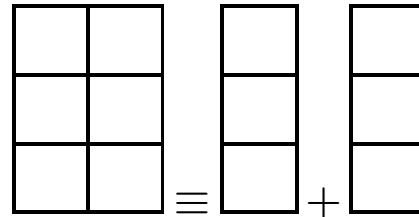
$N_q$	Young Diagram	Dimensionality	Interaction Energy	Energy $+ \frac{4}{3} N_q$
1		3	0	$\frac{4}{3}$
2		6	$1 - \frac{1}{3} = \frac{2}{3}$	$\frac{10}{3}$
2		3	$-1 - \frac{1}{3} = -\frac{4}{3}$	$\frac{4}{3}$
3		10	$3 \times 1 - 3 \times \frac{1}{3} = 2$	6
3		8	$1 + (-1) - 3 \times \frac{1}{3} = -1$	3
3		0	$3 \times (-1 - \frac{1}{3}) = -4$	0

- Add a constant self-energy per quark,  $\frac{4}{3}$  in these units, (just a book-keeping device: as long as we only compare states with the same number of quarks)
- 3-quark state, which is totally antisymmetric with respect to colour has the lowest energy: This is the baryon.
- All other three quark states have higher energy.

# Higher numbers of quarks

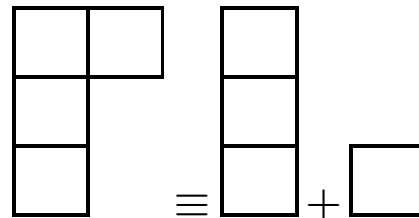
$N_q$	Young Diagram	Dimensionality	Interaction Energy	Energy $+\frac{4}{3}N_q$
4		3	$1 + 3 \times (-1) - 6 \times \frac{1}{3} = -4$	$\frac{4}{3}$
5		3	$2 \times 1 + 4 \times (-1) - 10 \times \frac{1}{3} = -\frac{16}{3}$	$\frac{4}{3}$
6		1	$3 \times 1 + 6 \times (-1) - 15 \times \frac{1}{3} = -8$	0

- Six quark state has the same energy as two baryons

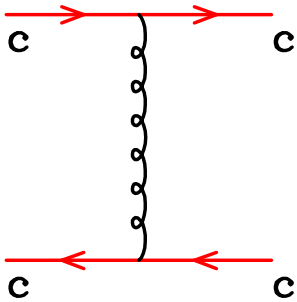


this crude approximation does not allow us to say whether the two baryon state is bound.

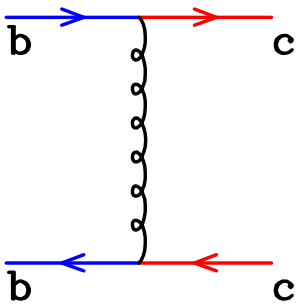
- No strong binding of a quark (or an antiquark) to a baryon



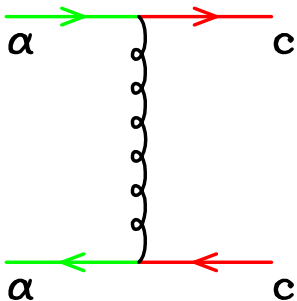
# Mesons



$$\rightarrow -(-2/\sqrt{6})^2 = -2/3$$



$$\rightarrow = -1$$



$$\rightarrow = -1$$

- For the diagonal interaction we have  $E_{c\bar{c} \rightarrow c\bar{c}} = -\frac{2}{3}$ .
- Note overall minus sign – just as in QED particle-antiparticle force is attractive.
- We have the off-diagonal interaction  $E_{c\bar{c} \rightarrow b\bar{b}} = E_{c\bar{c} \rightarrow a\bar{a}} = -1$

- Colour singlet meson is given by  $(c\bar{c} + b\bar{b} + a\bar{a})/\sqrt{3}$
- Overall is  $3 \times c\bar{c}/\sqrt{3} \times (c\bar{c} + b\bar{b} + a\bar{a})/\sqrt{3} = -8/3$
- Adding in self-energy of  $2 \times \frac{4}{3}$  we get 0
- Coloured meson gives energy of  $E_{c\bar{b} \rightarrow c\bar{b}} = \frac{1}{3}$ . Adding in the self energy of  $2 \times \frac{4}{3}$  we get 3.

- Antiquarks are represented by a column of  $N_c - 1$  boxes,  $\bar{3} = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array}$

■  $3 \otimes \bar{3} = 8 \oplus 1$ ,  $\begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \end{array} = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} + \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$

$N_q$	Young Diagram	Dimensionality	Energy $+\frac{4}{3}N_q$
2	$\begin{array}{ c } \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array}$	1	0
2	$\begin{array}{ c c } \hline \square & \square \\ \hline \square & \square \\ \hline \end{array}$	8	3

# Lagrangian of QCD

- Feynman rules for perturbative QCD follow from Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}^A F_A^{\alpha\beta} + \sum_{\text{flavours}} \bar{q}_a (i\not{D} - m)_{ab} q_b + \mathcal{L}_{\text{gauge-fixing}}$$

$F_{\alpha\beta}^A$  is field strength tensor for spin-1 gluon field  $\mathcal{A}_\alpha^A$ ,

$$F_{\alpha\beta}^A = \partial_\alpha \mathcal{A}_\beta^A - \partial_\beta \mathcal{A}_\alpha^A - gf^{ABC} \mathcal{A}_\alpha^B \mathcal{A}_\beta^C$$

Capital indices  $A, B, C$  run over 8 colour degrees of freedom of the gluon field. Third 'non-Abelian' term distinguishes QCD from QED, giving rise to triplet and quartic gluon self-interactions and ultimately to asymptotic freedom.

- QCD coupling strength is  $\alpha_S \equiv g^2/4\pi$ . Numbers  $f^{ABC}$  ( $A, B, C = 1, \dots, 8$ ) are structure constants of the SU(3) colour group. Quark fields  $q_a$  ( $a = 1, 2, 3$ ) are in triplet colour representation.  $D$  is covariant derivative:

$$(D_\alpha)_{ab} = \partial_\alpha \delta_{ab} + ig \left( t^C \mathcal{A}_\alpha^C \right)_{ab}$$
$$(D_\alpha)_{AB} = \partial_\alpha \delta_{AB} + ig (T^C \mathcal{A}_\alpha^C)_{AB}$$

- $t$  and  $T$  are matrices in the fundamental and adjoint representations of SU(3), respectively:

$$t^A = \frac{1}{2}\lambda^A, \quad [t^A, t^B] = if^{ABC}t^C, \quad [T^A, T^B] = if^{ABC}T^C$$

where  $(T^A)_{BC} = -if^{ABC}$ . We use the metric  $g^{\alpha\beta} = \text{diag}(1, -1, -1, -1)$  and set  $\hbar = c = 1$ .  $\mathcal{D}$  is symbolic notation for  $\gamma^\alpha D_\alpha$ . Normalisation of the  $t$  matrices is

$$\text{Tr } t^A t^B = T_R \delta^{AB}, \quad T_R = \frac{1}{2}.$$

- Colour matrices obey the relations:

$$\sum_A t_{ab}^A t_{bc}^A = C_F \delta_{ac}, \quad C_F = \frac{N^2 - 1}{2N}, \quad \sum_A t_{ab}^A t_{cd}^A = \frac{1}{2} \left[ \delta_{ad} \delta_{cb} - \frac{1}{N} \delta_{ab} \delta_{cd} \right]$$

$$\text{Tr } T^C T^D = \sum_{A,B} f^{ABC} f^{ABD} = C_A \delta^{CD}, \quad C_A = N$$

Thus  $C_F = \frac{4}{3}$  and  $C_A = 3$  for SU(3).

# Gauge invariance

- QCD Lagrangian is invariant under local gauge transformations. That is, one can redefine quark fields independently at every point in space-time,

$$q_a(x) \rightarrow q'_a(x) = \exp(it \cdot \theta(x))_{ab} q_b(x) \equiv \Omega(x)_{ab} q_b(x)$$

without changing physical content.

- Covariant derivative is so called because it transforms in same way as field itself:

$$D_\alpha q(x) \rightarrow D'_\alpha q'(x) \equiv \Omega(x) D_\alpha q(x) .$$

(omitting the colour labels of quark fields from now on). Use this to derive transformation property of gluon field  $\mathcal{A}$

$$\begin{aligned} D'_\alpha q'(x) &= (\partial_\alpha + igt \cdot \mathcal{A}'_\alpha) \Omega(x) q(x) \\ &\equiv (\partial_\alpha \Omega(x)) q(x) + \Omega(x) \partial_\alpha q(x) + igt \cdot \mathcal{A}'_\alpha \Omega(x) q(x) \end{aligned}$$

where  $t \cdot \mathcal{A}_\alpha \equiv \sum_A t^A \mathcal{A}_\alpha^A$ . Hence

$$t \cdot \mathcal{A}'_\alpha = \Omega(x) t \cdot \mathcal{A}_\alpha \Omega^{-1}(x) + \frac{i}{g} (\partial_\alpha \Omega(x)) \Omega^{-1}(x) .$$



- Transformation property of gluon field strength  $F_{\alpha\beta}$  is

$$t \cdot F_{\alpha\beta}(x) \rightarrow t \cdot F'_{\alpha\beta}(x) = \Omega(x) F_{\alpha\beta}(x) \Omega^{-1}(x) .$$

Contrast this with gauge-invariance of QED field strength. QCD field strength is not gauge invariant because of self-interaction of gluons. Carriers of the colour force are themselves coloured, unlike the electrically neutral photon.

- Note there is no gauge-invariant way of including a gluon mass. A term such as

$$m^2 \mathcal{A}^\alpha \mathcal{A}_\alpha$$

is not gauge invariant. This is similar to QED result for mass of the photon. On the other hand the quark mass term is gauge invariant.

# Feynman rules

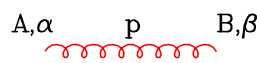
- Use free piece of QCD Lagrangian to obtain inverse quark and gluon propagators.
  - ★ Quark propagator in momentum space obtained by setting  $\partial^\alpha = -ip^\alpha$  for an incoming field. Result is in Table 1. The  $i\varepsilon$  prescription for pole of propagator is determined by causality, as in QED.
  - ★ Gluon propagator impossible to define without a choice of gauge. The choice

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\lambda} \left( \partial^\alpha \mathcal{A}_\alpha^A \right)^2$$

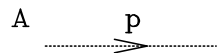
defines *covariant gauges* with gauge parameter  $\lambda$ . Inverse gluon propagator is then

$$\Gamma_{\{AB, \alpha\beta\}}^{(2)}(p) = i\delta_{AB} \left[ p^2 g_{\alpha\beta} - \left(1 - \frac{1}{\lambda}\right) p_\alpha p_\beta \right].$$

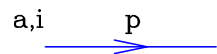
(Check that without gauge-fixing term this function would have no inverse.) Resulting propagator is in the table.  $\lambda = 1$  (0) is *Feynman (Landau) gauge*.



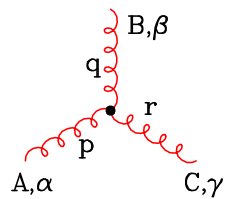
$$\delta^{AB} \left[ -g^{\alpha\beta} + (1-\lambda) \frac{p^\alpha p^\beta}{p^2 + i\epsilon} \right] \frac{i}{p^2 + i\epsilon}$$



$$\delta^{AB} \frac{i}{(p^2 + i\epsilon)}$$

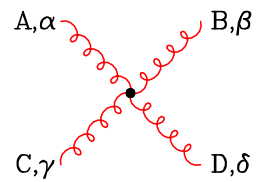


$$\delta^{ab} \frac{i}{(p^2 - m + i\epsilon)_{ji}}$$

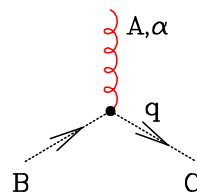


$$-g f^{ABC} [(p-q)^\gamma g^{\alpha\beta} + (q-r)^\alpha g^{\beta\gamma} + (r-p)^\beta g^{\gamma\alpha}]$$

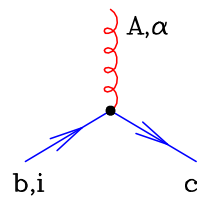
(all momenta incoming)



$$\begin{aligned} & -ig^2 f^{XAC} f^{XBD} [g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\delta} g^{\beta\gamma}] \\ & -ig^2 f^{XAD} f^{XBC} [g^{\alpha\beta} g^{\gamma\delta} - g^{\alpha\gamma} g^{\beta\delta}] \\ & -ig^2 f^{XAB} f^{XCD} [g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}] \end{aligned}$$



$$g f^{ABC} q^\alpha$$



$$-ig (t^A)_{cb} (\gamma^\alpha)_{ji}$$

- Gauge fixing explicitly breaks gauge invariance. However, in the end physical results will be independent of gauge. For convenience, we usually use Feynman gauge.
- In non-Abelian theories like QCD, covariant gauge-fixing term must be supplemented by a *ghost term* which we do not discuss here. Ghost field, shown by dashed lines in the above table, cancels unphysical degrees of freedom of gluon which would otherwise propagate in covariant gauges.
- Propagators determined from  $-S$ , interactions from  $S$ .

## Feynman rules – recipe

- Consider a theory which contains only a complex scalar field  $\phi$  and an action ( $iL$ ) which contains only bilinear terms,  $S = \phi^* (K + K') \phi$ .
- MOE: both  $K$  and  $K'$  are included in the free Lagrangian,  $S_0 = \phi^* (K + K') \phi$ . Using the above rule the propagator  $\Delta$  for the  $\phi$  field is given by

$$\Delta = \frac{-1}{K + K'}.$$

- JOE:  $K$  is regarded as the free Lagrangian,  $S_0 = \phi^* K \phi$ , and  $K'$  as the interaction Lagrangian,  $S_I = \phi^* K' \phi$ . Now  $S_I$  is included to all orders in perturbation theory by inserting the interaction term an infinite number of times:

$$\begin{aligned}\Delta &= \frac{-1}{K} + \left(\frac{-1}{K}\right) K' \left(\frac{-1}{K}\right) + \left(\frac{-1}{K}\right) K' \left(\frac{-1}{K}\right) K' \left(\frac{-1}{K}\right) + \dots \\ &= \frac{-1}{K + K'}\end{aligned}$$

# Alternative choice of gauge

- An alternative choice of gauge fixing is provided by the *axial gauges* which are fixed in terms of another vector which we denote by  $b$

$$\mathcal{L}_{\text{gauge-fixing}} = -\frac{1}{2\lambda} \left( b^\alpha \mathcal{A}_\alpha^A \right)^2,$$

The advantage of the axial class of gauge is that ghost fields are not required. However one pays for this simplicity because the gluon propagator is more complicated. The inverse propagator is

$$\Gamma_{\{AB, \alpha\beta\}}^{(2)}(p) = i\delta_{AB} \left[ p^2 g_{\alpha\beta} - p_\alpha p_\beta + \frac{1}{\lambda} b_\alpha b_\beta \right].$$

The inverse of this matrix gives the gauge boson propagator,

$$\Delta_{\{BC, \beta\gamma\}}^{(2)}(p) = \delta_{BC} \frac{i}{p^2} \left[ -g_{\beta\gamma} + \frac{b_\beta p_\gamma + p_\beta b_\gamma}{b \cdot p} - \frac{(b^2 + \lambda p^2) p_\beta p_\gamma}{(b \cdot p)^2} \right].$$

Notice the new singularities at  $b \cdot p = 0$ .

- What are the properties of these gauges which make them interesting? Let us specialize to the case  $\lambda = 0, b^2 = 0$ , (light-cone gauge).

$$\Delta_{\{BC, \beta\gamma\}}^{(2)}(p) = \delta_{BC} \frac{i}{p^2} d_{\beta\gamma}(p, b)$$

where

$$d_{\beta\gamma} = -g_{\beta\gamma} + \frac{b_\beta p_\gamma + p_\beta b_\gamma}{b \cdot p} .$$

In the limit  $p^2 \rightarrow 0$  we find that

$$b^\beta d_{\beta\gamma}(p, b) = 0, \quad p^\beta d_{\beta\gamma}(p, b) = 0 .$$

Only two physical polarization states, orthogonal to  $b$  and  $p$ , propagate. For this reason these classes of gauges are called physical gauges. In the  $p^2 \rightarrow 0$  limit we may decompose the numerator of the propagator into a sum over two polarizations:

$$d_{\alpha\beta} = \sum_i \varepsilon_\alpha^{(i)}(p) \varepsilon_\beta^{(i)}(p) .$$

In addition to the constraint  $\varepsilon_\beta^{(i)}(p) p^\beta = 0$ , which is always true, in an axial gauge we have the further constraint  $\varepsilon_\beta^{(i)}(p) b^\beta = 0$ .

# Running coupling

- Consider dimensionless physical observable  $R$  which depends on a single large energy scale,  $Q \gg m$  where  $m$  is any mass. Then we can set  $m \rightarrow 0$  (assuming this limit exists), and dimensional analysis suggests that  $R$  should be independent of  $Q$ .
- This is not true in quantum field theory. Calculation of  $R$  as a perturbation series in the coupling  $\alpha_S = g^2/4\pi$  requires renormalization to remove ultraviolet divergences. This introduces a second mass scale  $\mu$  — point at which subtractions which remove divergences are performed. Then  $R$  depends on the ratio  $Q/\mu$  and is not constant. The renormalized coupling  $\alpha_S$  also depends on  $\mu$ .
- But  $\mu$  is arbitrary! Therefore, if we hold bare coupling fixed,  $R$  cannot depend on  $\mu$ . Since  $R$  is dimensionless, it can only depend on  $Q^2/\mu^2$  and the renormalized coupling  $\alpha_S$ . Hence

$$\mu^2 \frac{d}{d\mu^2} R\left(\frac{Q^2}{\mu^2}, \alpha_S\right) \equiv \left[ \mu^2 \frac{\partial}{\partial \mu^2} + \mu^2 \frac{\partial \alpha_S}{\partial \mu^2} \frac{\partial}{\partial \alpha_S} \right] R = 0 .$$



■ Introducing

$$\tau = \ln \left( \frac{Q^2}{\mu^2} \right), \quad \beta(\alpha_S) = \mu^2 \frac{\partial \alpha_S}{\partial \mu^2},$$

we have

$$\left[ -\frac{\partial}{\partial \tau} + \beta(\alpha_S) \frac{\partial}{\partial \alpha_S} \right] R = 0.$$

This renormalization group equation is solved by defining running coupling  $\alpha_S(Q)$ :

$$\tau = \int_{\alpha_S}^{\alpha_S(Q)} \frac{dx}{\beta(x)}, \quad \alpha_S(\mu) \equiv \alpha_S.$$

Then

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = \beta(\alpha_S(Q)), \quad \frac{\partial \alpha_S(Q)}{\partial \alpha_S} = \frac{\beta(\alpha_S(Q))}{\beta(\alpha_S)}.$$

and hence  $R(Q^2/\mu^2, \alpha_S) = R(1, \alpha_S(Q))$ . Thus all scale dependence in  $R$  comes from running of  $\alpha_S(Q)$ .

- We shall see QCD is asymptotically free:  $\alpha_S(Q) \rightarrow 0$  as  $Q \rightarrow \infty$ . Thus for large  $Q$  we can safely use perturbation theory. Then knowledge of  $R(1, \alpha_S)$  to fixed order allows us to predict variation of  $R$  with  $Q$ .

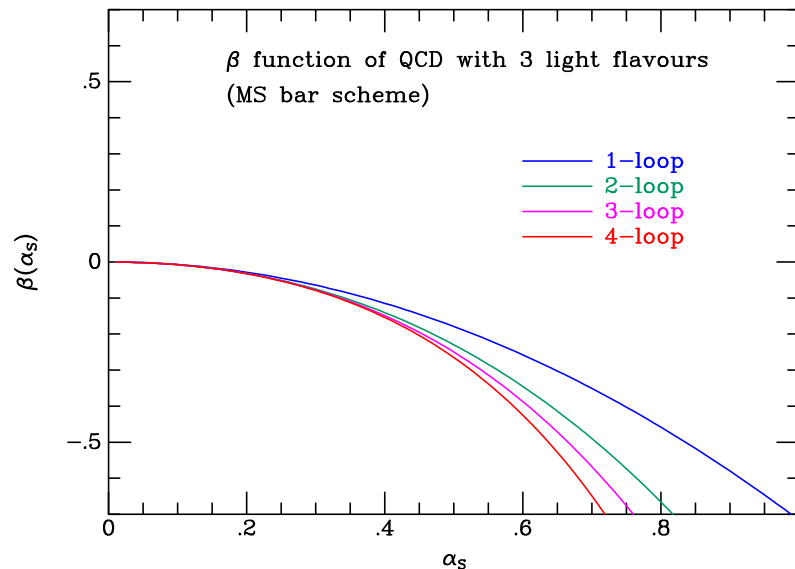
# Beta function

- Running of the QCD coupling  $\alpha_S$  is determined by the  $\beta$  function, which has the expansion

$$\beta(\alpha_S) = -b\alpha_S^2(1 + b'\alpha_S) + \mathcal{O}(\alpha_S^4)$$

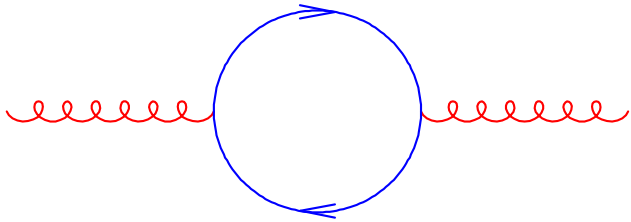
$$b = \frac{(11C_A - 2N_f)}{12\pi}, \quad b' = \frac{(17C_A^2 - 5C_A N_f - 3C_F N_f)}{2\pi(11C_A - 2N_f)},$$

where  $N_f$  is number of “active” light flavours. Terms up to  $\mathcal{O}(\alpha_S^5)$  are known.

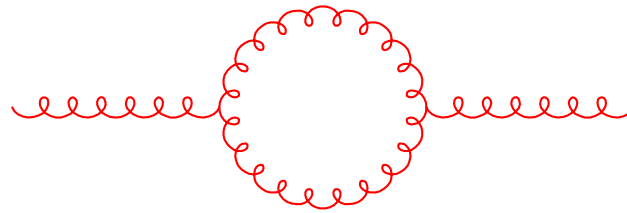


- ★ if  $\frac{d\alpha_S}{d\tau} = -b\alpha_S^2(1 + b'\alpha_S)$  and  $\alpha_S \rightarrow \bar{\alpha}_S(1 + c\bar{\alpha}_S)$ , it follows that  $\frac{d\bar{\alpha}_S}{d\tau} = -b\bar{\alpha}_S^2(1 + b'\bar{\alpha}_S) + \mathcal{O}(\bar{\alpha}_S^4)$
- ★ first two coefficients  $b, b'$  are thus invariant under scheme change.

# Asymptotic freedom



(a)



(b)

- Roughly speaking, quark loop diagram (a) contributes negative  $N_f$  term in  $b$ , while gluon loop (b) gives positive  $C_A$  contribution, which makes  $\beta$  function negative overall.
- QED  $\beta$  function is  $\beta_{QED}(\alpha) = \frac{1}{3\pi}\alpha^2 + \dots$   
Thus  $b$  coefficients in QED and QCD have opposite signs.
- From earlier slides,

$$\frac{\partial \alpha_S(Q)}{\partial \tau} = -b\alpha_S^2(Q) \left[ 1 + b'\alpha_S(Q) \right] + \mathcal{O}(\alpha_S^4).$$

Neglecting  $b'$  and higher coefficients gives  $\alpha_S(Q) = \frac{\alpha_S(\mu)}{1 + \alpha_S(\mu)b\tau}$ ,  $\tau = \ln\left(\frac{Q^2}{\mu^2}\right)$ .

# Asymptotic freedom

- As  $Q$  becomes large,  $\alpha_S(Q)$  decreases to zero: this is asymptotic freedom. Notice that sign of  $b$  is crucial. In QED,  $b < 0$  and coupling *increases* at large  $Q$ .
- Including next coefficient  $b'$  gives implicit equation for  $\alpha_S(Q)$ :

$$b\tau = \frac{1}{\alpha_S(Q)} - \frac{1}{\alpha_S(\mu)} + b' \ln\left(\frac{\alpha_S(Q)}{1 + b'\alpha_S(Q)}\right) - b' \ln\left(\frac{\alpha_S(\mu)}{1 + b'\alpha_S(\mu)}\right)$$

which can be solved numerically, (say using Newton's method).

- What type of terms does the solution of the renormalization group equation take into account in the physical quantity  $R$ ?  
Assume that  $R$  has perturbative expansion

$$R = \alpha_S + \mathcal{O}(\alpha_S^2)$$

Solution  $R(1, \alpha_S(Q))$  can be re-expressed in terms of  $\alpha_S(\mu)$ :

$$\begin{aligned} R(1, \alpha_S(Q)) &= \alpha_S(\mu) \sum_{j=0}^{\infty} (-1)^j (\alpha_S(\mu) b\tau)^j \\ &= \alpha_S(\mu) \left[ 1 - \alpha_S(\mu) b\tau + \alpha_S^2(\mu) (b\tau)^2 + \dots \right] \end{aligned}$$

Thus there are logarithms of  $Q^2/\mu^2$  which are automatically resummed by using the running coupling. Neglected terms have fewer logarithms per power of  $\alpha_S$ .

# Lambda parameter

- Perturbative QCD tells us how  $\alpha_S(Q)$  varies with  $Q$ , but its absolute value has to be obtained from experiment. Nowadays we usually choose as the fundamental parameter the value of the coupling at  $Q = M_Z$ , which is simply a convenient reference scale large enough to be in the perturbative domain.
- Also useful to express  $\alpha_S(Q)$  directly in terms of a dimensionful parameter (constant of integration)  $\Lambda$ :

$$\ln \frac{Q^2}{\Lambda^2} = - \int_{\alpha_S(Q)}^{\infty} \frac{dx}{\beta(x)} = \int_{\alpha_S(Q)}^{\infty} \frac{dx}{bx^2(1 + b'x + \dots)}.$$

Then (if perturbation theory were the whole story)  $\alpha_S(Q) \rightarrow \infty$  as  $Q \rightarrow \Lambda$ . More generally,  $\Lambda$  sets the scale at which  $\alpha_S(Q)$  becomes large.

- In leading order (LO) keep only first  $\beta$ -function  $b$ :

$$\alpha_S(Q) = \frac{1}{b \ln(Q^2/\Lambda^2)} \quad (\text{LO}).$$

- In next-to-leading order (NLO) include also  $b'$ :

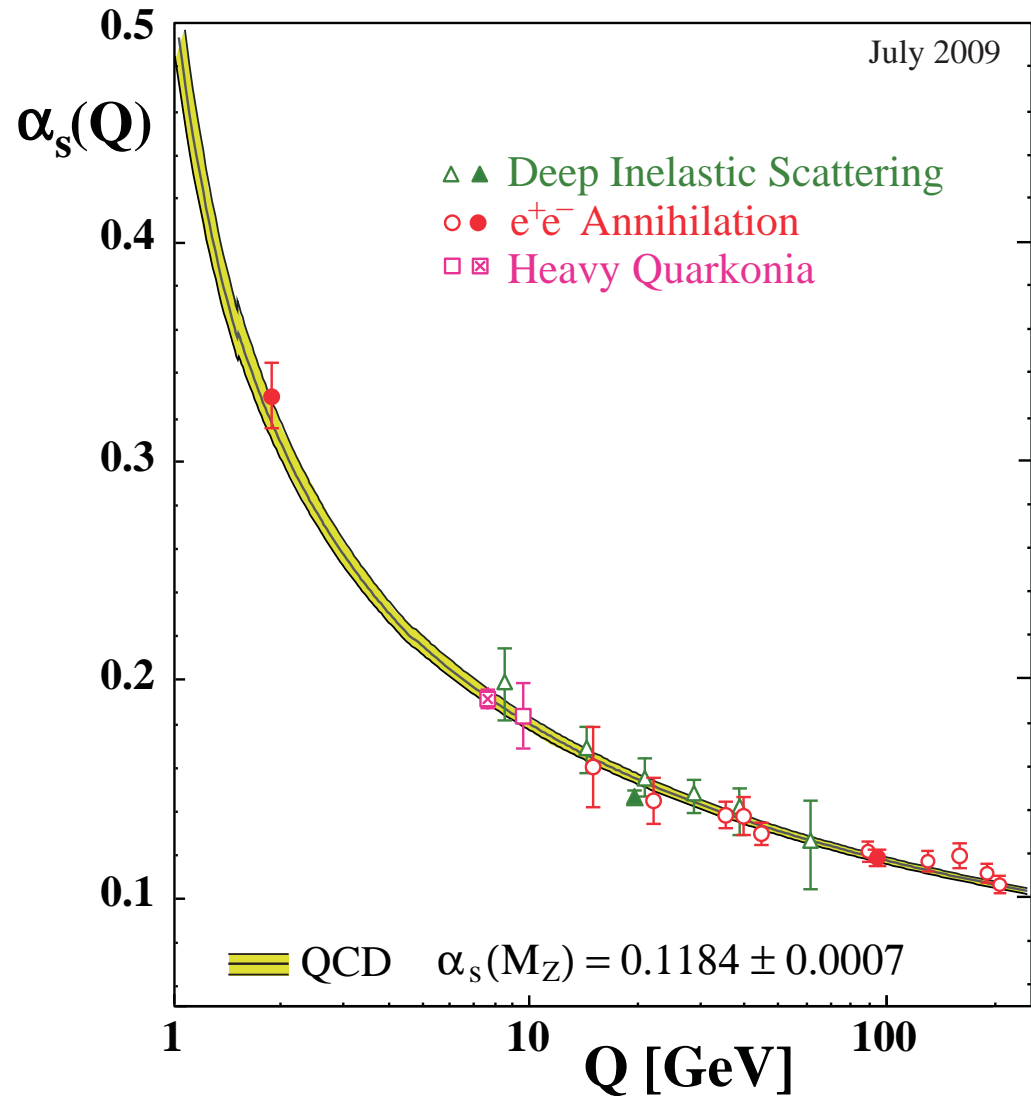
$$\frac{1}{\alpha_S(Q)} + b' \ln\left(\frac{b' \alpha_S(Q)}{1 + b' \alpha_S(Q)}\right) = b \ln\left(\frac{Q^2}{\Lambda^2}\right).$$

This can be solved numerically, or we can obtain an approximate solution to second order in  $1/\log(Q^2/\Lambda^2)$ :

$$\alpha_S(Q) = \frac{1}{b \ln(Q^2/\Lambda^2)} \left[ 1 - \frac{b'}{b} \frac{\ln \ln(Q^2/\Lambda^2)}{\ln(Q^2/\Lambda^2)} \right] \quad (\text{NLO}).$$

This is Particle Data Group (PDG) definition.

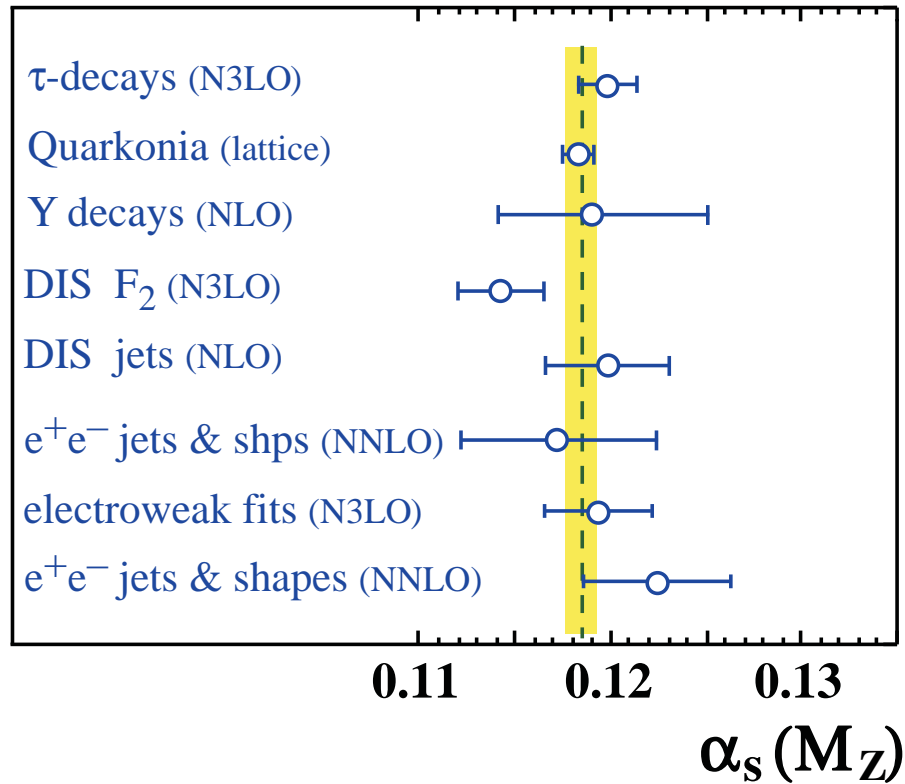
- Note that  $\Lambda$  depends on number of active flavours  $N_f$ . ‘Active’ means  $m_q < Q$ . Thus for  $5 < Q < 175$  GeV we should use  $N_f = 5$ . See ESW for relation between  $\Lambda$ 's for different values of  $N_f$ .
- The use of  $\Lambda$  is full of pitfalls; however from a parametric point of view it is a fundamental quantity in QCD. The scale of masses of hadrons, protons, neutrons etc are determined primarily by  $\Lambda$  QCD. Thus QCD is responsible for the mass of the observed universe.



- Measurements of  $\alpha_S$  are reviewed in ESW. A more recent compilation from arXiv:0908.1135v1 is shown above. Evidence that  $\alpha_S(Q)$  has a logarithmic fall-off with  $Q$  is persuasive.



# $\alpha_S$ at $m_Z$



$$\alpha_S(M_Z) = 0.1184 \pm 0.0007, \quad \text{Bethke, arXiv:0908.1135v1}$$

# Recap

- Colour degree of freedom is motivated by  $R e^+ e^-$ , baryon spectroscopy.
- QCD is an SU(3) gauge theory of quarks (3 colours) and gluons (8 colours, self interacting)
- Renormalization of dimensionless observables depending on a single large scale implies that the scale dependence enters through the running coupling.
- The scale  $\Lambda$  is a fundamental parameter of QCD, but its practical use is fraught with difficulties. It is better to  $\alpha_S(M_Z)$  to specify the size of  $\alpha_S$ .
- Asymptotic freedom implies that some high scale quantities can be calculated in perturbation theory.
- $\alpha(M_Z) \simeq 0.118$  in five flavour  $\overline{MS}$ -renormalization scheme.